Chapter 3

ELECTROMAGNETIC PROCESSES AS RELATED TO COSMIC RAY PHENOMENA

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3.1 Introduction

Various phenomena induced during the propagation of cosmic rays in interstellar space are mostly considered as the results of the interactions of cosmic ray particles with gases or electromagnetic waves ambient in this space. At present, there are four known kinds of primary interactions in particle physics. They are described in the order of their weakness as follows;

1. Gravitational interaction,
2. Weak interaction,
3. Electromagnetic interaction,
4. Strong interaction.

Among them, the interaction by gravity is not important in cosmic ray phenomena except for the strong gravitational fields generated near blackholes and neutron stars. The weak interaction plays an important role in the phenomena associated with neutrinos. It is now known that there exists the possibility for numerous high-energy neutrinos to be released into outer space in association with supernova explosions. Since these high-energy neutrinos should be considered to be one of the members constituting cosmic rays, several plans have been proposed worldwide to investigate high-energy neutrino astronomy by making observations of these neutrinos. As regards cosmic ray research in general, however, the effect of the weak interaction is always negligible, which may be seen in their propagation. Sometimes, the neutrinos resulting from the decay of pions and muons produced from the strong interaction become the subject of the observations.

In comparison with the weak interaction, the coupling constant of the electromagnetic interaction \( \frac{e^2}{\hbar c} \approx 137 \) is relatively large and the Coulomb force resulting from this interaction can propagate to a great distance. Thus, various kinds of electromagnetic phenomena are produced in interstellar space as a result of the action of this force. Electromagnetic waves resulting from this interaction cover a wide frequency range from radio waves, via X-rays, to \( \gamma \)-rays. At present, all these waves from space are being investigated to dissolve the secret of the universe.
Since the strong interaction is easily seen in interstellar space through cosmic ray collisions with interstellar gases, this interaction has to be taken into account in the studies of the propagation of cosmic rays. Through the fragmentations of cosmic ray nuclei due to their collisions just mentioned, light nuclei are yielded from the nuclear component of cosmic rays. Since many mesons are also simultaneously yielded from these collisions, high-energy γ-rays and electrons are produced secondarily from the decay of these mesons. These phenomena are now considered as essentially important in the consideration of cosmic ray propagation in galactic space and of the structure of this space. However, our consideration will be here limited only to various processes as related to the electromagnetic interactions which seem to occur in galactic space, because they seem most important and useful to the understanding of cosmic ray phenomena. Though the elementary processes of these interactions will be mainly dealt with in this chapter, some processes other than those related to these interactions will be considered if they are important to cosmic ray research.

3.1.1 The unit of energy

In human society, energy is measured by the unit defined as Joule (J) or kilo-Joule (kJ), since this unit is well fitted to the human scale. An energy of 1 J can raise the temperature of 1 cm³ of water by 0.24°C. In the realm of particle physics, however, the unit defined as the electron volt (eV) is mostly used, because Joule units are too large to measure the energy of particles. The kinetic energy gained by an electron or a proton accelerated in an electric potential difference of one volt corresponds to the magnitude of 1 eV. When expressed by the unit J, 1 eV is transformed as

\[ 1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg} = 1.6 \times 10^{-19} \text{ J} \]  

(3.1.1)

Since a particle of energy \(10^{20}\) eV has an energy of about 6 J in the human scale, the temperature of 1 cm³ of water could be raised by about 1°C if the water would perfectly absorb this energy. When the unit eV seems inconvenient because of its smallness, other units such as keV, MeV, GeV and TeV are used. They are expressed as follows; 1 keV = \(10^3\) eV, 1 MeV = \(10^6\) eV, 1 GeV = \(10^9\) eV and 1 TeV = \(10^{12}\) eV. For instance, it can be said that the particle accelerator at the National Institute of High-Energy Physics in Tsukuba is able to accelerate a proton to an energy of 12 GeV.

A particle of mass \(m\) has rest energy \(mc^2\). Therefore, it is often convenient to express the mass in terms of energy. The masses and energies of the electron, proton and pi-meson are shown in Table 3.1.1 with the physical quantities important to the elementary processes in particle interactions. Since the rest energies for the electron, proton and pi-meson are, respectively, 0.5 MeV, 1 GeV and 140 MeV, it is possible to estimate the order of magnitude of the energies for various particle interactions.

3.1.2 The unit of length

Due to the uncertainty principle in quantum mechanics, it is impossible to fix the position of a particle with a definite momentum. If the uncertainty of the order of mc
is allowed in the measurement of the momentum, the size, \( r \) of an elementary particle is roughly estimated as

\[
  r \sim \frac{\hbar}{mc}.
\]  

(3.1.2)

The numerical values for the sizes of several particles are summarized in Table 3.1.2. It is clear that particle size becomes larger with increase of mass. Since its mass is much heavier, the size of a proton is relatively smaller, as seen from this Table. However, the real size of a proton is larger compared with that estimated above, because of its internal structure. In consequence, the size of a pi-meson gives the order of the size of a proton.

Next, consider the size of an electron from the classical point of view. In order that the charge of an electron (\( e \)) is confined with an area of the radius \( r \), it is necessary to consume energy of amount \( e^2 / r \). When this energy is assumed to be equal to the rest energy of electron, the following result is obtained for the electron radius

\[
  r_0 = \frac{e^2}{mc^2} = 2.818 \times 10^{-13} \text{ cm}.
\]  

(3.1.3)
This is called the classical radius of electron, which often appears in the calculation for the electromagnetic interactions.

### 3.1.3 The cross-section and the mean-free paths

The cross-section and the mean-free paths are important physical quantities together with those which indicate the rates for various phenomena to occur due to the interactions between the elementary particles. The origin of the term “cross-section” lies in the fact that the efficiency of the reaction process which is produced by a particle depends on the “effective” area extended by this particle. For instance, as will be discussed later, the cross-section for the Thomson scattering which occurs as a result of the scattering of photons by an electron is expressed as $6.65 \times 10^{-25}$ cm$^2$. This means that the effective area of an electron is observed to be this when viewed from photons, but it should not be thought that photons, which have impinged upon the area of $6.65 \times 10^{-25}$ cm$^2$ surrounding the free electron, are only scattered by this electron. Although the momentum and the location of the electron cannot be specifically determined together in quantum mechanics, it is allowed to say that photons passing by some area around the virtual location of the electron are scattered with some probability. Thus, the cross-section for the scattering is calculated by summing up all of the effective areas experienced by each of the photons to be scattered. The unit of the cross-section is given by the term “barn” in the unit of $10^{-24}$ cm$^2$, since the expression by cm$^2$ is not appropriate to the smallness of the cross-section. Thus, the cross-section of the Thomson scattering is expressed as 665 mb. As regards the photons being scattered to some specific direction from the free electron, the “differential” cross-section is in use which gives the cross-section to a specific angular distance of the scattering.

When the magnitude of the cross-section $\sigma$ is numerically given, the “mean free path” of a photon, which gives the mean distance for this photon to be scattered once in the motion, is calculated by assuming that the number of target electrons per 1 cm$^2$ is $1/\sigma$. This mean-free path is expressed in the unit of g/cm$^2$ in general. Since the number of any molecules in a mole is given by the Avogadro number $N (=6.02 \times 10^{23})$, the number of electrons for 1 g/cm$^2$ in the matter which has the atomic number $Z$ and the mass number $A$ is expressed as

$$\frac{NZ}{A}.$$  

Therefore, the mean-free path $l$(g/cm$^2$) is obtained as

$$l = \frac{1}{\sigma} \frac{A}{NZ}.$$  \hspace{1cm} (3.1.4)

In the case of Thomson scattering, if we assume as $Z=A/2$ for the matter usually available, this path is estimated as, by taking $\sigma=6.65 \times 10^{-25}$ cm$^2$ into account,
\[ l \approx 5 \text{ g/cm}^2. \]

Namely, this result indicates that a photon is usually scattered once while it is passing through matter with a thickness of 5 g/cm\(^2\).

Let us consider now the proton-proton collision, as the size of proton has been briefly discussed earlier. Since this collision occurs due to the strong interaction, the range is limited to the domain close to protons. Thus, the cross-section of this collision is estimated as

\[ \pi \left( \frac{\hbar}{m \pi c^2} \right)^2 \approx 6 \times 10^{-26} \text{ cm}^2 = 60 \text{ mb}. \]

According to the accelerator experiments, the cross-section of this collision is given as \( \sim 40 \text{ mb} \). Therefore, the mean-free path of a proton in hydrogen gas is estimated as about 42 g/cm\(^2\).

Since, as mentioned above, the mean-free path defines a physical quantity felt more practical as compared to the cross-section, this quantity is often made use of in cosmic ray research. The Avogadro number is about equal to the reciprocal of the cross-section as expressed in the unit of barn (10\(^{-24}\) cm\(^2\)). Thus a cross-section larger than the order of mb (10\(^{-27}\) cm\(^2\)) generally gives a mean-free path practical in the research. In the cases where the cross-section is smaller than this order, it is estimated that no reaction usually occurs in matter with thickness of about 1 kg/cm\(^2\).

### 3.1.4 The Lorentz transformation

In considering some of the elementary processes produced by light or high-speed particle, it is sometimes more convenient to take some coordinate system other than the laboratory one, like the one in which the particle to be dealt with is at rest, for instance. In so doing, the Lorentz transformation must be applied to transform from the one system to the other that is moving with a constant velocity as viewed from the first system. This transformation becomes equivalent to the Galilei transformation in any case in which this velocity is negligibly small compared with the speed of light.

#### a) The coordinate transformation

Let us assume now that the coordinate system \( K' \) (denoted by the coordinates; \( x', y', z', t' \)) is moving with a constant speed \( v = \beta c \) along the \( x \)-axis with respect to the other system \( K \) (denoted by the coordinates; \( x, y, z, t \)) (Fig. 3.1.1). In the case where \( \beta \leq 1 \), the formulae for the Galilei transformation are obtained as

\[ x' = x - vt, \quad y' = y, \quad z' = z \quad \text{and} \quad t' = t. \]

(3.1.5)

In contrast, the Lorentz transformation is expressed as

\[ x' = \frac{x - \beta ct}{\sqrt{1 - \beta^2}}, \quad y' = y, \quad z' = z \quad \text{and} \quad ct' = \frac{ct - \beta x}{\sqrt{1 - \beta^2}}, \]

(3.1.6)
where the quantity $c$ is the speed of light and $\gamma = 1/\sqrt{1 - \beta^2}$ is often called the Lorentz factor. Now, let us observe from the system $K$ the moving body being fixed in the system $K'$. The length of this body $l = x'_1 - x'_2$ as seen in the system $K'$ along the $x$-axis is expressed in the system $K$ as, via the Lorentz transformation,

$$x_1 - x_2 = (x'_1 - x'_2)\sqrt{1 - \beta^2} = l\sqrt{1 - \beta^2}. \quad (3.1.7)$$

This length is observed to be contracted by the factor $\sqrt{1 - \beta^2}$. This is the so-called Lorentz contraction. The passage of time $\tau'$ associated with this body in motion is transformed as $\tau$ in the rest frame $K$; this time $\tau$ is obtained, by substituting the relation $x = \beta ct$ into the last of Eq. (3.1.6) as

$$\tau = \frac{\tau'}{\sqrt{1 - \beta^2}} = \gamma \tau'. \quad (3.1.8)$$

This means that the passage of time is slowed down by $\gamma$-times. It is known that $\mu$-mesons, which are produced from $\pi$-mesons, decay into electrons and neutrinos in a mean life time $\tau' \approx 2.21 \times 10^{-6}$ sec. It seems that the flight paths of these mesons are only $c\tau' = 663$ m even if they move with the speed of light, but high-energy $\mu$-mesons ($\gamma \approx 100$) can mostly reach the ground without decay since they are generated in the upper atmosphere at about 20 to 30 km altitude. This is a result of the slowing down in the passage of time due to the Lorentz transformation.

b) The transformations of momentum and energy

When a particle of mass $m$ is moving with the velocity $v^* = \beta^* c$, its momentum and energy are, respectively, given by

$$P = \frac{mc\beta^*}{\sqrt{1 - \beta^*^2}} \quad \text{and} \quad E = \frac{mc^2}{\sqrt{1 - \beta^*^2}}, \quad (3.1.9)$$
where $E$ is defined by the sum of the rest energy $mc^2$ and the kinetic energy. In order to resolve the momentum into three components of the space coordinates ($x$, $y$, $z$), it is only necessary to multiply the momentum $P$ with each of the direction cosines.

The three components of the momentum and the energy $(cP_x, cP_y, cP_z, iE)$ form the four-dimensional vectors. They follow the Lorentz transformation like the four-dimensional coordinate components ($x$, $y$, $z$, $ict$), since both of them form the four-dimensional vectors. In other words, when the three momenta and the energy as viewed from the frame moving with the speed $v=\beta c$ along the $x$-axis are given as $cP'_x$, $cP'_y$, $cP'_z$ and $iE'$, respectively, they are expressed by using the physical quantities in the rest frame, as

$$cP'_x = \frac{cP_x - \beta E}{\sqrt{1 - \beta^2}}, \ P'_y = P_y, \ P'_z = P_z \ \text{and} \ E' = \frac{E - \beta P_x c}{\sqrt{1 - \beta^2}}.$$ \hspace{1cm} (3.1.10)

It should be remarked that there is no change in the components of the momentum perpendicular to the $x$-axis, which is the coordinate to be transformed.

In the case where the particle is at rest in the coordinate system $K'$, it is observed to move with a velocity $v=\beta c$ in the system $K$. Thus,

$$P_x = \frac{mc\beta}{\sqrt{1 - \beta^2}}, \ P_y = 0, \ P_z = 0 \ \text{and} \ E = \frac{mc^2}{\sqrt{1 - \beta^2}}.$$  

When the above transformation is applied to the result shown in Eq. (3.1.10), the following result is reduced in the coordinate system $K'$;

$$P'_x = 0, \ P'_y = 0, \ P'_z = 0 \ \text{and} \ E' = mc^2.$$ \hspace{1cm} (3.1.11)

The momentum and the energy of a photon, which is being dealt with quantum mechanically, are respectively expressed as $hf/c$ and $hf$, where $f$ is the frequency of this photon. When they are transformed to the frame $K'$, the following results are obtained.

The momentum: $hf'_x/c = h/c \left( \frac{f_x - \beta f}{\sqrt{1 - \beta^2}} \right) \ hf'_y/c$ and $hf'_z/c = hf_z/c$.

The energy: $hf' = h \left( \frac{f - \beta f_x}{\sqrt{1 - \beta^2}} \right)$.

(3.1.12)

When we observe the frequency in the coordinate system $K'$, it is transformed as follows:

$$f' = f \left[ \frac{1 - \beta f_x/f}{\sqrt{1 - \beta^2}} \right].$$ \hspace{1cm} (3.1.13)
This explains the so-called Doppler effect. When the direction into which the photon is propagating has an angle \( \theta \) with respect to the \( x \)-axis, the above result is reduced to

\[
f' = \gamma f (1 - \beta \cos \theta).
\]  
(3.1.14)

In the case where the photon propagates along the \( x \)-axis, this result is further reduced to

\[
f' = \gamma f (1 - \beta).
\]  
(3.1.15)

It is known that in our universe, the more distant is a galaxy, the higher is its receding velocity. When the Doppler shift is observed from any of the galaxies in the far distance, the light from the galaxies is Doppler-shifted to the red side in wavelength. By writing the shift of the wavelength \( \lambda \) by \( \Delta \lambda \), the parameter for the Doppler shift \( Z \) is defined as

\[
Z = \frac{\Delta \lambda}{\lambda}.
\]  
(3.1.16)

If the frequency of the red-shifted wave with wavelength \( \lambda + \Delta \lambda \) is given by \( f' \), this parameter is calculated as, by using the relation \( f \lambda = c \),

\[
1 + Z = \frac{\lambda + \Delta \lambda}{\lambda} = \frac{f}{f'} = \frac{1}{\gamma (1 - \beta)} = \sqrt{\frac{1 + \beta}{1 - \beta}}.
\]  
(3.1.17)

Thus,

\[
Z \approx \sqrt{\frac{1 + \beta}{1 - \beta}} - 1 \approx \beta, \text{ when } \beta \ll 1
\]

\[
\beta = \frac{(Z + 1)^2 - 1}{(Z + 1)^2 + 1} \approx Z, \text{ when } Z \ll 1.
\]

Although the parameter \( Z \) is almost proportional to \( \beta \) when \( Z \) is very small, it is necessary to use the exact formula for \( Z \) when \( Z \) becomes larger than 0.1.

Some examples in which the Lorentz transformation is applied will be discussed in the following. Let us first consider the case in which a photon of energy \( hf \) encounters an electron of energy \( E \) from the direction with angular distance \( \theta \) as measured from the direction of the electron motion (Fig. 3.1.2). Since the Lorentz factor \( \gamma \) of this electron is given by

\[
\gamma = \frac{E}{m_e c^2},
\]  
(3.1.18)

the photon energy \( hf^* \), expressed in the rest frame referred to the electron, is expressed as
$hf^* = \gamma hf(1 + \beta \cos \theta)$.  \hfill (3.1.19)

If the Lorentz factor $\gamma$ is large, visible light is seen as $\gamma$-rays, whose frequencies are much higher than those of visible light. If necessary, it is also possible to calculate the direction of the photon incidence in the frame referred to the electrons. As regards the elastic scattering of this photon by the electron, the photon of energy $\gamma^2 hf$ is observed in the laboratory system as a result of the Lorentz transformation. Consequently, the electron has to lose this amount of energy during the scattering. The process just mentioned is called the inverse Compton effect, which will be later described in more detail in Subsection 3.2.4. This is also an effective method of producing mono-energetic $\gamma$-rays by illuminating an energetic electron beam with visible light.

