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Hydrodynamics, magnetohydrodynamics, and astrophysical plasmas

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1 Introduction

It is not uncommon to encounter the assertion that the dynamical evolution of the large-scale bulk velocity **v** of a collisionless gas is not described by the hydrodynamic (HD) equations. A favorite "reason" is the statement that pressure cannot be defined in the absence of collisions. Then in some quarters it is believed that the large-scale bulk dynamics of a swirling magnetized ionized gas must be described in terms of the electric current density **j** and the electric field **E** rather than **B** and **v**, thereby rejecting magnetohydrodynamics (MHD). This point of view arises from the conviction that **j** is the cause of **B**, and, therefore, the more fundamental field variable.

As a matter of fact, HD is based on the concepts of conservation of particles, momentum and energy, so HD cannot be avoided whether there are interparticle collisions (which conserve particles, momentum and energy) or not. So the basic HD equations cannot be avoided.

The dynamics of a swirling magnetized plasma follows from the equations of Newton, Maxwell, Lorentz, etc., from which one discovers that the dynamical theory cannot be formulated in terms of **j** and **E** in tractable form. The dynamical equations become nonlinear global integro-differential equations, of little use in a time varying system. Unfortunately the response often is to maintain the supremacy of **j** and **E** and turn away from Newton and