Since the momentum and energy constitute a four-dimensional vector, the summation of their squares is a scalar quantity and so becomes invariant with respect to the Lorentz transformation. As for the particle of mass $m$, the following relation is clearly satisfied,

$$E^2 - (P_x^2 + P_y^2 + P_z^2)c^2 = (mc^2)^2. \hfill (3.1.20)$$

For the case consisting of two particles, by defining their momenta and energies with suffixes 1 and 2, it follows that the following expression is also invariant with respect to the Lorentz transformation:

$$(E_1 + E_2)^2 - [(P_{x1} + P_{x2})^2 + (P_{y1} + P_{y2})^2 + (P_{z1} + P_{z2})^2]c^2. \hfill (3.1.21)$$

Since the term related to the momenta becomes zero in the center-of-mass system for these two particles, the above relation (3.1.21) is only given by the square of the summation of the energies of these two particles as referred to the center-of-mass system. Thus, it becomes possible to calculate the energy to be expended in producing the secondary particles. Now, we shall consider the case in which a proton with Lorentz factor $\gamma$ collides with the target proton at rest as shown in Fig. 3.1.3. In order to calculate the energy that can be expended to produce other particles as a result of the collision, the following technique is convenient, since the energy and the momentum of the colliding protons are respectively expressed as, in the laboratory system,
(a) Laboratory System

(b) Center-of-Mass System: $2\gamma*^2 - 1 = \gamma$

Fig. 3.1.3. Relation between the laboratory and the center-of-gravity systems.

\[
E_1 = m\gamma c^2, \quad P_x = m\gamma \beta c, \quad P_y = P_z = 0
\]
\[
E_2 = mc^2, \quad P_x = P_y = P_z = 0.
\]

Thus,

\[
(E_1 + E_2)^2 - \sum_{x,y,z} (P_x + P_x)^2 c^2 = 2(mc^2)^2(\gamma + 1).
\]  \hspace{1cm} (3.1.22)

If the Lorentz factor in the center-of-mass system is denoted by $\gamma^*$, the above result is reduced to

\[
(2m\gamma^* c^2)^2.
\]

From this result, the following relationship is obtained:

\[
2\gamma^*^2 - 1 = \gamma
\]  \hspace{1cm} (3.1.23)

It becomes clear from Eq. (3.1.23) that the energy to be expended in creating other particles is given by $2mc^2(\gamma^* - 1)$. To create a proton and antiproton pair, it follows that $\gamma^* > 2$; thus, $\gamma > 7$. Therefore, it is clear that, in the laboratory system the minimum energy of the incident proton is about equal to 5.6 GeV.

\(\hspace{1cm}\)

c) The Lorentz transformation of electromagnetic field

Electric and magnetic fields, being denoted by $E$ and $H$, respectively, are the physical quantities that are both derived from functions called the scalar and the vector potentials denoted by $\phi$ and $A$, respectively (see Subsection 3.2.1). Since these two potentials form the four-dimensional vectors, the Lorentz transformation of $E$ and $H$ is derived from the transformation of these vectors as shown in the last Subsection b. When the coordinate system $K'$ moving along the $x'$-axis with velocity $\beta c$ is considered, the formulae for the transformation is given as

\[
E'_x = E_x \quad \quad \quad \quad \quad \quad H'_x = H_x
\]
\[
E'_y = \gamma(E_y - \beta H_z) \quad \quad \quad \quad \quad \quad H'_y = \gamma(H_y + \beta E_z)
\]
\[
E'_z = \gamma(E_z + \beta H_y) \quad \quad \quad \quad \quad \quad H'_z = \gamma(H_z - \beta E_y).
\]  \hspace{1cm} (3.1.24)

It should be here remarked that both $E$ and $H$ never vary with the transformation in the direction of the motion of the coordinate system.* On the other hand, the lateral

* $E$ and $H$ constitute the four-dimensional anti-symmetric tensor.
components of $E$ and $H$ perpendicular to the direction of the motion are enhanced by the Lorentz factor. Furthermore, it is interesting that the lateral components of the electric field $E$ in the moving system are enhanced only by the lateral components of the magnetic field $H$, whereas these components of the magnetic field $H$ in the moving system are enhanced by those of the electric field $E$. For simplicity, let us assume here that there only exists the $z$-component of magnetic field $H_z$ in the rest system $K$ (no electric field). In this case, in the moving frame $K'$, the $y'$-component of electric field is induced and the $z'$-component of magnetic field is enhanced by the Lorentz factor $\gamma$. Namely, such relations are expressed as

$$
E' = E' = 0 , \quad H' = H' = 0
$$
$$
E' = \gamma \beta H_z , \quad H' = \gamma H_z .
$$
(3.1.25)

The reason why an electric field appears in the moving frame $K'$ is understood as described below. We now assume that the electron with Lorentz factor $\gamma$ is moving in the $x$-$y$ plane in the rest system $K$. This electron is moving circularly under the action of the Lorentz force $e\beta H_z$ in the $x$-$y$ plane. Next, consider the Lorentz transformation to the coordinate system $K'$ moving with velocity $\beta c$, with which the electron is assumed to be moving. In this case, the electron has to be necessarily at rest in this system, but receives the action of the force $e\gamma \beta H_z$ due to the electric field $\gamma \beta H_z$ induced in the $y'$-direction as shown by Eq. (3.1.25). The gain of its momentum after $\Delta t'$ sec in this direction is thus given by $e\gamma \beta H_z \Delta t'$. The momentum change in the $y$-direction as referred to the rest system is calculated as $e\beta H_z \Delta t$ as a result of the action of the Lorentz force. Because of the relation $\Delta t = \gamma \Delta t'$, it follows that both momenta in the rest and the moving systems are given by $P_y$. Therefore, the electric field $E'$ must exist in the moving system $K'$.

### 3.2 Electromagnetic Interactions

In the cosmic space, there are many electromagnetic processes produced by cosmic rays. For instance, the processes in which a photon is scattered by charged particles are Thomson- and Compton-scattering. When a photon collides with charged particles moving with relativistic velocity, the energy of the photon is increased on average. This is the process called the inverse Compton effect. While passing through a region where a weak magnetic field is applied, the orbit of an electron is deflected. As a result, this electron emits electromagnetic radiations, which are defined as the cyclotron emissions or the synchrotron emissions according to their frequency spectra. The loss of electron energy due to ionization and radiation is also important in the interaction between charged particles.

In addition to these processes, phenomena such as the photoelectric effect, the black-body radiation from bright stars, and electron-positron pair creation from high-energy $\gamma$-rays are also substantially electromagnetic.

Due to the recent progress in quantum electrodynamics, the cross-sections to the elementary processes related to electromagnetic interactions are now calculated with
higher accuracies, though these calculations are usually very complex. In consequence, the elementary processes due to these interactions are now the important means of understanding the nature of the various phenomena which are observed in the cosmic space.

There exist some elementary processes in which the quantum mechanical effect plays no important role. For instance, such a case is that in which the energy of photons being emitted from charged particles is negligibly low as compared with the particle energy. In this case, the wave properties of the particles have no influence on the processes, since the de Broglie wavelength is much smaller than that of photons. Therefore, the classical treatment gives exact results in such cases.

3.2.1 Radiations from a charge particle in acceleration

a) Liénard-Wiechert Potentials

A charged particle at rest never emits electromagnetic waves, though it generates a Coulomb electric field around it. A charged particle moving uniformly as the result of the Lorentz transformation into the coordinate system moving with a constant velocity in a vacuum also does not radiate electromagnetic wave. In contrast, charged particles always emit electromagnetic waves when accelerated. To deal with this process, we may begin with the Maxwell equations. By introducing the scalar and the vector potentials, $\phi$ and $A$, as usually done, and using the Lorentz conditions as

$$\text{div } A + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0,$$

the equations for $\phi$ and $A$ are expressed as

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -4\pi \rho$$  \hspace{1cm} (3.2.1)

$$\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A = -\frac{4\pi}{c} \rho \nu.$$  \hspace{1cm} (3.2.2)

where $\rho$ and $\nu$ are, respectively, the charge density and the velocity of the electron. The electric and magnetic fields are respectively given by

$$E = -\nabla \phi - \frac{1}{c} \frac{\partial A}{\partial t}$$  \hspace{1cm} (3.2.3)

and

$$H = \text{rot } A.$$  \hspace{1cm} (3.2.4)

*Charged particles emit electromagnetic waves during their passage through dispersive media, due to their interaction with such media.
The formulae as expressed by Eqs. (3.2.1) and (3.2.2) indicate that these potentials, \( \varphi \) and \( A \), both propagate with the velocity of light. Here, consider a charged particle moving with velocity \( v \) along the \( z \)-axis. When we consider \( \varphi \) and \( A \) at the vectorial position \( r \) from the particle at time \( t' \), these two potentials are respectively expressed as

\[
\varphi = \left. \int \frac{\rho}{r} \, dV \right|_{t=t-(t'/c)} \tag{3.2.5}
\]

and

\[
A_z = \left. \int \frac{\rho v}{rc} \, dV \right|_{t=t-(t'/c)}. \tag{3.2.6}
\]

Though these above formulae give \( \varphi \) and \( A_z \) at time \( t \) at the observing point, they show that the physical quantity which influences both \( \varphi \) and \( A_z \) is the charge distribution at the time earlier by \( r/c \), which is necessary for the electromagnetic fields to propagate between the positions of the charge and of the observation. When the charges are distributed spatially, the effects from their respective locations are additive. In order to calculate these effects, it is only necessary to deal with the wave front which converges to the point of observation with the speed of light. To integrate Eqs. (3.2.5) and (3.2.6), we shall here consider the volume element \( dV \) as a spherical shell which is perpendicular to the position vector \( r \) as shown in Fig. 3.2.1. If the thickness of this shell is assumed to be \( \Delta r \), it takes time \( \Delta t = \Delta r/c \) until the spherical wave converging to the outer boundary of the shell passes through this thickness and appears in the inner boundary. Since the charged particles move by a distance \( v \Delta t \) during this time, a part of these particles, as shown in the following, invade inside the inner boundary of this shell:

\[
\rho \left( \frac{rv}{r} \right) \, dsdt = \rho \left( \frac{rv}{r} \right) \, ds \frac{dr}{c} = \rho \left( \frac{rv}{rc} \right) \, dV,
\]

Fig. 3.2.1. Contraction of wave packet.
where $ds$ is the surface element of this spherical shell. Thus, the charge element due to encounter with the spherical wave converging with the speed of light is given as

$$de = \rho \left(1 - \frac{(rv)}{rc}\right) dV = \rho(1 - \beta \cos \theta) dV,$$

(3.2.7)

where $v/c = \beta$. By integrating Eq. (3.2.7), we obtain the result

$$\int \rho dV = \frac{e}{(1 - \beta \cos \theta)}.$$

(3.2.8)

Comparing this result with Eqs. (3.2.5) and (3.2.6), the potentials, $\varphi$ and $A_z$, are reduced to

$$\varphi = \frac{e}{r(1 - \beta \cos \theta)},$$

(3.2.9)

and

$$A_z = \frac{ev}{re(1 - \beta \cos \theta)}.$$

(3.2.10)

Since these equations were independently obtained by Liénard in 1889 and Wiechert in 1901, they are now called the Liénard-Wiechert potentials. It should be noted that, in the classical electromagnetic theory, they are very important in calculating the electromagnetic fields generated by a moving charged particle.

**b) Radiation from a charged particle**

Using the scalar and the vector potentials, $\varphi$ and $A$, we can calculate the electromagnetic fields from Eqs. (3.2.3) and (3.2.4). In these equations, however, the differentiation has to be done with respect to the time and the space at the point of observation, whereas the Liénard-Wiechert potentials are expressed using the time and the space as referred to the charged particles at the point of observation. In differentiating Eqs. (3.2.3) and (3.2.4), therefore, it is necessary to make a coordinate system transform. Because this operation is somewhat complicated, readers should refer to some other book on it.* Assuming here that the velocity of the charged particle $v$ is very small as compared to that of light, the term of the first order as regards both $\beta$ and $1/r$ is only considered, because of the complexity of mathematical treatment. The electromagnetic fields expressed in the polar coordinate system are written as

$$E_\theta = \frac{e}{rc} \hat{\beta} \sin \theta$$

(3.2.11)

*e.g., refer to Heitler\textsuperscript{11} in the references.
and

\[ H_\phi = \frac{e}{rc} \dot{\beta} \sin \theta, \quad (3.2.12) \]

where "\( \cdot \)" indicates the time derivative. Since all other field components such as \( E \), corresponding to the Coulomb field and the magnetic field induced by the former are of the order of \( 1/r^2 \), they are all neglected here. The directions \( \theta \) and \( \phi \) are orthogonal from each other, so that the Pointing vector, which gives the energy flow per unit time and solid angle, is given as

\[ \frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\psi}|^2 \sin^2 \theta. \quad (3.2.13) \]

By integrating with respect to the whole solid angle, the total emissivity is obtained as

\[ P = \frac{2e^2}{3c^3} |\dot{\psi}|^2. \quad (3.2.14) \]

It follows from these two equations that, in the case where the charged particle is being accelerated, the frequencies of the emitted electromagnetic waves are equal to those of this particle in motion. Hence, these results shown in Eqs. (3.2.13) and (3.2.14) have to be considered as the most fundamental to determine the characteristics of the electromagnetic emissions from a charged particle in the state of acceleration. The cause for electromagnetic waves to be emitted may be explained as follows: since the particle responsible for generating the potentials is being accelerated, both the electric and the magnetic fields, being transmitted to the point of observation from the time and space occupied by the particle, are forced to lose phase-matching and are then released from this particle.

Next, we shall consider the assumption made to introduce Eq. (3.2.13) and then clarify the limitation of its application. The most important is the quantum mechanical limitation. This means that the energy of the photon to be emitted has to be negligibly small compared to the kinetic energy of the charged particle. Since the effect of the wave nature of this particle becomes important when the above conditions are not fulfilled, the classical theory fails to deal with this process.

The next limitation is associated with the assumption made as \( \beta \ll 1 \) to introduce Eq. (3.2.13). This assumption cannot be applied to any particle moving with a speed close to that of light. In this case, we should first take the coordinate system in which the particle is almost at rest by taking the Lorentz transformation. After introducing the same equation as Eq. (3.2.13), we should go back to the laboratory system by taking again the Lorentz transformation. Thus, it becomes possible to obtain the Poynting vector with respect to the relativistic case.
3.2.2 Thomson scattering

a) The cross-section of Thomson scattering

There are two phenomena in which photons are scattered by free electrons. They are, respectively, Thomson and Compton scatterings. First, let us consider Thomson scattering. This scattering can be dealt with classically. Really speaking, a good enough approximation can be reached if the energy of the photon to be scattered is of the order of 10 KeV or less, since the rest energy of an electron is about 500 KeV as shown in Table 3.1.2. For instance, the energy of a visible light photon is of the order of 1 eV, so that the classical treatment is thought to be a fully satisfied approximation.

When a free electron encounters electromagnetic waves, the equation of motion of this electron is given by

\[ m \ddot{X} = eE, \]  \hspace{1cm} (3.2.15)

where \( E \) is the electric vector of these waves and \( X \) is the position vector in the same direction as \( E \). Since the electron becomes accelerated as given by Eq. (3.2.15), the radiation intensity emitted in unit time is calculated from Eq. (3.2.14).

\[ P = \frac{2e^2}{3c^3} \frac{e^2E^2}{m^2} = \frac{2}{3} \left( \frac{e^2}{mc^2} \right)^2 E^2c. \]  \hspace{1cm} (3.2.16)

Since the energy influx per unit area for a second is given by

\[ \left( \frac{H^2}{8\pi} + \frac{E^2}{8\pi} \right)c = \frac{E^2c}{4\pi}, \]

Eq. (3.2.16) is rewritten as

\[ P = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 \frac{E^2c}{4\pi}. \]  \hspace{1cm} (3.2.17)

Thus, this result indicates that only the fraction corresponding to the factor \( 8\pi/3 \times \left( e^2/mc^2 \right)^2 \) for the above influx is scattered by electron. By using the classical radius of the electron \( r_0 \) in the expression for the cross-section of Thomson scattering \( \sigma_T \), this cross-section is expressed as

\[ \sigma_T = \frac{8\pi}{3} r_0^2 = 6.65 \times 10^{-25} \text{ cm}^2. \]  \hspace{1cm} (3.2.18)

The differential cross-section is also derived from Eq. (3.2.13) as follows:

\[ d\sigma_T = 2\pi r_0^2 \sin^2 \theta \ d(\cos \theta). \]  \hspace{1cm} (3.2.19)

In this scattering, the incident electromagnetic waves are scattered into the direction
perpendicular to the electric field of these waves, but the frequencies of the scattered waves remain the same as those of the incident waves.

Once the cross-section is determined, the mean-free path for the scattering to occur one time on average can be calculated. As described earlier, its numerical value is generally of the order of 5 g/cm² in normal matter.

Radio waves and visible light pass through air with thickness 1 kg/cm² without Thomson scattering, for the electronic binding energy of the nitrogen or the oxygen molecules is much higher than that of the energy of each photon of radio waves and visible light, so they are not considered as free electrons. On the other hand, these photons are mostly reflected away from metals with many free electrons without even reaching a depth of about 5 g/cm². This situation reflects upon the fact that the wavelengths of those incident photons are significantly longer than the mean intervals (≈10⁻¹⁸ cm) of free electrons in the metals. In fact, all of the electrons within the region of the order of those wavelengths, being numbered as N, are enforced to collectively oscillate immediately after those photons have reached the metallic surface. For this collective motion of these electrons, the magnitude of the “equivalent” charge corresponding to e in Eq. (3.2.17) has to be replaced by Ne. Thus, the cross-section apparently becomes N²στ, which is, of course, very large. In the case of X-rays with energy of 10 KeV or more, Thomson scattering therefore occurs even in air or metals.

b) The Eddington limit

Thomson scattering plays a very important role in many astrophysical phenomena, though it should be considered the simplest of the phenomena related to electromagnetic interactions.

Let us consider the case in which radiation of total intensity \( L \) (erg/sec) is being emitted from the whole surface of a star. The radiation intensity per unit area measured at the distance \( r \) from the center of this star is, therefore, obtained as \( L/4\pi r^2 \). These radiations are scattered by the Thomson scattering process by free electrons ambient in the plasmas located inside the outer portion of the star. Since they are scattered toward the direction perpendicular to their incident direction on average, the component of their momenta along their direction is eventually transferred to the electrons. The momentum thus transferred is deduced as \( hf/c \) per scattering in the picture of light quanta. The energy of the radiation being scattered in one second is equal to \( L\sigma_T/4\pi r^2 \) per electron, so that the momenta to be transferred in one second is calculated as \( L\sigma_T/4\pi r^2 c \). Namely, the factor \( L\sigma_T/c \) can be thought of as the pressure pushing electrons outward. The same result as this is also derived when the radiation pressure in the classical electromagnetic theory is taken into consideration. In this case, electrons are enforced to be pushed outward, but they tend to move with protons due to the electric fields induced by their movement. When we denote the mass of a star by \( M \), the gravitational force of the star to a proton is given by \( GMm_p/r^2 \), where \( G \), \( m_p \), and \( r \) are, respectively, the gravitational constant, the proton mass and the distance between the center of the star and the proton. Therefore, it follows that the force for the proton to be pulled toward the center of the star is necessarily given as
\[
\frac{GMm_p}{r^2} - \frac{L\sigma_T}{4\pi r^2 c}.
\] (3.2.20)

In order that the plasma gas is always constrained to the star, the above equation must be positive, namely,

\[
L < \frac{4\pi GMm_p c}{\sigma_T} = L_c.
\] (3.2.21)

If \( L \) becomes greater than \( L_c \), it becomes impossible for the star to continue to radiate in a stable state, because the light energy is expended to accelerate particles in the plasma state over the surface of the star. The quantity \( L_c \), which defines the upper limit of the radiation from the star of a given mass, is now called the Eddington limit. If the stellar mass is measured in units of solar mass \( M_\odot \), this limit is expressed as

\[
L_c \approx 10^{38} \left( \frac{M}{M_\odot} \right) \text{ (erg/sec)}.
\] (3.2.22)

It is now thought that the radiation mechanism of X-rays from the so-called X-ray stars is perhaps related to the transformation of the kinetic energy of particles accelerated by a strong gravitational field into X-ray energy in most cases. Even if the X-ray intensity becomes so strong as to go beyond \( L_c \) as a result of the increase of the plasma contracting towards the star, this plasma is unable to continue to contract, since the radiation pressure finally begins to overcome the gravitational pull on the plasma. Therefore, it is thought that no X-ray star emits X-rays of intensity higher than \( L_c \).

For the case of the sun, the total flux of the radiation from the solar surface is estimated to be \( 3.8 \times 10^{13} \) erg/sec, which is by the fourth order of magnitude lower than the Eddington limit. Therefore, it is of no use to discuss such problems as considered in the last paragraph.

### 3.2.3 Compton scattering

When the energy of an X-ray photon is close to the rest energy of an electron, the classical treatment cannot be applied to the interactions between such photons and electrons. Furthermore, it should be considered that the electron has its own magnetic moment of spin 1/2, in addition to the mass and charge. This necessarily requires that electrons behave as particles with the wave nature deduced from the Dirac equations.

Let us first begin with an elementary treatment from the dynamical point of view. By assuming that the incident photon of energy \( W \) has become the photon of energy \( W' \) after being scattered by an electron into the direction \( \theta \) (Fig. 3.2.2), and defining the energy of the recoiled electron by \( E \), it follows that

\[
W' = W - E.
\] (3.2.23)

When we take into account the relation of momentum conservation, the following
result is deduced:

\[ W' = \frac{Wmc^2}{mc^2 + W(1 - \cos \theta)} \]  \hspace{1cm} (3.2.24)

From this equation, some important results are obtained as follows:

1) Forward scattering: in the case where the scattering angle \( \theta \) is equal to 0 \((\theta = 0)\), it follows that \( W' = W \). Thus, the energy of the photon never changes in this case. The energy \( W' \) tends to become lower as the angle \( \theta \) increases.

2) In the case where \( W' \approx mc^2 \), it follows that \( W' = W \). In this case, the energy of the photon never changes, too. This is the case of Thomson scattering.

3) In the case where \( W' \gg mc^2 \), the energy of the photon at three angles is given as follows:

For the angle, \( \theta = 0 \) (forward), \( W' \approx W \),
\[ \theta = \pi/2 \) (perpendicular), \( W' = mc^2 \), and
\[ \theta = \pi \) (backward), \( W' = \frac{mc^2}{2} \). \] \hspace{1cm} (3.2.25)

In this case, therefore, the energy of the photon to be scattered at a right angle is \( mc^2 \) irrespective of the initial energy, while the photon of energy \( mc^2/2 \) is scattered backward.

Although the cross-section is estimated to be of the order of \( \pi r^2 \) as will be shown later, the exact formulation was derived by Klein and Nishina in 1929 by using the Dirac equation.

The so-called Klein-Nishina formula for the cross-section is expressed as, using the quantities \( W \), \( W' \) and \( \theta \),

\[ \sigma(W, W')dW' = \pi r^2 \frac{mc^2}{W} \frac{dW'}{W'} \left[ 1 + \left( \frac{W'}{W} \right)^2 - \left( \frac{W'}{W} \right)^2 \sin^2 \theta \right]. \] \hspace{1cm} (3.2.26)

In the case where \( W \) is much lower than \( mc^2 \), it becomes clear by integrating for \( W' \)
that this formula is the same as the Thomson cross-section. In this integration, however, the upper and the lower limits of the energy have to be taken as $W$ and $W/(1+(2W/mc^2))$, respectively.

In the case where $W'>mc^2$, on the contrary, the result obtained by integrating Eq. (3.2.6) over $W'$ from $W$ to $mc^2/2$ gives the cross-section expressed as follows:

$$\sigma_c \approx \pi r_0^2 \frac{mc^2}{W} \left[ \ln \left( \frac{2W}{mc^2} \right) + \frac{1}{2} \right].$$ \hspace{1cm} (3.2.27)

This indicates that $\sigma_c$ decreases almost proportionally to $1/W$ with energy. Since the cross-section for scattering, therefore, becomes smaller in the case where $W \gg mc^2$, the phenomena as related to the passage of high-energy $\gamma$-rays through matter are identified as electron-positron pair creation.

In the quantum-mechanical treatment of Compton scattering, the mathematical formulation is obtained by referring to the diagrams shown in Fig. 3.2.3. In doing so, two cases are taken into consideration; they are, respectively, the case where the incident photon is virtually absorbed by an electron and then reemitted, and that in which the electron first emits a photon virtually and then absorbs this photon. Since, quantum-mechanically, the electron size is of the order of $\hbar/mc$, its cross-section is roughly estimated as $\pi(\hbar/mc^2)^2$, but this cross-section is not identified as that for cross-section, for the strength of the coupling between the electron and photon becomes important for the incident photon to be absorbed. For the electromagnetic interaction, this strength is given by the constant for the hyper-fine structure $\alpha = e^2/\hbar c$. This means that, among the photons hitting the cross-section of the electron, only the fraction $1/137$ is absorbed. Therefore, this constant plays some role in the reemission of photons. As a result, the magnitude of the cross-section for Compton scattering is roughly given as

$$\sigma = \pi \left( \frac{e^2}{\hbar c} \right)^2 \left( \frac{\hbar}{mc^2} \right)^2 = \pi r_0^2.$$  

In order to obtain the exact results as shown by Eqs. (3.2.6) and (3.2.7), it is
necessary to calculate the cross-section quantum-mechanically. However, the reason why the cross-section is of the order of \( \pi r_0^2 \) becomes clear from the consideration made above.

3.2.4 Inverse Compton scattering

In the case of the Compton effect, X-rays or \( \gamma \)-rays are scattered by electrons at rest. Even when electrons are moving with high velocity, Thomson or Compton scattering also occurs when photons collide with electrons. In the case where the energy of the electron is relativistic, it follows that the energy of the scattered photon becomes higher than that of the incident photon. This situation is similar to the case in which a ball moving slowly is thrown away by being hit strongly (see Fig. 3.2.4).

Let us assume that a photon is incident in the direction of a moving electron with an angle \( \theta \) from this direction, and that the energy, the velocity and the Lorentz factor of this electron are, respectively, denoted by \( E, \beta c \) and \( \gamma(=E/mc^2=1/\sqrt{1-\beta^2}) \). If we further assume the energy of the incident photon is \( hf \), the energy (\( hf^* \)) of this photon as referred to the rest system of the electron is given by

\[
hf^* = \gamma hf(1 + \beta \cos \theta) .
\]

(3.2.28)

In the case where photons are incident with various angles, their mean energy may be taken as \( hf^* \approx \gamma hf \).

When the energy of the photon is as low as \( \gamma hf \ll mc^2 \), the scattering becomes Thomson-type. Thus, the energy of the scattered photon is given as \( hf^* \). Taking the scattered angle as \( \varphi \), we now transform the rest system to the laboratory system in which the electron is moving. By defining the energy of the scattered photon in the latter system by \( hf' \), the following equation is derived:

\[
hf' = \gamma hf^*(1 + \cos \varphi) \approx \gamma^2 hf .
\]

(3.2.29)

The result obtained from the mean value calculated by taking into account the exact angular distribution is given as

![Diagram](image)

Fig. 3.2.4. The inverse Compton effect.
\[ hf' = \frac{4}{3} \gamma^2 hf. \quad (3.2.30) \]

In other words, the incident photon is scattered as a photon of energy higher by the order of \((E/mc^2)^2\). For this reason, the inverse Compton effect is thought to be an important process for the X-ray and \(\gamma\)-ray emissions in cosmic space, since light is transformed into X-rays or \(\gamma\)-rays. In fact, an electron of energy 10 GeV (\(\gamma \approx 2 \times 10^6\)) can transform a photon of visible light (~1 eV) to a \(\gamma\)-ray photon of energy of several 100 MeV or more.

On the other hand, it is clear that the electron has lost the energy \((4/3)(E/mc^2)^2 hf\) due to the inverse Compton scattering. When the electron passes through space with the energy of photons \(E_{\text{ph}}(nhf)\), it loses the energy given by

\[- \frac{\text{d}E}{\text{d}t} = \frac{4}{3} \sigma_T c \left( \frac{E}{mc^2} \right)^2 E_{\text{ph}} = 2.66 \times 10^{-14} \left( \frac{E}{mc^2} \right)^2 E_{\text{ph}} . \quad (3.2.31)\]

When we consider the case of an electron with energy \(E=10\) GeV and for \(E_{\text{ph}}=1\) eV/cm\(^3\), it follows that the energy loss rate for this electron is given as about 1 eV per day. Therefore, this electron would have lost an energy more than half of that originally owned by it during \(3 \times 10^7\) years. This indicates that photons ambient in galactic space should be considered as substantial obstacles for high-energy electrons moving in this space; namely, the inverse Compton effect is an important process for electrons to lose their own energy in the Galaxy.

The calculation as described above can only be applied to the case where the energy of this incident photon is as low as \(\gamma hf \ll mc^2\). When this condition is not satisfied, it is, however, necessary to calculate this process by referring to the exact cross-section for Compton scattering, though the calculation is rather complex (e.g., see Ref. 2.1). Since the energy of the photon to be scattered in the system referred to the electron is estimated to be of the order of \(mc^2\) in the case where \(\gamma hf \gg mc^2\), the photon energy \(hf'\) becomes of the order of

\[ hf' \approx \gamma mc^2 = E^*. \quad (3.2.32)\]

As is clearly seen from Eq. (3.2.27), the scattering cross-section also decreases with energy, and the electron releases most of its energy to the photon at once.

### 3.2.5 Synchrotron radiation

Inverse Compton scattering is thought to be an efficient process to produce

*If the formula given as \(hf = (4/3)\gamma^2 hf\) is used in Eq. (3.2.30), this is reduced to \(hf' \approx \gamma mc^2\) for the case where \(\gamma hf \gg mc^2\). This result, therefore, seems to be contradictory, since the energy of the photon to be scattered becomes higher than that of the electron. However, this is not so, because, in this calculation, the energy of the photon to be scattered is assumed as equal to that of the incident photon. If we consider the effect of the electron recoil exactly, the result as described in the text is necessarily obtained which has no contradiction from the physical point of view.
X-rays and \( \gamma \)-rays. As a similar process to this, there is synchrotron radiation which emits photons ranging from radio to \( \gamma \)-ray wavelengths. While moving in a magnetic field, an electron emits electromagnetic waves by being accelerated by the Lorentz force. First, let us deal with this process in the non-relativistic realm, in which the energy of the electron is less than its rest energy. Taking the intensity of the applied magnetic field to be \( H \) gauss, the electron is deflected by the action of the Lorentz force and moves in a circular orbit. Its radius \( \rho \) is given as

\[
\rho = \frac{mv}{eH_\perp}, \tag{3.2.32}
\]

where \( H_\perp \) is the component of the magnetic field perpendicular to the direction of the electron motion. The period of this gyration \( T \) is calculated as

\[
T = \frac{2\pi \rho}{v} = \frac{2\pi mc}{eH_\perp}. \tag{3.2.33}
\]

Thus, this period is independent of its speed. The angular frequency \( \omega \) is thus given by

\[
\omega = \frac{2\pi}{T} = \frac{eH_\perp}{mc} = 17.6 \, H_\perp \, \text{MHz}, \tag{3.2.34}
\]

where, on the far right-side, the field intensity is given in the unit of gauss. This is called the cyclotron frequency. Substituting the term \( veH_\perp/mc \) into the acceleration term of Eq. (3.2.14), the total energy radiated from the electron per second is calculated as

\[
P = \frac{2e^2}{3c^2} v^2 \left( \frac{eH_\perp}{mc} \right)^2 = \sigma_T c \beta^2 \frac{H_\perp^2}{4\pi}. \tag{3.2.35}
\]

Based on the results shown in Eqs. (3.2.34) and (3.2.35), cyclotron radiation can be qualitatively interpreted as follows; since the energy of the emitted photon is \( \hbar \omega \), it could be considered that the electron scatters away a virtual photon of energy \( \hbar \omega \) due to the inverse Compton process. The distance which this electron moves during a second is equal to its speed \( v \), so that the number of such virtual photons in one cm\(^3\) is obtained as

\[
\beta \frac{H_\perp^2}{4\pi} \hbar \omega = \frac{1}{4\pi} \beta \frac{H_\perp mc}{\hbar} = 4.29 \times 10^8 \beta H_\perp/\text{cm}^3. \tag{3.2.36}
\]

In the case of an electron of relativistic energy, the energy of the scattered photon becomes of the order of \( \gamma^2 \hbar \omega \) as a result of the inverse Compton scattering. Thus, the radiated power per second is calculated as
\[ P = \sigma_\gamma c^2 \beta^3 \frac{H_\perp^2}{4\pi} \]  \hspace{1cm} (3.2.37)

Since the magnetic field is given as, taking the angle $\theta$ between the velocity and the magnetic vectors,

\[ H_\perp = H \sin \theta \]  \hspace{1cm} (3.2.38)

it follows that $\langle H_\perp^2 \rangle = 2/3 H^2$. Substituting this result into Eq. (3.2.37), this equation is rewritten as*

\[ P = \frac{4}{3} \sigma_\gamma c^2 \beta^3 \frac{H^2}{8\pi} \]  \hspace{1cm} (3.2.39)

Since the term $H^2/8\pi$ denotes the energy density of the magnetic field, it is clear that this formula has the same form as that of the inverse Compton effect. Like this effect, the emitted photons are concentrated into the angular direction of the order of 1/$\gamma$ as regards the direction of the electron motion.

One of the characteristics of synchrotron radiation, as understood from the consideration in the footnote on Page 57, is that this radiation is polarized into the direction $(\nu \times \mathbf{H})$. The consideration described above is very qualitative, but the results based on the rigorous calculation were given independently by Vladimilskii and Schwinger in 1949.\(^{22}\)

The frequency distribution of the emitted photons in the frequency range $(f, f+df)$ per second is given as, according to their results,

\[ \sqrt{3} \frac{\varepsilon^2}{\hbar c} \frac{eH_\perp}{2\pi mc} F(f \mid f_e) \, h \, df \]  \hspace{1cm} (3.2.40)

*Eq. (3.2.39) is also obtained as follows: Since the electron moves with the speed near the light in the laboratory system, a question is raised on the applicability of Eq. (3.2.14) without any modification. Let us, therefore, take up the coordinate system in which the electron is at rest. The intensity of the electric field $E_\perp$ in this system is reduced, from the Lorentz transformation, as

\[ E_\perp = \gamma \beta H_\perp \].

Eq. (3.2.14) is expressed in this system, as

\[ P' = \frac{2\alpha^2}{3c^3} \left( \frac{\gamma \beta H}{m} \right)^2 = \sigma_\gamma c^2 \beta^3 \frac{H^2}{4\pi} \].

Next, return this formula to the laboratory system. Since the energy and the time become $\gamma$-times together, its formula does not change and so is given as

\[ P = \sigma_\gamma c^2 \beta^3 \frac{H^2}{4\pi} \].
where \( f_c = (3/4\pi)(eH/mc)\gamma^2\beta^2 \) and

\[
F(x) = x \int_x^\infty K_{5/3}(\eta) d\eta. \tag{3.2.41}
\]

Here, the function \( K \) is the modified Bessel function of imaginary variable.* Integrating Eq. (3.2.41) with respect to \( f \), the following result is obtained:

\[
\int_0^\infty x d\nu \int_x^\infty K_{5/3}(\eta) d\eta = \frac{8\pi}{3\sqrt{3}}. \tag{3.2.42}
\]

Thus, the total radiation power per second is calculated as

\[
P = \frac{2e^2}{3c^3} \gamma^2\beta^2 H_c^2 = \sigma_1 c^2 \gamma^2 \beta^2 \frac{H_c^2}{4\pi}. \tag{3.2.43}
\]

It follows that Eq. (3.2.35) derived from the qualitative consideration also gives the correct result.

The energy distribution of the electromagnetic waves generated by the synchrotron radiation mechanism is obtained by integrating Eq. (3.2.41). The result after integration is shown in Fig. 3.2.5. The integrated result is series-expanded in the regions near \( x=0 \) and \( x\gg 1 \) and is given as follows:

* \[ K_{5/3}(x) = \frac{1}{2} \left( \frac{x}{2} \right)^{5/3} \int_0^x \xi^{2/3} \exp \left[ -\left( \frac{x^2}{4\xi} + \frac{1}{\xi} \right) \right] d\xi \]

\[ = \frac{1}{2} \left( \frac{2}{x} \right)^{5/3} \int_0^x \xi^{2/3} \exp \left[ -\left( \xi + \frac{x^2}{4\xi} \right) \right] d\xi. \]
\[ F(x) = x \int_{x}^{\infty} K_{5/3}(x')dx' \approx \begin{cases} \frac{4\pi}{\sqrt{3\Gamma(1/3)}} \left( \frac{x}{2} \right)^{1/3} + \ldots, & (x \ll 1) \\ \sqrt{\frac{\pi}{2}} x^{1/2} e^{-x} + \ldots, & (x \gg 1) \end{cases} \] (3.2.44)

Since \( F(x) \) has a broad peak around \( x = 0.29 \) as seen in Fig. 3.2.5, it follows that almost monochromatic electromagnetic waves are generated from the synchrotron mechanism. The peak frequency \( f_m \) is thus given as
\[
\begin{align*}
f_m &= 0.29 \times \frac{3}{4\pi} \left( \frac{E}{mc^2} \right)^2 \frac{eH_\perp}{mc} \\
&= 1.22 \left( \frac{E}{mc^2} \right)^2 H_\perp \text{ MHz .} \hspace{2cm} (3.2.45)
\end{align*}
\]

The intensity of magnetic field inside an electron accelerator is of the order of \( 10^4 \) gauss. When an electron has been accelerated to an energy of about 1 GeV, it thus follows that the peak frequency \( f_m = 5 \times 10^{16} \) Hz. This is comparable to that of soft X-rays of wavelength 60 Å. Since their energy distribution is given by Eq. (3.2.44), blue lights of shorter wavelengths are efficiently emitted. This is the reason that this radiation mentioned above is named synchrotron radiation. This radiation is now often used to produce strong beams of soft X-rays. Since this radiation can be made as a monochromatic polarized X-ray source in this case, it has been used in the research on the properties of matter. The device called the SOR (Synchrotron Orbit Radiation) at the Institute of High-Energy Physics at Tsukuba has been manufactured for this purpose.

3.2.5.1 Non-thermal galactic radio emissions

There exists a magnetic field of about \( 3 \times 10^{-6} \) gauss in the Galaxy. The electrons, as a component of cosmic rays, radiate electromagnetic waves due to the synchrotron mechanism in this magnetic field. Taking an electron of energy of about 5 GeV, it follows that
\[
f_m = 1.22 \times 10^6 \times (10^6)^2 \times 3 \times 10^{-6} = 366 \text{ MHz} . \hspace{2cm} (3.2.46)
\]

Although a part of the galactic radio emissions is generated by thermal process, the radio waves near 100 MHz are mainly emitted from the synchrotron mechanism. In other words, this mechanism is thought to be responsible for the galactic radio emissions in this frequency range. There is a magnetic field of the order of \( 10^{-4} \) gauss in the Crab nebula, a supernova remnant. In this nebula, an electron of the order of \( 10^{12} \) eV can emit photons of 6000 Å in wavelength, because \( f_m = 4.9 \times 10^{14} \) Hz. The extended light source surrounding the Crab nebula was once explained by Ginzburg and Shklovsky in 1950 by referring to the synchrotron radiation mechanism. The observation which had shown the light emissions to be polarized well confirmed that this source originates from synchrotron radiation. It has now been confirmed that
X-ray emissions from this nebula are also generated by synchrotron radiation from very high-energy electrons.

3.2.5.2 The energy loss of high-energy electrons

Like the inverse Compton scattering, the synchrotron radiation also plays some role in the energy loss of high-energy electrons in galactic space. The energy losses of high-energy electrons in this space due to synchrotron radiation and inverse Compton scattering are written as follows: namely, taking the sum of the energy density of the magnetic field and that electromagnetic waves as \( \varepsilon \), it follows that \( \varepsilon = (H^2 / 8\pi) + \varepsilon_{\text{ph}} \). Thus, the rate of the energy loss of the electron is given as

\[
- \frac{dE}{dt} = bE^2, \quad \text{and} \quad b = \frac{4}{3} \sigma_{\text{sc}} \frac{\varepsilon}{(mc^2)^2}.
\]

The time for the electron to lose most of its energy is also given by

\[
T = \frac{1}{bE} = \frac{3 \times 10^8}{\varepsilon E} \text{(Yrs)},
\]

where \( \varepsilon \) and \( E \) are, respectively, measured in units of eV/cm\(^3\) and GeV. Since the order of magnitude of \( \varepsilon \) is 1 eV in the Galaxy, it is clear that an electron of 1 GeV would have lost most of its energy during about \( 10^8 \) years. Because \( H^2 / 8\pi \approx \varepsilon_{\text{ph}} \) in the Galaxy, both inverse Compton scattering and synchrotron radiation contribute to the energy loss of electrons to the same order of magnitude. However, the energy of scattered photons becomes relatively higher, since the incident photons are usually optical photons or microwaves in the case of inverse Compton scattering. Conversely, the wave frequencies corresponding to the incident photons are relatively lower in the case of synchrotron radiation (see Eq. (3.2.45)). In consequence, the energy of photons to be emitted becomes lower, but an enormous number of photons are emitted instead, so as to make the amount of released energy almost equal to that due to inverse Compton scattering.

3.2.6 Photoelectric effect

Thomson scattering and Compton scattering, which play an important role in the higher energy region, have been considered in the last subsection.

Let us here consider again the photoelectric effect, which plays an important role in the energy range where Thomson scattering occurs efficiently. This effect is the phenomenon in which an electron bound to an atom absorbs an incident photon and then becomes a free electron. Taking the energy of the incident photon and that of the free electron as \( hf \) and \( E \), respectively, it follows that

\[
E = hf - I,
\]

(3.2.47)

where \( I \) is the binding energy of the electron. It has long been thought that the photoelectric effect is important experimental evidence in the light-quantum
hypothesis in the development of the quantum theory, since the photon interacts with matter as a quantum of energy $hf$. Consider a K-shell electron inside an atom of charge number $Z$. The orbital radius and the binding energy for this electron are respectively given as

$$a = \frac{\hbar c}{e^2} \left(\frac{\hbar}{m} \frac{1}{Z} \frac{1}{mc} \right) = \frac{137}{Z} \frac{\hbar}{mc}$$

(3.2.48)

and

$$I = \frac{Ze^2}{2a} = \frac{Z^2}{2} \left(\frac{1}{137}\right)^2 mc^2.$$

(3.2.49)

From these results, it follows that the wavelength and the speed of the bound electron in the realm of the old quantum theory are of the order of $a=(137/Z)\hbar/mc$ and $Ze/137$, respectively. Furthermore, the time necessary for the electron to move by one wavelength is estimated to be $(137/Z)^2(\hbar/mc^2)$. While the energy of an incident photon is of the order of $I$, the oscillatory period of the electron is calculated as

$$\frac{\hbar}{I} = 2 \left(\frac{137}{Z}\right)^2 \frac{\hbar}{mc^2},$$

(3.2.50)

the order of which is almost the same as the time for the electron to move by one wavelength. Since the wave nature of the electron influences the result, the classical treatment cannot be made use of in this treatment.

The total cross-section calculated for the photoelectric effect for two K-shell electrons as induced by an incident photon of energy $hf$ is given by

$$\sigma = 4\sqrt{2}\sigma_T \frac{Z^5}{137^4} \left(\frac{mc^2}{hf}\right)^{7/2},$$

(3.2.51)

where $\sigma_T$ denotes the cross-section for Thomson scattering ($=6.65 \times 10^{-25}$ cm$^2$). This cross-section is proportional to the term $Z^5/hf^{7/2}$. The reason that this cross-section is proportional to $Z^5$ is as follows: the density of the electron wave function is proportional to $1/a^3$, which varies with $Z^3$. The other factor $Z^2$ comes from the Fourier component resulting from the integration after the superposition of all the wave functions for free electrons and incident photons and the wave functions of bound electrons.

As seen from Eq. (3.2.51), the cross-section of the photoelectric effect becomes very large in the case where the atomic number $Z$ is large and also where the energy of the incident photon is less than $mc^2$ (i.e., 510 keV). This effect is also the most important absorption process in the X-ray range. When the energy of incident photons is low, the atoms are ionized by losing the electrons in their outer orbits with large principal quantum number $n$, since the ionization energy is given as $I=$
\((Z^2/137)(1/n^2)mc^2\) in such cases. When the energy of incident photons is high, however, the atoms become ionized by losing the K-shell electrons. In the cases where the energy of incident photons is higher than the bound energy of the K-shell electrons, we may assume that these electrons are considered as free electrons. Therefore, the Thomson and the Compton scattering processes become important in these cases. The mean free path of the photoelectric effect for various matters is shown in Fig. 3.2.6. and Table 3.2.1.

3.2.7 Blackbody radiation

The electromagnetic radiation from a blackbody in thermal equilibrium is denoted as the blackbody radiation. When the temperature of matter is expressed by \(T\), the kinetic energy of the electrons constituting matter, from which photons are emitted, is necessarily given as \(kT\) in order of magnitude, where \(k\) is the Boltzmann constant. Here, the classical treatment is not applicable, since the energy of photons to be emitted becomes of the order of \(kT\) on average. The energy radiated from a blackbody in the frequency range \((f, f+df)\) for unit area and unit solid angle per second is derived as

\[
B(T)df = \frac{2hf^3}{c^2} \times \frac{1}{e^{hf/kT} - 1} \; df \; (\text{erg/cm}^2 \text{ sec sr}). \tag{3.2.52}
\]

Here only the calculated result has been shown. This formula is now well known as the formula for the Planckian radiation.
Table 3.2.1. The mean free paths of photoelectric effect for various matter.

<table>
<thead>
<tr>
<th>Matter</th>
<th>10 keV</th>
<th>20 keV</th>
<th>50 keV</th>
<th>100 keV</th>
<th>200 keV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beryllium</td>
<td>3.15 g/cm²</td>
<td>31.2 g/cm²</td>
<td>656 g/cm²</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>Air</td>
<td>0.24</td>
<td>2.2</td>
<td>43</td>
<td>416</td>
<td></td>
</tr>
<tr>
<td>NaI</td>
<td>0.0084</td>
<td>0.062</td>
<td>0.11</td>
<td>0.71</td>
<td>4.90</td>
</tr>
<tr>
<td>Lead</td>
<td>0.0125</td>
<td>0.005</td>
<td>0.193</td>
<td>0.198</td>
<td>1.25</td>
</tr>
</tbody>
</table>
Blackbody radiation can be understood as follows: first consider a cavity occupying unit volume in the blackbody, which is naturally filled by blackbody radiation. Since the volume element of phase space is calculated as $4\pi(h\nu/c)^3(dh\nu/c)$, the number of unit cells of magnitude $h^3$ is $(4\pi f^3/c^3)df$ in this volume. The distribution of photons follows the law of Bose-Einstein statistics, so that the number of photons per unit cell is given by $2/(e^{h\nu/kT} - 1)$, in which the factor 2 appears because the spin of the photon is given by 1. Taking into account that the energy and the speed of a photon are expressed as $h\nu$ and $c$, respectively, the energy radiated into unit area and unit solid angle is calculated as

$$h\nu^2/(e^{h\nu/kT} - 1) \cdot \frac{f^2}{c^3} \, df = \frac{2hf^3}{c^2} \left(e^{h\nu/kT} - 1\right)df.$$ 

Thus, Eq. (3.2.52) has been derived by the procedure as mentioned above.

When $h\nu$ is much higher than $kT$, it follows that

$$B(T)df = \frac{2hf^3}{c^2} e^{-h\nu/kT}df. \quad (3.2.53)$$

This is the well known as the Wien formula.

When $h\nu$ is much lower than $kT$, contrary to the above case, the energy of the photon being radiated is necessarily lower as compared to the kinetic energy of the electron. In consequence, the radiation formula in this case can be dealt with from the classical theoretic point of view. This formula is derived from Eq. (3.2.52) as shown by

$$B(T)df = \frac{2kTf^2}{c^2} \, df, \quad (3.2.54)$$

which is called the formula of Rayleigh-Jeans, since this had been independently obtained by Rayleigh and Jeans.

The energy distribution of the Planckian radiation is shown in Fig. 3.2.7 as a function of the blackbody temperature. Some important results derived from the formula shown in Eq. (3.2.52) will be described in the following.

a) The frequency at the maximal luminosity

By the derivation of Eqs. (3.2.52) and (3.2.53) the relation $h\nu_m = 2.82 \, kT \approx 3 \, kT$ is derived for the frequency at the maximal intensity and thus the wavelength, $\lambda_m'$, corresponding to this frequency is given as $\lambda_m' \equiv c\nu / 3 \, kT$. If the spectral distribution of the Planckian radiation is expressed using the wavelength $\lambda$, this distribution is shown as

$$\frac{2hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda kT} - 1} \, d\lambda. \quad (3.2.55)$$

In this expression, the wavelength, $\lambda_m$, at the maximal intensity is given as
Fig. 3.2.7. The energy distribution on the thermal radiation (it should be remarked that this distribution is dependent on temperature).

\[ \lambda_m = \left( \frac{hc}{4.96kT} \right) \approx \left( \frac{hc}{5kT} \right). \] Since the temperature of the solar photosphere is 6000 K, one obtains \( \lambda_m \approx 4800 \text{ Å}. \) A star of surface temperature \( 10^7 \text{ K} \) emits X-rays of energy \( h\nu_m \approx 3 \text{ keV}. \) Near the ground, since the temperature is at 300 K this wavelength \( \lambda_m \) is about 10 \( \mu \text{m}, \) which corresponds to infrared radiation.

\textbf{b) The energy radiated from unit area (The Stefan-Boltzmann law)}

By integrating the Planck formula with respect to solid angle and frequency, the following result is obtained:

\[
2\pi \int_0^1 \cos \theta \, d(\cos \theta) \int_0^\infty \frac{2h\nu^3}{c^2} \frac{df}{e^{\nu/kT} - 1} = \sigma T^4, \tag{3.2.56}
\]

where \( \sigma \) is defined as

\[
\sigma = \frac{2\pi}{c^2\hbar^2} \int_0^\infty \frac{x^3 \, dx}{e^x - 1} = \frac{2\pi^2k^4}{15c^2\hbar^3} = 5.67 \times 10^{-5} \text{ erg/cm}^2 \text{ sec K}^4.
\]
The expression of Eq. (3.2.56) is now called the law of Stefan-Boltzmann.

Let us now assume that the distance to a star \((D)\) and its surface temperature \(T\) from its radiation spectrum are known. If the radiation flux of this star at the earth \(Q\) is measured, the radius of this star \(R\) is always derived from the relation

\[
2\pi \frac{R^2}{D^2} \int_0^1 \cos \theta \, d(\cos \theta) \int_0^\infty B(T) \, df = \frac{R^2}{D^2} \sigma T^4 = Q.
\]

It thus follows that \(R = D(Q/\sigma T^4)^{1/2}\). In the case of the sun, since \(Q, T\) and \(D\) are, respectively, given as

\[
Q = 2 \text{ cal/cm}^2\cdot\text{min} = 1.38 \times 10^6 \text{ erg/cm}^2\cdot\text{sec},
\]

\[
D = 1.5 \times 10^8 \text{ km} \quad \text{and} \quad T = 5778 \text{ K},
\]

one obtains the radius \(R\) to be \(R = 7 \times 10^{10} \text{ cm}\), which is almost equal to the exact radius of \(6.96 \times 10^{10} \text{ cm}\).

c) The energy density and the number density of photons

The energy of blackbody radiation in unit volume (1 cm\(^3\)) is given as

\[
u = \frac{4\pi}{c} \int_0^\infty B(T) \, df = aT^4,
\]

(3.2.57)

where

\[
a = \frac{4\sigma}{c} = 7.56 \times 10^{-15} \text{ erg/cm}^3 \text{ K}^4 = 4.73 \times 10^{-7} \text{ eV/cm}^3 \text{ K}^4.
\]

(3.2.58)

The number density of photons is further given as

\[
n = \frac{4\pi}{c} \int_0^\infty \frac{B(T)}{hf} \, df = \frac{8\pi}{c^3} \int_0^\infty \frac{f^2 \, df}{e^{hf/kT} - 1} = bT^3,
\]

(3.2.59)

where \(b = 16\pi(k/hc)^3 \zeta(3)\), in which \(\zeta\) is the Riemannian function \(\zeta\) and \(\zeta(3) = 1.202\). Thus, it follows that \(b = 20.2/\text{cm}^3 \text{ K}^3\) and \(n = 20.2 \text{ T}^3/\text{cm}^3 \text{ K}^3\). It may be here remarked that, though rough, the number density \(n\) is also estimated by dividing the energy density \(\nu\) by the characteristic energy of photons given by \(3kT\).

Using the results shown in Eqs. (3.2.57) and (3.2.59), the energy density of the blackbody radiation at 2.7 K, being ambient in the universe is calculated as 0.25 eV/cm\(^3\), from which the number density of photons is estimated to be about 400/cm\(^3\). Various physical quantities of the blackbody radiation at different temperatures are summarized in Table 3.2.2.
<table>
<thead>
<tr>
<th>Object</th>
<th>Temperature (K)</th>
<th>$\nu_{\text{max}}$ (Hz)</th>
<th>$\lambda_{\text{max}}$ (nm)</th>
<th>$\nu$ (eV/cm$^2$)</th>
<th>$n$ (cm$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cosmic Background Radiation</td>
<td>2.7</td>
<td>6.57x10$^{-4}$</td>
<td>1.59x10$^{-1}$</td>
<td>0.251</td>
<td>398</td>
</tr>
<tr>
<td>Ground Surface</td>
<td>300</td>
<td>7.30x10$^{-2}$</td>
<td>1.76x10$^{-1}$</td>
<td>3.83x10$^9$</td>
<td>5.45x10$^9$</td>
</tr>
<tr>
<td>Sun</td>
<td>6000</td>
<td>1.46</td>
<td>1.79x10$^1$</td>
<td>4.80x10$^9$</td>
<td>6.13x10$^9$</td>
</tr>
<tr>
<td>White Dwarf</td>
<td>10$^3$</td>
<td>243</td>
<td>8.50x10$^3$</td>
<td>4.73x10$^9$</td>
<td>4.73x10$^9$</td>
</tr>
<tr>
<td>X-Ray Star</td>
<td>10$^3$</td>
<td>5.88x10$^9$</td>
<td>5.1x10$^3$</td>
<td>4.73x10$^9$</td>
<td>4.73x10$^9$</td>
</tr>
</tbody>
</table>

$h_{\text{max}}=\frac{2.82K^4}{c^2}$, $\lambda_{\text{max}}=\frac{c}{h}$, $\nu_{\text{max}}=\frac{1}{\lambda_{\text{max}}}$, $\nu=\frac{h}{\epsilon}$, and $n=\frac{J}{\epsilon}$.

$k=1.38x10^{-23}$ erg/K, and $h=6.63x10^{-34}$ erg/sec.
3.2.8 Ionization losses

While a charged particle is moving in matter or plasmas, it necessarily loses its energy by scattering electrons due to the Coulomb interaction. This process for the particle to lose its energy is called the ionization loss. Assume that a particle with charge $Ze$ is moving with speed $v$ along the $x$-axis and that an electron is located at a distance $b$ from the path of the particle. The Coulomb force on the electron along the $x$-axis before the particle approaches the electron is cancelled out by that from the particle moving away from the electron (see Fig. 3.2.8). Here we consider only the Coulomb force perpendicular to the $x$-axis for simplicity. Denoting the perpendicular component of this force at the position of the electron by $E_n$, the impulse to be given to the electron is expressed by

$$e \int_{-\infty}^{\infty} E_n \, dt = \frac{e}{v} \int_{-\infty}^{\infty} E_n \, dx . \quad (3.2.60)$$

Furthermore, because $\text{div } E = 4\pi Ze$, this impulse is rewritten as

$$\frac{e}{v} \int_{-\infty}^{\infty} E_n \, dx = \frac{2Ze^2}{bv} . \quad (3.2.61)$$

Since the momentum change of the electron $\Delta P$ is equal to this impulse, the energy given to this electron $\Delta E$ becomes equal to

$$\Delta E = \frac{(\Delta P)^2}{2m} = \frac{2Z^2e^4}{mb^2v^2} . \quad (3.2.61')$$

Assuming the electron density per unit volume is $n_e$, this charged particle loses energy of amount given as, per unit length,

$$- \frac{dE}{dx} = n_e \int 2\pi bdb \Delta E = \frac{4\pi n_e Z^2e^4}{mv^2} \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) . \quad (3.2.62)$$

As the factor $b$ becomes longer, the effect of the Coulomb force on the electron becomes smaller as understood from Eq. (3.2.61). In the case where the electron is in

![Fig. 3.2.8. The passage of a charged particle through matter.](image-url)
the bound state, where the ionization potential is given by $I$, this electron is never ionized unless the force acts within a time interval shorter than $(I/h)^{-1}$. Since this time interval for collisions is of the order of $b/\nu$ when the collision parameter is equal to $b$, it follows that

$$b_{\text{max}} = \frac{h \nu}{I}.$$  

(3.2.63)

In the case of a plasma, the numerical value corresponding to $I/h$ could be taken as the plasma frequency $\omega_p = \sqrt{4\pi n_e^2 / m}$.

Equation (3.2.62) shows that the electron acquires energy without limit as the factor $b$ becomes smaller, but such a conclusion does not hold even in the classical treatment. Referring to the conservation laws for both energy and momentum, the maximal energy being acquired by the electron is given as $2mv^2$ for the general case in which the mass of the incident charged particle is significantly heavier than that of the electron. Combining this result with Eq. (3.2.62), the order of $b_{\min}$ is given as

$$b_{\min} = \frac{Z e^2}{mv^2} = \frac{Ze^2}{\hbar c} \frac{c}{\nu} \frac{h}{mv} = \frac{Z}{137} \frac{c}{\nu} \frac{h}{mv}.$$  

(3.2.64)

As regards the small value of $b$, there is some problem about the classical treatment itself. Since the speed of the electron is given as $v$ as viewed from the frame referred to the charged particle, the electron must have a wave-mechanical nature with wavelength $\hbar/mv$. It therefore becomes impossible to apply Eq. (3.2.64) to the case of small $b$, because of the nature of the electron just mentioned. The quantum mechanical condition obtained as $b_{\min} = \hbar/mv$, which is determined from the procedure mentioned above, is clearly different from the value as seen in Eq. (3.2.64). In general, the value $b_{\min} = \hbar/mv$ is small compared with that as given in Eq. (3.2.64).

The rate of energy loss derived from the procedure as described in the foregoing paragraph for the general case is given by

$$- \frac{dE}{dx} = \frac{4\pi n_e Z^2 e^4}{mv^2} \ln \left( \frac{mv^2}{I} \right).$$  

(3.2.65)

Taking the unit of thickness ($x$) of mass as g/cm$^2$, it follows that $n_e = NZ/ A_0$, where $Z$, $A_0$ and $N_0$ are the atomic number and the atomic mass of matter and the Avogadro number, respectively. As a result, the following formula is derived:

$$- \frac{dE}{dx} = 4\pi \frac{NZ}{A_0} Z^2 \frac{e^4}{mc^2} \ln \left( \frac{mv^2}{I} \right)$$

$$= 4\pi \frac{NZ}{A_0} Z^2 \left( \frac{e^2}{mc^2} \right)^2 \frac{mc^2}{\beta^2} \ln \left( \frac{mv^2}{I} \right) \text{g/cm}^{-2}.$$  

(3.2.66)
Characteristic is that the ionization loss is proportional to $Z^2/\beta^2$ of the charged particle. The reason that this relation has appeared in Eq. (3.2.66) is as follows: the term $Z^2$ appears due to the strong Coulomb force acting on the electron, but the term $1/\beta^2$ appears as a result of the time of interaction becoming longer between the electron and the charged particle of slow speed. In other words, this situation is similar to that in which a film is strongly exposed for enhanced illumination or is given a long exposure time.

Since the factor $Z_0/A_0$ is always nearly equal to 1/2 except for hydrogen when the length is expressed in the unit of g/cm$^2$ as in Eq. (3.2.66), it becomes clear that the rate of energy loss is almost independent of the nature of matter. This situation occurs because there exist almost the same number of electrons in the same quantity of any matter, in which the same number of protons with electrons are always found. The numerical expression of Eq. (3.2.66) by taking into account $Z_0/A_0=1/2$ is obtained as

$$\frac{-dE}{dx} \approx \frac{1.5Z^2}{\beta^2} \text{ MeV/g}\cdot\text{cm}^{-2}. \quad (3.2.67)$$

Since the thickness of the atmospheric layer is about 1 kg/cm$^2$ from the top to ground level, a charged particle with unit charge loses 1.5 GeV by the ionization loss alone in passing through this layer.

The formula for the ionization loss has been derived as mentioned above from the qualitative point of view, but, really speaking, it is necessary to deal with this process more exactly and also to consider the energy loss due to the excitation of atoms, the relativistic effect and the polarization effect in matter altogether.

The exact formula derived by taking into account the processes mentioned above is shown here, but it is, of course, clear that this formula is almost the same as derived earlier in the text. The rate of the energy loss is given, for any kind of matter, by

$$-\frac{dE}{dx} = 4\pi \frac{NZ_0}{A_0} \frac{Z^2e^4}{mv^2} \left[ \ln\left( \frac{2m\beta^2}{(1-\beta^2)^{1/2}} \right) - 2\beta^2 - \delta \right] \text{ (g/cm}^2\text{)}, \quad (3.2.68)$$

where $\delta$ is the coefficient to express the polarization effect of matter, but usually negligible. However, in the relativistic range, this term cannot be neglected in matter of high density. The numerical values of $\delta$ may be referred to the references 1.3 and 2.3.

In a fully ionized plasma, this rate is given by

$$-\frac{dE}{dx} = 4\pi \frac{NZ_0}{A_0} \frac{Z^2e^4}{mv^2} \left[ \ln\left( \frac{2m\beta^2}{\sqrt{1-\beta^2} \hbar \omega_p} \right) - \beta^2 + 1 \right], \quad (3.2.69)$$

where $\omega_p=\sqrt{4\pi n_e e^2/m}$ is the plasma frequency.

All the considerations given up to now can only be applied to the cases where the charged particle is not identified with the electron. In fact, for the case of electrons, the factor of the $\ln$ term varies slightly.
3.2.8.1 Landau's fluctuation

Some remarks will be here considered on ionization loss. When a charged particle makes a head-on collision with an electron in matter, it loses an energy $2mv^2$ at maximum as described earlier. Since the chance for such head-on collisions is rare, the frequency for charged particles to lose the large amount of energy as in the case head-on collision is very small for the case where the thickness of matter is not so large. When the charged particle has given most of its energy to an electron, however, such an electron, which has acquired this energy, is observed as a delta-ray.

In general, because of the rare chances for head-on collisions, there may exist fluctuations of the amount by which charged particles lose energy even if they pass through matter of constant thickness. This phenomenon is now called the Landau fluctuation. When we try to determine the atomic number $Z$ of a charged particle from the measurements of its ionization loss, this fluctuation greatly affects this procedure.

3.2.8.2 The ionization losses of low-speed particles

Equation (3.2.68) cannot be applied to cases where the speed of the charge particles is not so high. From Eq. (3.2.44), the speed of K-shell electrons inside atoms of atomic number $Z$ is estimated as $\beta = Ze^2/\hbar c = Z/137$ in order of magnitude. When the speed of the incident charged particle has approached the magnitude just mentioned, the "effective" charge of this particle decreases as a result of the capture of electrons in matter if the charge of the incident particle is positive. For example, since, as regards the iron nucleus, this effect becomes efficient if its energy is several ten MeV/nucleon or less, the rate of its energy loss becomes lower in this energy range. Furthermore, the bound states of electrons in atoms of matter have to be taken into account. See reference 2.3.2. as regards numerical results.

3.2.9 Bremsstrahlung

While passing through matter, an electron with high speed tends to be deflected away from its original direction due to the action of the Coulomb force from nuclei in matter. In consequence, it emits radio waves because of acceleration due to this force. This process is called "bremsstrahlung". This is the most important phenomenon by which high-energy electrons lose their energy in matter, and also the important process for such electrons to emit X-rays and $\gamma$-rays. In a high-temperature plasma, electrons also generate bremsstrahlung through their deflections by the electric fields of ions and other electrons there. This radiation is usually denoted as "thermal bremsstrahlung" since it is generated by electrons in the plasma in thermal equilibrium, and is the important process for the optical and X-ray emissions from high-temperature plasmas (Fig. 3.2.9). This thermal bremsstrahlung is also called the free-free transition radiation, since it is emitted from free electrons in plasmas.

When an electron passes by an atomic nucleus of charge $Ze$ with the collision parameter $b$, it acquires a speed of $\Delta v = 2Ze^2/bmv$ in the direction perpendicular to its original direction, as considered in the last section. Defining the acceleration given while passing by the atomic nucleus with $\gamma$, the total power of radiation from the electron is calculated, by applying Eq. (3.2.14), as
\[ \Delta v = \frac{2Ze^2}{bmv} \]

Fig. 3.2.9. Bremsstrahlung mechanism.

\[ \frac{2e^2}{3c^3} \int_{-\infty}^{+\infty} |\dot{y}|^2 \, dt \quad (3.2.70) \]

In order to examine the spectral distribution of the emitted electromagnetic waves, it is necessary to obtain the distribution of the frequency components after \( \dot{y} \) is Fourier-transformed. From the relation described in the foot note,* it follows that

\[
\int_{-\infty}^{+\infty} |\dot{y}|^2 \, dt = 4\pi \int_{0}^{\infty} d\omega \left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \dot{y} e^{i\omega t} \, dt \right|^2
\]

\[
= \frac{1}{\pi} \int_{0}^{\infty} d\omega \left| \int_{-\infty}^{+\infty} \dot{y} e^{i\omega t} \, dt \right|^2.
\]  

(3.2.71)

In order that \( \dot{y} \) has a large value, since the time interval has to be about equal to \( b/\nu \) or less, in which the effect of the electric field is very strong, the following relations must be accordingly satisfied:

\[
\begin{align*}
\text{for} \quad \omega < \frac{\nu}{b}, & \quad \int_{-\infty}^{\infty} \dot{y} e^{i\omega t} \, dt \approx \Delta \dot{y} = \Delta \nu \\
\omega > \frac{\nu}{b} & \quad \approx 0 \\
\end{align*}
\]  

(3.2.72)

*Defining the Fourier component of \( f(t) \) with \( F(\omega) \), the following formula is obtained:

\[ F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt. \]

Thus,

\[
\int_{-\infty}^{\infty} |f|^2 \, dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\omega) F(\omega') e^{i(\omega + \omega')t} \, d\omega d\omega' \, dt
\]

\[
= 2\pi \int_{-\infty}^{\infty} \delta(\omega + \omega') F(\omega) F(\omega') d\omega d\omega'
\]

\[
= 4\pi \int_{0}^{\infty} |F(\omega)|^2 \, d\omega.
\]

This relation is called "Perceval's theorem."
Thus, the radiation power for each frequency is deduced for the case $\omega < v / b$ as

$$\frac{2e^2}{3\pi c^3} |\Delta \nu |^2 d\omega = \frac{8}{3\pi} \frac{Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{c}{v} \right)^2 \frac{1}{b^2} \hbar d\omega . \quad (3.2.73)$$

Since there is a great variety in $b$, by integrating the above formula with respect to $b$ after being multiplied with $2\pi b \hbar$, it follows that

$$\frac{16}{3} \frac{Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{c}{v} \right)^2 \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) \hbar d\omega . \quad (3.2.74)$$

The differential cross-section for the photons to be emitted is obtained, by dividing the above formula with $\hbar \omega$ or $hf$, as

$$\sigma(v, f) df = \frac{16}{3} \frac{Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{c}{v} \right)^2 \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) \frac{df}{f} , \quad (3.2.75)$$

where the term $\ln(b_{\text{max}}/b_{\text{min}})$ is called the screening factor. In the cases for the bremsstrahlung in the ranges of X-rays, light and radio waves, the factor $(\sqrt{3/\pi}) \ln(b_{\text{max}}/b_{\text{min}})$ is called the “Gaunt” factor, which is determined by the properties of plasmas or matter, the energy of the electron and the frequencies of the X-rays.

3.2.9.1 Characteristics of bremsstrahlung

We here consider the characteristic nature of bremsstrahlung on the basis of Eqs. (3.2.74) and (3.2.75), both of which have been derived from the qualitative point of view. One characteristic is that the cross-section and the radiation power are both proportional to $Z^2 / m^2$. This situation occurs since the quantity of bremsstrahlung is proportional to the square of the acceleration. Thus, it follows that the cross-section becomes very large in matter with high atomic numbers. Since the processes such as Thomson and Compton scattering and the ionization loss are all proportional to the number density of electrons in matter, their cross-sections are necessarily in proportion with the charge number $Z$ per atom. However, the bremsstrahlung is proportional to $Z^2$ and also to $1/m^2$. This indicates that the bremsstrahlung is essentially important for particles with small mass like the electron, but is not so important for protons or other nuclei. Indeed, it has never been observed that protons may play an important role in the bremsstrahlung process.

The other important characteristic is that the differential cross-section is proportional to $df / f$ with respect to frequency. This means that, though photons of low energy are emitted in large numbers, the total amount of energy emitted is almost the same frequencies.

The exact treatment of bremsstrahlung has been made by many authors. Here, two cases will be considered; they are the non-relativistic case from low-energy electrons (thermal bremsstrahlung) and the relativistic case related to the bremsstrahlung from high-energy electrons in cosmic rays.
3.2.9.2 Bremsstrahlungs in the non-relativistic domain

When the classical treatment is applied to the factor $b_{\text{min}}$ in Eq. (3.2.75), $b_{\text{min}}$ is determined by the position where the velocity change $\Delta v$ lateral to the direction of the electron motion becomes almost equal to the initial velocity $v$. In this case, it follows from Eq. (3.2.61) that

$$b_{\text{min}} \approx \frac{Ze^2}{mv^3} = \frac{\hbar}{mv} \frac{Ze^2}{hc} \frac{c}{v}.$$  \hspace{1cm} (3.2.76)

Since the wavelength associated with the electron $\hbar/mv$ becomes larger compared with $(Ze^2/hc)(c/v)$ for the case where the latter is less than 1, it is necessary to take $b_{\text{min}} = \hbar/mv$. In order to determine the factor $b_{\text{max}}$, the changing rate of time characteristic to the change of the electron speed has to be smaller than the frequency of the radiated electromagnetic waves. Thus,

$$b_{\text{max}} \approx \frac{v}{\omega},$$

where $\omega$ is the frequency of these waves. The Gaunt factor $g \approx (\sqrt{3}/\pi)\ln(b_{\text{max}}/b_{\text{min}})$ in Eq. (3.2.74) is given as follows, in accordance with the magnitude of $Ze^2/hv$:

For the case $Ze^2/hv > 1$, the factor may be given as

$$g \approx \frac{\sqrt{3}}{\pi} \ln \left( \frac{mv^3}{Ze^2 \omega} \right).$$  \hspace{1cm} (3.2.77)

According to the Born approximation, this factor is calculated as

$$g \approx \frac{\sqrt{3}}{\pi} \ln \left( \frac{\gamma^2 mv^2}{Ze^2 \omega} \right),$$  \hspace{1cm} (3.2.77')

where $\gamma = 1/\sqrt{1-\beta^2}$. This result is almost the same as given in Eq. (3.2.77).

For the case $Ze^2/hv < 1$,

$$g = \frac{\sqrt{3}}{\pi} \ln \left( \frac{mv^2}{h\omega} \right).$$  \hspace{1cm} (3.2.78)

From the Born approximation for this case, it follows that

$$g = \frac{\sqrt{3}}{\pi} \ln \left( \frac{2\gamma^2 mv^2}{h\omega} \right) - \left( \frac{v}{c} \right)^2.$$  \hspace{1cm} (3.2.78')

In these two cases, the wave functions of electrons before and after collision are approximated as plane waves in the Born approximation method. In the real case, the wave functions of electrons are not approximated like this, because the Coulomb
electric field has to be taken into account. The exact form derived by including the
effect of this field is given as
\[ g \approx \frac{\sqrt{3}}{\pi} \frac{v_i}{v_f} \left( \frac{1 - \exp(-2\pi \xi_i)}{1 - \exp(-2\pi \xi_f)} \right) \ln \left( \frac{v_i + v_f}{v_i - v_f} \right), \]  
(3.2.79)

where \( v_i \) and \( v_f \) are, respectively, the velocities of the electron before and after
collision, and \( \xi_i = Ze^2/hv_i \) and \( \xi_f = Ze^2/hv_f \).

When using the relations
\[ \hbar \omega = \frac{1}{2} m (v_i^2 - v_f^2) \text{ and } m \nu^2 = \frac{1}{4} \]
in Eq. (3.2.77), the term \( \ln((v_i + v_f)/(v_i - v_f)) \) is easily derived. The two factors with the
exponents in Eq. (3.2.79) represent the effect of the Coulomb fields on the electron
wave functions. Referring to Eq. (3.2.75), the differential cross-section is reduced to
\[ \sigma(f)df = \frac{16}{3} \frac{Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{c}{v_i} \right)^2 \frac{v_i}{v_f} \left(1 - \exp(-2\pi \xi_i)\right) \ln \left( \frac{v_i + v_f}{v_i - v_f} \right) \cdot \frac{df}{f}. \]  
(3.2.80)

This is the exact representation of bremsstrahlung from a low-energy electron.

1) Thermal bremsstrahlung

Both electrons and ions in a high-temperature plasma in thermal equilibrium
follow the Maxwellian distribution law corresponding to the temperature of this
plasma. Because of their small mass, electrons move much faster than ions. For this
reason, using Eq. (3.2.80), it becomes possible to study the bremsstrahlung from this
plasma by assuming approximately that the electrons collide with ions at rest.
Denoting the number densities of ions and electrons with \( N_i \) and \( N_e \) and taking the
velocity distribution of electrons as \( f(v)dv \), which is identified as Maxwellian, the
number of photons emitted into unit solid angle per second is given by
\[ q_u(f)df = \frac{N_i N_e}{4\pi} \int \sigma(f) v_i f(v_i) dv_i \]
\[ = \frac{8}{3\sqrt{6\pi}} \frac{N_i N_e}{\hbar c} \frac{Z^2 e^2}{mc^2} \left( \frac{e^2}{mc^2} \right)^2 \cdot c \cdot \left( \frac{mc^2}{kT} \right)^{1/2} g(f, T) e^{-hf/kT} \frac{df}{f} (\text{m}^2\cdot\text{sec sr}), \]  
(3.2.81)

where \( g(f, T) \) is the Gaunt factor averaged over \( T \) and approximately given by
\[ g(f, T) = \frac{\sqrt{3}}{\pi} e^{(hf/kT)} K_0 \left( \frac{hf}{2kT} \right). \]
In this expression, \( K_0 \) is the modified Bessel function.* As to the more exact expression of this factor, see the reference 1.8 and 2.6. By integrating Eq. (3.2.80) over solid angle after being multiplied with the photon energy \( hf \), the total energy emitted by the bremsstrahlung from unit volume per second is given by

\[
Q_{\text{ff}} = 4\pi \int q_{\text{ff}} h f d\Omega = \frac{32}{3} \sqrt{\frac{\pi}{6}} \frac{N_e N_i Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 c(m c^2 k T)^{1/2} g(T)
\]

\[
= 1.4 \times 10^{-27} N_e N_i Z^2 T^{1/2} g(T) \text{ (erg/cm}^3\text{·sec).} \quad (3.2.82)
\]

This result gives the total power emitted from the hot plasma by the bremsstrahlung. Assuming the chemical composition of the plasma as being almost the same as that of the galactic matter which almost consists of 90% protons and 10% heliums, the average value of \( Z^2 \) is deduced to be 1.4. Since the magnitude of \( g(T) \) becomes about 1.2 for a plasma of temperature of 10^6 K as shown in reference 2.6, it follows that

\[
Q_{\text{ff}} = 2.4 \times 10^{-27} N_e^2 T^{1/2} \text{ (erg/cm}^3\text{·sec).} \quad (3.2.83)
\]

The energy maintained in 1 cm\(^3\) of the plasma is expressed by \((3/2)k T (N_e + N_i) = 3 N_e k T\). Thus, characteristic time for the plasma to cool down is given by

\[
\tau = \frac{3k T}{Q_{\text{ff}}} = \frac{1.8 \times 10^{11} T^{1/2}}{N_e} \text{ (sec).} \quad (3.2.84)
\]

It is clear that this cooling time becomes longer as the density of the plasma decreases.

2) Free-free absorption

The formula for thermal bremsstrahlung given by Eq. (3.2.83) gives the quantity of photons emitted by the collision between electrons and ions. As an inverse process, there exists the process that electrons absorb photons in the vicinity of ions. Since these electrons are never captured by ions while this process occurs, this process is called the free-free absorption. Let us consider now a high-temperature plasma, the volume of which is assumed to have been made infinitely large without change of its density. Since this plasma becomes opaque to photons and then becomes a blackbody in its final state, the radiation found in this plasma becomes that of the blackbody \((B(f))\) (see Eq. (3.2.52)). Taking the rate of absorption by free electrons as \( K_f / cm \) by referring to the Kirchhoff law, it follows that

\[
K_f B(f) = h f q_{\text{ff}} . \quad (3.2.85)
\]

* \( K_0 \left( \frac{hf}{2kT} \right) = \ln \left( \frac{4k T}{hf} \right) - 0.577 \left( \frac{hf}{2kT} \ll 1 \right)
\]

\[
= \left( \frac{k T}{hf} \right)^{1/2} e^{-hf/2kT} \left( \frac{hf}{2kT} \gg 1 \right).
\]
Hence,
\[
K_f = \frac{\hbar q_{\text{ff}}}{|\mathbf{B}(f)|} = (\frac{c^2}{2\hbar^3})(e^{\frac{hf}{kT}} - 1)(\hbar q_{\text{ff}}) = 3.7 \times 10^8 f^{-3}(1 - e^{-\frac{hf}{kT}}) N_e N_i Z^2 q_{\text{ff}}(f, T)/\text{cm} .
\] (3.2.86)

Since the absorption rate is proportional to \( f^{-3} \), it becomes significant in the low frequency range. In the case where \( hf/kT \ll 1 \), this rate is given by
\[
K_f = 3.7 \times 10^8 f^{-2} \left( \frac{h}{k} \right) \frac{1}{T^{3/2}} N_e N_i Z^2 q_{\text{ff}}(f, T)/\text{cm}
\]
\[
= 1.77 \times 10^{-2} \frac{1}{f^2 T^{3/2}} N_e N_i Z^2 q_{\text{ff}}(f, T)/\text{cm} .
\] (3.2.87)

Let us now consider the galactic radio emission around \( f \sim 1 \) MHz with the assumption that \( N_e N_i \approx 0.1 \) and \( T \approx 10^4 \) K in the galactic plasma. In this case, we obtain
\[
K_f \approx 1.77 \times 10^{-22} Z^2 q_{\text{ff}}(f, T)/\text{cm} .
\] (3.2.88)

The value of \( q_{\text{ff}} \) becomes of the order of 20 as seen from Eq. (3.2.79). Using 1.4 as the average value of \( Z^2 \), it follows that
\[
K_f = 5.0 \times 10^{-21}/\text{cm} .
\] (3.2.89)

This means that this radio emission would have been almost absorbed within the distance of several 10 pc. In fact, the results of the galactic radio observations show that, in galactic space, the non-thermal radio emission generated by high-energy electrons from the synchrotron mechanism is highly absorbed due to this process and thus the radio intensity is very low at frequencies less than about 10 MHz (see the discussion in Chapter 5).

3.2.9.3 Bremsstrahlung in the relativistic domain

When the energy of an incident electron is highly relativistic, bremsstrahlung becomes very important as the process by which this electron loses its energy. This process also plays an essential role in the interpretation of cosmic ray phenomena. The exact calculation of this process has been made by Bethe and Heitler.\(^{11,27}\)

Although the term \( \ln(b_{\text{max}}/b_{\text{min}}) \) as related to the ratio of two impact parameters is a cause for some difficulty, they have referred to the Thomas-Fermi gas model as that of real atoms in their calculations. In the case where the energy of an incident electron is enormously high, only the part of the effective charges screened off by outer-shell electrons efficiently acts on the electron. Thus, \( b_{\text{max}} \) and \( b_{\text{min}} \) are given by the radius of this model, \( 137 (\hbar/mc)(1/Z^{1/3}) \), and the nuclear radius, respectively. As a result, the differential cross-section is calculated as, taking into account the relation \( hf/E = \nu \),
\[ \sigma(E, \nu) d\nu = 4 \frac{Z^2 e^2}{\hbar c} \left( \frac{e^2}{mc^2} \right)^2 \frac{d\nu}{\nu} \cdot \left[ \left( 1 + (1 - \nu^2) - \frac{2}{3} (1 - \nu) \right) \ln(184 Z^{-1/3}) + \frac{1}{9} (1 - \nu) \right]. \]  

(3.2.90)

As already described before, this cross-section is proportional to the square of \( Z \), but inversely proportional to the square of the mass of the incident particle. In order to estimate the total amount of energy loss of the incident particle during its passage through matter, we take, as the unit of length of this matter, the following quantity

\[ \frac{1}{X_0} = 4 \frac{Z^2 e^2}{\hbar c} \frac{N}{A} \left( \frac{e^2}{mc^2} \right)^2 \ln(184 Z^{-1/3}), \]  

(3.2.91)

and the rate of energy loss is given by

\[ \frac{N}{A} \int_0^\infty \sigma(E, f) h\nu df = \frac{E}{X_0}. \]  

(3.2.92)

In other words, the original energy of the incident particle is reduced to half during its passage through the thickness given by \( X_0 \). Using this thickness \( X_0 \) as the unit of length as shown in Eq. (3.2.91), the amount of bremsstrahlung can be given as independent from the properties of the matter under consideration. The unit defined in Eq. (3.2.91) is now denoted as the “radiation unit”. This radiation unit is summarized for various matters in Table 3.2.3.

### 3.2.10 Electron pair creation

In deriving the cross-section of bremsstrahlung, the classical formula for

<table>
<thead>
<tr>
<th>Matter</th>
<th>( Z )</th>
<th>( A )</th>
<th>g/cm(^2)</th>
<th>Radiation Unit* (cm)</th>
<th>Critical Energy (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>1</td>
<td>1.003</td>
<td>63.05</td>
<td>7 012(m)</td>
<td>305.8</td>
</tr>
<tr>
<td>Carbon</td>
<td>6</td>
<td>12.011</td>
<td>42.70</td>
<td>18.9</td>
<td>125.0</td>
</tr>
<tr>
<td>Alminum</td>
<td>13</td>
<td>26.981</td>
<td>24.01</td>
<td>8.90</td>
<td>14.3</td>
</tr>
<tr>
<td>Iron</td>
<td>26</td>
<td>55.847</td>
<td>13.84</td>
<td>1.76</td>
<td>20.5</td>
</tr>
<tr>
<td>Tungsten</td>
<td>74</td>
<td>183.85</td>
<td>6.76</td>
<td>0.351</td>
<td>7.94</td>
</tr>
<tr>
<td>Lead</td>
<td>82</td>
<td>207.19</td>
<td>6.36</td>
<td>0.561</td>
<td>7.59</td>
</tr>
<tr>
<td>Air**</td>
<td></td>
<td></td>
<td>36.8</td>
<td>285(m)</td>
<td>80.3</td>
</tr>
</tbody>
</table>

* Since the effect from the radiations of electrons outside of the nuclei and others are taken into account, the numerical results on the radiation unit are different from those obtained from eq. (3.2.91). The state of 1 STP is assumed on gases.
** The atmospheric gas consists of nitrogen (75.52%), oxygen (23.14%) and Argon (1.3%) in weight.
electromagnetic radiation has been so far used to understand the physical context easily. In order to estimate roughly the cross-section based on the quantum mechanical consideration, it is enough to calculate it by referring to the diagram as shown in Fig. 3.2.10. The magnitude of the electromagnetic interaction due to the Coulomb field by the nucleus of charge $Ze$ is given by $Z^2e^2/hc$. Since the effective cross-section for the incident electron to be scattered by this Coulomb field is of the order of $(e^2/hc)^2(h/mc^2)^2$, their product becomes

$$\frac{Z^2e^2}{hc} \left( \frac{e^2}{hc} \right)^2 \left( \frac{h}{mc^2} \right)^2 = Z^2e^2 \left( \frac{e^2}{mc^2} \right)^2.$$  

(3.2.93)

This result corresponds to the magnitude of the cross-section due to bremsstrahlung as shown in Eq. (3.2.74).

As considered in the last section, in the free-free-absorption as the inverse process of bremsstrahlung, the energy of a free electron is raised after it absorbs a light quantum. When the energy of the incident photon is very high, however, electron-pair creation becomes important as the inverse process of bremsstrahlung. When the energy of incident $\gamma$-rays is higher than twice the rest energy of an electron, i.e.,

$$2mc^2 = 1.02 \text{ MeV},$$

an electron-positron pair can be created in the Coulomb field of the atomic nucleus (Fig. 3.2.11). According to the Dirac theory of electrons, this process corresponds to that in which the electron in a negative energy state is transferred to a positive energy state after the absorption of high-energy $\gamma$-rays. For this reason, after being transferred to this state, this electron is observed as a positron. Although it seems likely that high-energy $\gamma$-rays can generate an electron-positron pair even in a vacuum, such a process would never happen since some kind of field has to exist

Fig. 3.2.10. The diagrams on the bremsstrahlung processes.
which fulfills the conservation laws of both energy and momentum. For this reason, this process only occurs near the atomic nucleus where a strong Coulomb field prevails.

As in the case of bremsstrahlung, the diagram shown in Fig. 3.2.11 is obtained for electron-positron pair creation. This suggests the similarity between bremsstrahlung and the creation of a electron-positron pair. From the comparison of Fig. 3.2.10 with Fig. 3.2.11, it follows that bremsstrahlung and electron-positron pair creation are similar to each other in the pattern of interaction, and that their cross-sections are almost equal to each other. The exact formula for the cross-section has been calculated by Bethe and Heitler. Now, the differential cross-section for the electron-positron pair creation due to high-energy $\gamma$-rays is given by

$$
\sigma(h\nu, u) du = \frac{4Z^2e^4}{hc} \left( \frac{e^2}{mc^2} \right)^2 du \ln(184 \ Z^{-1/3}) \left[ u^2 + (1 - u)^2 + \frac{2}{3} u(1 - u) \right] - \frac{1}{9} u(1 - u),
$$

(3.2.94)

where $u$ is the ratio of the energy ($E$) of the created electron or positron to that of the incident $\gamma$-rays ($h\nu$) ($u = E/h\nu$). This result corresponds to the result shown in Eq. (3.2.90). It is natural that the above result is symmetric between a positron and an electron; in other words, Eq. (3.2.94) is symmetric with respect to $u$ and $(1-u)$. From the comparison of the above result with the cross-section in Eq. (3.2.90), it follows that these two cross-sections have the same form except for the relation to $\nu$ and $u$. The total cross-section is obtained by integrating Eq. (3.2.94) over $u$ from 0 to 1. The probability for the electron-positron pair creation per 1 g/cm$^2$ of matter is given by

$$
\frac{N}{A} \int_0^1 \sigma du \approx \frac{7}{9} \frac{1}{X_0}.
$$

(3.2.95)
It thus follows that the creation of an electron-positron pair occurs about once per unit radiation length.

3.2.10.2 Electron Showers

After entering matter, high-energy electrons or $\gamma$-rays generally produce bremsstrahlung or electron-positron pairs, both of which are repeated many times in the matter. This phenomenon is now known as the electron shower. These showers are considered to be the most important process among the high-energy electromagnetic interactions, and also apply to the interpretations of cosmic ray phenomena and their experimental results.

Now, let us consider a high-energy electron incident on some matter. This electron emits high-energy $\gamma$-rays at once due to bremsstrahlung, by which about half of the electron energy is given to $\gamma$-rays during the passage through one radiation unit of the matter. The $\gamma$-rays thus emitted produce an electron-positron pair after the passage through one radiation length on an average, too. Furthermore, since this pair emits $\gamma$-rays again due to bremsstrahlung, the number of electrons and positrons increases in a geometric progression. Thus, the mean energy of these electrons gradually decreases through these processes. When the amount of energy lost by ionization becomes about equal to that from bremsstrahlung through these processes, the efficiency for the generation of $\gamma$-rays becomes small so that the production of electrons ceases as they are almost stopped in the matter. As mentioned above, it is clear that the amount of the ionization loss per unit radiation length is an important factor in the development of the electron showers. This quantity is now defined as the critical energy of matter, the numerical values of which are shown in Table 3.2.3 for several different matters.

Whenever the radiation length and the critical energy of matter and the energy of incident particles are all given, the pattern of the development of the electron showers is necessarily determined. More exactly, it is not necessary to know both the energy of the incident particle and the critical energy together, but only the ratio of these two energies. Since the cross-sections for bremsstrahlung and electron-positron pair creation are given in Eqs. (3.2.90) and (3.2.94), respectively, it is most important to construct the exact theory of electron showers on the basis of these cross-sections.

The theory that considers only electrons and $\gamma$-rays of energy higher than the critical energy by neglecting ionization loss is now called the theory of the approximation A. Since the cross-sections for bremsstrahlung and electron-positron pair creation are both expressed by the ratio of the energy of primary particles to that of secondary particles, the exact results are analytically obtained for shower theory. The theory of Landau and Rummer is representative of this type of theory.\(^2\text{7)}\)

The theory which considers the effect of ionization loss becomes complicated because the terms containing the particle energy cannot be expressed as homogeneously. The representative theory, being called that of the approximation B, is that which has been developed by Tamm and Belenky and Snyder and Serber.\(^2\text{7)}\) In this theory, the energy loss of particles takes place only through ionization loss. Now, measure the depth of matter in radiation units, and denote the number of shower electrons at depth $t$ by $N(E_0, t)$, where $E_0$ is the initial energy of the incident particle.
Since the amount of energy lost between the depth \((t, t+dt)\) by ionization loss is given by \(\varepsilon dt\) for an electron, the total energy being lost by the electron showers is given by

\[
E_0 = \varepsilon \int_0^\infty N(E_0, t) \, dt.
\]  
(3.2.96)

The integration \(\int_0^\infty N(E_0, t) \, dt\) gives the sum of all the lengths of the ranges for the shower electrons, and is called the track-length. Since the energy loss rate due to bremsstrahlung is proportional to the energy of the electrons, the depth characteristic to electron showers seems to be also proportional to the logarithm of the energy, \(\ln(E_0/\varepsilon)\).

In consequence, the maximum value of shower electrons, \(N_{\text{max}}\), is estimated as, using Eq. (3.2.96),

\[
N_{\text{max}} \approx \frac{E_0/\varepsilon}{\ln(E_0/\varepsilon)} \propto \frac{E_0}{\varepsilon}.
\]  
(3.2.97)

It is certain that this number is almost proportional to the energy of the incident particle. The shower curves deduced from the theory of the approximation B are shown in Fig. 3.2.12.

So far the one-dimensional development of electron showers has been considered. However, the size of the showers is really widened laterally because of the Coulomb scattering of the electrons produced. As for the analysis of the observed shower phenomena, the question as to how these electrons are distributed around the

![Fig. 3.2.12. The development of the electron shower for the case that electrons are incident (Approximation B) (Snyder, 1949).](image-url)
shower axis becomes important. The notable theory on the three-dimensional development of the electron showers has been developed by Nishimura and Kamata.  

3.2.10.3 Processes of electron pair creation

1) Electron-positron pair creation by charged particles

Since a charged particle of charge $e$, moving with high speed, is associated with a Coulomb field for itself, it is possible to think that the number of photons given by $e^2 / hc$ are moving virtually with this particle. It seems that these virtual photons induce the creation of electron-positron pairs by the action of the Coulomb field from the atomic nuclei in matter. This process is expressed in the diagrams shown in Fig. 3.2.13, from which it follows that the cross-sections of these processes are by the factor $e^2 / hc$ less than that for the electron-positron pair creation considered earlier. Bhabha and Nishina, Tomonaga and Kobayashi have done the calculation for these processes, but the more exact calculation has been later done by Murota, Ueda and Tanaka based on quantum electrodynamics. According to their results, the cross-section of the electron-positron pair creation by muons ($\mu$-mesons) is given by

$$d^2\sigma = \frac{2Z^2}{3\pi} \left( \frac{e^2}{mc^2} \right)^2 \left( \frac{e^2}{hc^2} \right)^2 \frac{du}{u} \frac{dv}{v} \left[ \phi_s + \phi_b \left( \frac{m}{\mu} \right)^2 \right],$$  \hspace{1cm} \text{(3.2.98)}$$

where $u$ and $v$ are, respectively, the parameters defined as follows:

$$u = \frac{E_x + E_y}{E} \quad \text{and} \quad v = \frac{E_x - E_y}{E_x + E_y},$$  \hspace{1cm} \text{(3.2.99)}$$

where $E_x$, $E_y$ and $E$ are the energies of two electrons generated and the energy of the incident particle, respectively. In Eq. (3.2.98), $\mu$ is the mass of muons and the two

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{electron-positron_pair_creation.png}
\caption{The electron-positron pair creations by muons.}
\end{figure}
parameters, \( \phi_a \) and \( \phi_b \) are the cross-sections corresponding to \( a \) and \( b \) in Fig. 3.2.13, respectively. It is clear that the term containing \( \phi_b \) is by the order of \( (m/\mu)^3 \) smaller than \( \phi_a \), since the acceleration of muons becomes effective. The functional forms of \( \phi_a \) and \( \phi_b \) are very complicated because it is necessary to consider the effects such as the screening by outer-shell electrons and the size of nuclei in matter. The calculations applicable to practical problems have been made by Kokoulin and PETRUKHIM (see reference 2.8 for details).

In the case where high-energy \( \mu \)-mesons (muons) penetrate deep underground, the contribution of the electron-positron creation becomes of the same order as that of the energy loss by bremsstrahlung and thus is considered one of the important processes by which mu-mesons lose their energy. Despite the fact that the process for charged particles directly to produce pairs of electrons and positrons is the electromagnetic interaction of higher orders as compared to bremsstrahlung, the reason that this process is important for the energy loss of muons is based on the following reason: for bremsstrahlung, photons are emitted in proportion to the square of the acceleration of the incident particles. Hence, the energy loss becomes inversely proportional to the square of the mass of muons. Whereas, the physical quantity corresponding to this acceleration in the part of \( \phi_a \) which mainly contributes to the direct creation of electron-positron pairs, is identified as the acceleration given to the produced electron and positron pairs.

The ratio between the cross-sections of bremsstrahlung and of electron-positron pair creation is roughly given by

\[
\frac{1}{(m\mu)^2} : \frac{e^2}{hc} \cdot \frac{1}{(me)^2} = \left( \frac{me}{m\mu} \right)^2 : \frac{e^2}{hc} \approx 10^{-4} : 10^{-2}.
\]

This means that the contribution from the electron-positron pair creation can not be neglected. Contrary to the case for muons, the probability of the direct creation of the pairs of electrons and positrons for the case of electrons becomes of the order of \( e^2/hc \) compared with that for bremsstrahlung. Therefore, the process of electron-positron pair creation is generally negligible for the case of electrons.

2) **Electron-positron pair creation by the collision between photons**

When high-energy \( \gamma \)-rays pass through a region where the photon density is high, pairs of electrons and positrons are created as a result of the collision of the \( \gamma \)-rays with ambient photons. Although this cross-section may seem to be very small at first sight (this process is described in the diagram shown in Fig. 3.2.14), it is estimated to be of the order

\[
\frac{e^2}{hc} \left( \frac{\hbar}{mc^2} \right)^2 \approx \left( \frac{e^2}{mc^2} \right)^2 \approx 10^{-25} \text{ cm}^2.
\]  

(3.2.100)

The cross-section is not so small in this process, but is of the same order as the cross-section for Thomson scattering. Let us now consider the collision between two photons. If the energies of these two photons are assumed to be \( E_1 \) and \( E_2 \),
respectively, their total energy $Q$ in the center of gravity system is given, if the scattering angle is $\theta$ by* (see Fig. 3.2.15)

$$Q^2 = 2 \ h f_1 \ h f_2 (1 \ - \ \cos \theta) .$$  \hspace{1cm} (3.2.101)

When $Q$ is higher than $2mc^2$, it is possible for the creation of electron and positron pairs to occur. If we further assume that the velocities of the created electrons and positrons are $\nu(\beta=\nu/c)$ in the center of gravity system, it follows that $mc^2/\sqrt{1-\beta^2}=Q/2$. Thus, the cross-section is given by\(^5\,9\)

$$\sigma = \frac{\pi}{2} \left( \frac{e^2}{mc^2} \right)^2 (1-\beta)^2 \left[ (3-\beta^2) \ln \left( \frac{1+\beta}{1-\beta} \right) - 2\beta (2-\beta^2) \right] . \hspace{1cm} (3.2.102)$$

This cross-section is maximum around $\beta=0.7$ as shown in Fig. 3.2.16(a) and the maximum cross-section is obtained as

$$\sigma = 1.43 \times 10^{-25} \text{ cm}^2 . \hspace{1cm} (3.2.103)$$

If the Lorentz factor of the produced electrons is taken as $\gamma$, the cross-section in the high-energy region where $\gamma \gg 1$ is given by

$$\sigma = \pi \left( \frac{e^2}{mc^2} \right)^2 \frac{1}{\gamma^2} \left[ 2 \ln(2\gamma) - 1 \right] .$$

* $Q^2 = (hf_1 + hf_2)^2 - (hf_1 - hf_2 \cos \theta)^2 - (hf_2 \sin \theta)^2 = 2hf_1hf_2(1-\cos \theta)$.
It is therefore clear that the cross-section for this process tends to decrease with the electron energy.

Although the origin of galactic γ-ray bursts is not clear, it is known that their energy spectra are extended up to MeV energies. Since the photon density seems to be very high in and near their source regions, it seems probable that many positrons are produced due to the process of electron-positron pair creation.

3) γ-ray lines associated with electron-positron annihilation (0.5 MeV γ-ray lines)

The process in which two γ-ray photons are emitted as a result of the collision of an electron with a positron is considered as the inverse process of electron-positron pair creation. Since this process proceeds in the way reversed from that described in Fig. 3.2.14, it seems that the magnitude of this cross-section should be of the order of $\pi(e^2/(mc^2))^2$, too. In fact, the cross-section to produce two photons as a result of the annihilation by the collision of a positron with Lorentz factor $\gamma(=1/\sqrt{1-\beta^2})$ with an electron in matter is given by

$$\sigma = \pi \left( \frac{e^2}{mc^2} \right)^2 \frac{1}{\gamma + 1} \left[ \frac{\gamma^2 + 4\gamma + 1}{\gamma^2 - 1} \ln(\gamma + \sqrt{\gamma^2 - 1}) - \frac{\gamma + 3}{\sqrt{\gamma^2 - 1}} \right].$$

(3.2.104)

This result is similar to Eq. (3.2.102), which describes the process reversed from that considered above. When $\gamma$ is greater than 1, the above result can be expanded with respect to $\gamma$ and reduced to

$$\sigma = \pi \left( \frac{e^2}{mc^2} \right)^2 \frac{1}{\gamma} [\ln(2\gamma) - 1].$$

(3.2.105)
Let us consider a positron of energy $5 \text{ MeV}$. This positron passes through a distance of about $3 \text{ g/cm}^2$ (see Eq. (3.2.67)) before it stops after losing its energy by ionization loss. The number of electrons contained in this distance is about $10^{24}$. Since the cross-section given by Eq. (3.2.105) is of the order of $5 \times 10^{-26} \text{ cm}^2$, the probability that a positron is annihilated into two photons before it stops only reaches about $5\%$ at most. Most of the positrons are efficiently annihilated just before they cease to move in the matter. The cross-section near the region where most of the positrons are annihilated is calculated as

$$\sigma = \pi \left( \frac{e^2}{mc^2} \right)^2 \frac{1}{\beta} .$$

If the positron crosses the distance $\beta c \Delta t$ in time $\Delta t$ in the matter whose electron number density is assumed to be $N_e/\text{cm}^3$, the probability of annihilation becomes

$$N_e \sigma \beta c \Delta t = \pi N_e \left( \frac{e^2}{mc^2} \right)^2 c \Delta t .$$

When we assume the mass density to be $\sim 1 \text{ g/cm}^3$, it follows that $N_e = (Z/A)N = 3 \times 10^{23}$. Since the probability as mentioned above is deduced to be $\sim 10^9 \Delta t$ for this case, the positron decays into two photons with a life of the order of $10^{-9}$ seconds. In this case, these photons are observed as the emission lines consisting of two $\gamma$-rays photons of energy $0.51 \text{ MeV}$, because this positron is almost at rest.

Before positrons are annihilated with electrons, the bound state of electrons with positrons is often produced, as in the case of the hydrogen atom. This neutral electron-positron pair is called the positronium, in which there exist two ground states. In one of them, the spins of the electron and positron are oppositely directed from each other, while, in the other case, these two spins are parallel. The former and the latter states are defined as the $^1S$ and $^3S$ states, respectively. The positronium in the $^1S$ state is able to be annihilated to two photons since its total angular momentum is zero, while the positronium in the $^3S$ state has to decay into at least three photons since its total angular momentum is 1. Since this latter case is a three-body decay, the $\gamma$-rays emitted are not the line emissions of $0.5 \text{ MeV}$ energy, but are continuously distributed in energy from 0 to $0.5 \text{ MeV}$. The lifetimes for the positronium in the $^1S$ and $^3S$ states to decay into photons have been calculated to be of the order of $10^{-10}$ sec and $10^{-7}$ sec, respectively. The condition that the three-body decay of the positronium in the $^3S$ state can be observed is that this $^3S$ state is not destroyed as a result of the collision with other particles within the time $10^{-7}$ sec. The continuous spectra of $\gamma$-ray emissions from the positronium in this state cannot be expected unless the electron density is less than $10^{15}/\text{cm}^3$ in number.*

*Denoting the speed of the electron in matter and the electron density as $\beta c$ and $N/\text{cm}^3$, respectively, and assuming the cross-section of the collision of electrons with the positronium in the $^3S$ state as $\sigma_0 = 10^{-16}$ cm$^2$, the condition that such a collision does not occur within $10^{-7}$ sec is given by $\sigma_0 N \beta c 10^{-7} \leq 1$. When taking $\beta = 10^{-3}$, it follows that $N \leq 10^{15}/\text{cm}^3$. 
In solar flares or in the cases of two photon annihilation which occurs in high-temperature plasma, thermal broadening of the 0.5 MeV γ-ray lines from hot plasmas can be expected. Furthermore, it seems possible that, for the cases of two photon annihilation, the 0.5 MeV lines are shifted to the lines of several 10 KeV energy due to the red shift by the strong gravitational force as seen on the surface of neutron stars. As mentioned above, the observations of these 0.5 MeV lines are now thought of as an important means to search for the cause* of the production of positrons in space and the physical state of the areas where these positrons are annihilated with electrons.\(^2\)\(^{10}\)

### 3.2.11 Cherenkov radiation

Synchrotron radiation and bremsstrahlung have so far been considered as processes related to the emission of photons from charged particles. These processes of the emission of photons are well understood on the basis of the picture in which the photons are emitted from the high-energy charged particles being accelerated. The mechanism of Cherenkov radiation, which will be considered here, is slightly different from those of the radiations considered earlier.

Whilst a high-energy charged particle is moving in a medium of refractive index \(n\), its speed occasionally becomes higher than the speed of light in this medium, defined as \(c/n\). This medium is polarized along the path of the particle and then an oscillatory motion is induced due to the generation of a restoring force in the medium. As shown in Fig. 3.2.17, the effects of these motions induced in each part along the path of the particle in the medium are superposed along the cone making an angle \(\theta=\cos^{-1}(1/n\beta)\) from this particle path, and such resultant effects are radiated as light waves into the direction perpendicular to the surface of this cone. This is now known as Cherenkov radiation, which was discovered in 1926 by Mallet who first observed a blue light from a transparent substance located beside naturally radioactive matter. The detailed nature of this light was examined by Cherenkov in 1935 and the mechanism of this radiation was verified theoretically by Frank and Tamm\(^2\)\(^{11}\) in 1936.

*In addition to the pair creation as mentioned before, the processes such as the \(\beta^+\)-decay from radioactive substances and the following decay of \(\mu^-\) particles as \(\mu^- \rightarrow e^- + \nu_e + \nu_\mu\) are also considered as a source of positrons.
The Cherenkov light is generated only when the refractive index of matter $n$ is greater than $1/\beta(n>1/\beta)$, in other words, when the speed of a charged particle is higher than that of light in this matter. As described before, the direction in which this light is emitted makes an angle

$$\theta = \cos^{-1}\left(\frac{1}{n\beta}\right)$$

(3.2.108)

with respect to the particle path. In analogy, this situation is very similar to that of shock waves generated from a material body moving with supersonic speed. The number of photons emitted in the frequency range $(f, f+df)$ from the passage of a particle of charge $Ze$ in a length of 1 cm is given by

$$N(f) \ df = \frac{2\pi Z^2 e^2}{hc} \left(1 - \frac{1}{n^2 \beta^2}\right) df^* .$$

(3.2.109)

*Using the vector potential $A$ and the scalar potential $\phi$ in the Maxwell equation, $A$ and $\phi$ fulfill the following equations:

$$\nabla^2 \phi - \frac{n^2}{c^2} \phi = -4\pi \rho$$

$$\nabla^2 A - \frac{n^2}{c^2} A = -\frac{4\pi}{c} j .$$

When the charged particle moves along the z-axis with speed $v$ in the cylindrical coordinate system, the charge density $\rho$ and the current intensity $j_z$ are respectively given as

$$\rho = e\delta(r)\delta(Z - vt)$$

$$j_z = e\sigma(r)\delta(Z - vt) .$$

After being Fourier-transformed, using the relation

$$\delta(Z - vt) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(z - vt)}d\omega ,$$

the $\omega$-components for $\phi$ and $A_\phi$ are obtained as

$$\nabla^2 \phi_\omega + (n^2 \beta^2 - 1) \frac{\omega^2}{v^2} \phi = -e\delta(r)$$

$$\nabla^2 A_\omega + (n^2 \beta^2 - 1) \frac{\omega^2}{v^2} A_\omega = -e\delta(r) .$$

For the solutions of the waves propagating outward, the following are derived:

$$\phi_\omega, A_\omega \propto H_0^{(1)}\left[(n^2 \beta^2 - 1)^{1/2} \frac{\omega}{v} r\right] \propto \exp\left[i(n^2 \beta^2 - 1)^{1/2} \left(\frac{\omega}{v}\right) r\right].$$

Since the electric and the magnetic fields, $E$ and $H$ are derived from these solutions, the Poynting vector $(1/4\pi)[EH]$ can be calculated. From this vector, the energy distribution of emitted photons is derived as $h\nu/N(h\nu)df$, which gives that of Cherenkov radiation as shown in Eq. (3.2.109).
This radiation has the characteristics that the number of photons being emitted per unit frequency is constant in the range where the refractive index of matter is independent of wave frequency. When the dependence of the radiated power on wave frequency and wavelength is taken into account, as in real cases, the number of emitted photons per unit length is expressed as follows, based on the exact theoretical treatments,

\[
\text{the number of emitted photons } \propto Z^2 df \\
\propto Z^2 \frac{d\lambda}{\lambda^3}
\]

and

\[
\text{the radiated power } \propto Z^2 f df \\
\propto Z^2 \frac{d\lambda}{\lambda^3}
\]

where \( \lambda \) denotes the wavelength. As understood by these expressions, the energy of the emitted photons tends to concentrate towards shorter wavelengths.

Let us calculate the number of photons being emitted for the case of water, for instance. Since the refractive index \( n \) of water is given by \( n = 1.33 \), the number of these photons in the range from 4000 to 8000 Å (visible waves) in wavelength is estimated, for a charged particle of atomic number \( Z = 1 \), as

\[
\int N d \left( \frac{1}{\lambda} \right) = \frac{2\pi e^2}{hc} \left[ 1 - \frac{1}{n^2} \right] \left[ \frac{1}{\lambda_{4000}} - \frac{1}{\lambda_{8000}} \right] \approx 250 \text{ photons/cm}.
\]

(3.2.110)

This quantity of photons is much lower than the number of photons from the scintillators being used to detect charged particles.* As regards the Cherenkov radiation, however, the counters using this radiation are constructed to select the speeds of charged particles, since this radiation cannot be produced unless the speed of the charged particles is higher than the speed of light in the medium under consideration. Because the emitted power is, furthermore, proportional to \( Z^2 \), these counters are widely used to select heavy charged particles in cosmic rays.

The cases in which the refractive index \( (n) \) of matter is relatively high have so far been considered. On the contrary, for the cases in which the refractive index \( n \) is small and nearly equal to one, as in gaseous media, the limiting speed of charged particles capable of generating Cherenkov light approaches the speed of light. Taking air \( (n = 1.0003) \), for instance, this limiting speed \( \beta \) is given as

\[
\beta = \frac{1}{n} \approx 1 - 0.0003 = 1 - \frac{1}{2\gamma^2}.
\]

(3.2.111)

*For the case of NaI, the number of photons is estimated to be \( 10^7 / \text{cm} \).
where, in this case, the Lorentz factor $\gamma$ becomes about 40. Thus, the energies of the proton and electron must be of the order of $40 \text{ GeV}$ and $20 \text{ MeV}$, respectively, for them to produce Cherenkov radiation. For this reason, gas Cherenkov counters are used to distinguish the energies of high-energy nuclear components of primary cosmic rays, and the observations of Cherenkov light from the air due to the electrons in extensive air showers can be performed.

3.2.12 Physical processes in a strong magnetic field

Since the intensity of the ambient magnetic field in the Galaxy is, in general, relatively weak, the energy of photons emitted by the process of synchrotron radiation is less than that of electrons which emit these photons. For this reason, this process can be dealt within the frame of the classical theory. For such cases as neutron stars, however, the classical theory is not necessarily useful because the intensity of their surface field often reaches $10^{12}\sim 10^{13}$ gauss or more. When the curvature radius of the orbital electron becomes of the order of the electron radius, $\hbar/mc$, the classical orbital motion is never fulfilled.

Assuming the momentum of an electron to be of the order of $mc$, its orbital radius is given as

$$r = \frac{mc^2}{eH} .$$  \hspace{1cm} (3.2.112)

In order for this radius to become the same order of magnitude as the quantity $\hbar/mc$, the intensity of the magnetic field must be taken as

$$H \equiv H_0 = \frac{m^2 c^3}{e\hbar} = 4.414 \times 10^{13} \text{ gauss} .$$  \hspace{1cm} (3.2.113)

The field intensity given by $H_0$ corresponds to that of a magnetic field in which the quantum mechanical effect should be taken into account.

3.2.12.1 Cyclotron radiation and the Landau level

Let us first consider an electron of momentum $P=\hbar \mathbf{v}$, moving in a magnetic field. When the de Broglie wavelength $\lambda$ of this electron approaches the order of the orbital radius $r=mc^2\beta/eH$, a quantum mechanical effect appears in the motion of the electron. Accordingly, the electron is unable to take any of the circular orbits freely, but can only take quantized orbits as in the case of the hydrogen atom. Assuming that this electron is moving circularly in the plane perpendicular to the direction of the magnetic field, and calculating its orbital motion as in the classical atomic model considered by Bohr, it follows that the quantized energy of the electron is given as $n(H/H_0)mc^2$,\textsuperscript{*} where $n=0, 1, 2, \ldots$. The exact solution of the wave equations, being obtained quantum mechanically, also gives the quantum levels as $n(H/H_0)mc^2$. After

\textsuperscript{*}Take the direction of the magnetic field along the z-axis in the cylindrical coordinate system. In this case, $A_z=(1/2)rH_z$. Thus it follows that $P_z=(e/c)A_z=m\mathbf{v}$. If we take the Larmor radius $mc^2/eH$ in place of $r$, we obtain $P_z=(m\mathbf{v}/2)$. The result given in the text is thus obtained using the relation $P_z=e\hbar$. 


being extended to the relativistic case, the quantized energy of the electron is given by

\[ E = \left[ mc^2 + P_\parallel c^2 + 2n \frac{H}{H_q} mc^2 \right]^{1/2}, \quad n = 0, 1, 2 \ldots \]  \hspace{1cm} (3.2.114)

These are called the Landau levels, in which \( P_\parallel \) gives the energy of the electron along the magnetic field. In the case of non-relativistic electrons, the above result is also written as

\[ E_{\text{kin}} = \frac{P_\parallel^2}{2m} + n \frac{H}{H_q} mc^2. \]  \hspace{1cm} (3.2.115)

When the transition occurs from the state of \( n \) to a lower state, a quantum of energy corresponding to multiples of \((H/H_q)mc^2 = 1.16 \times 10^{-16} \text{ eV}\) is released. When the field intensity \( H \) is \( \sim 10^{12} \text{ gauss} \), the energy of this quantum becomes about 10 KeV. Observations of such line emissions have been reported from some special X-ray stars and \( \gamma \)-ray bursters.

When the quantum number \( n \) is very large, the variation with \( n \) can be assumed as almost continuous. Consequently, the classical theory can be applied without any modification. The treatment of synchrotron radiation mentioned here corresponds to the emission of a photon due to the transition from the orbit of quantum number \( n \) to that of the other quantum number, as viewed from quantum mechanics.

In addition to the condition given by Eq. (3.2.113) on the limitation of the classical theory to the problems related to a strong magnetic field, the problem as discussed in the following also seems to happen in the case where the Lorentz factor is enormously large. In the coordinate system in which the electron is at rest, as considered in Subsection 3.2.5, this electron is thought of as scattering the photon of energy \( \gamma \beta \hbar \times (eH/mc) \). Since this electron further scatters another photon during an oscillation period as given by \((\gamma \beta eH/mc)^{-1}\), the total energy of scattered photons is obtained as

\[ \left( \frac{2}{3} \sigma_T \gamma^2 \beta^2 \frac{H^2}{8\pi} \right) \left( \frac{eH}{mc} \right) \approx \frac{e^2}{\hbar c} \gamma^2 \frac{H}{H_q} mc^2. \]  \hspace{1cm} (3.2.116)

When the magnitude of the above expression is larger than the rest energy of an electron \( mc^2 \), namely when

\[ \frac{e^2}{\hbar c} \gamma^2 \frac{H}{H_q} \geq 1, \text{ i.e., } \gamma^2 H \geq 5 \times 10^{15} \text{ gauss}, \]

it is impossible to deal with the problem from the classical theoretical point of view. As regards this case, see reference 2.11.

3.2.12.2 Electron-pair creation by \( \gamma \)-rays

While traversing a strong magnetic field, a high-energy \( \gamma \)-ray photon is able to
produce an electron-positron pair as mentioned in Subsection 3.2.10. This process proceeds as shown in Fig. 3.2.18. At first, the incident γ-ray photon produces a virtual pair of an electron and a positron, and then this virtually created electron absorbs a virtual photon in the magnetic field before finally becoming a real electron. Taking the density of virtual photons to be \((H^2/8\pi)[\hbar(eH/mc)]\) in the magnetic field, the probability that such a process may occur for 1 cm is of the order of

\[
\left( \frac{e^2}{\hbar c} \right)^2 \left( \frac{h}{mc} \right)^2 \frac{Hmc}{\hbar} = \frac{e^2}{\hbar c} \left( \frac{mc}{\hbar} \right) \frac{H}{H_q} . \tag{3.2.117}
\]

The exactly calculated result is here given as \(2, 10, 2.11\)

\[
\frac{1}{2} \frac{e^2}{\hbar c} \frac{mc}{\hbar} \frac{H}{H_q} \cdot T(x) , \text{ where } x = \frac{1}{2} \frac{hf}{mc^2} \frac{H}{H_q} . \tag{3.2.118}
\]

The values of \(T(x)\) are given as follows:

\[
T(x) = 0.60 \, x^{-1/3} \quad \text{for } x \gg 1
\]

and

\[
= 0.46 \left[ -\frac{4}{3x} \right] \quad \text{for } x \ll 1 .
\]

If we take γ-rays of energy 2 MeV \((hf/mc^2 \approx 4)\), as an example, and consider the two cases of the intensity of the magnetic field given as \(H = 10^{12}\) gauss and \(H = 4 \times 10^{12}\) gauss, respectively, it follows that \(x = 0.05\) and 0.2 since \(H_q = 4.4 \times 10^{13}\) gauss. Therefore, the probability of the occurrence of electron-positron pair creation is estimated to be \(2.85 \times 10^{-6}\) and 7.04 for the cases mentioned above. The mean-free paths are respectively given by 3.5 km and 0.15 cm for these two cases, while the free-flight times are correspondingly given by \(10^{-5}\) sec and \(10^{-11}\) sec. These results are highly suggestive that electron-positron pair creation is not negligible for high-energy γ-rays passing through the space where a strong magnetic field prevails.

### 3.2.12.3 Fission of photons

The process in which an incident photon is divided into two photons due to the

![Fig. 3.2.18. The creation of electron-positron pair in a strong magnetic field.](image-url)
action of a strong magnetic field has also been investigated. Since this process is similar to that as shown in Fig. 3.2.19 for photon-photon scattering, its probability is estimated as

$$\left( \frac{e^2}{hc} \right)^3 \left( \frac{mc}{h} \right) \frac{H}{H_q} .$$  \hspace{1cm} (3.2.119)

The exact result calculated for this process gives the probability of this process as

$$\tau^{-1} = \frac{5}{3} \frac{1}{(144\pi)^2} \left( \frac{e^2}{hc} \right)^3 \left( \frac{mc}{h} \right) \left( \frac{H}{H_q} \right)^2 \frac{hf}{mc} .$$  \hspace{1cm} (3.2.120)

For the case in which the energy of the photon is $hf = 2$ MeV as in the case considered earlier, the mean-free paths are respectively given by $3.96 \times 10^{24}$ cm and $6.18 \times 10^{22}$ cm with respect to the intensities of magnetic fields of $H = 10^{12}$ gauss and $4 \times 10^{12}$ gauss. It therefore seems that this process never becomes important as long as the magnetic field is not so strong.

Fig. 3.2.19. Fission of a photon in a strong magnetic field.

REFERENCES

I. General references

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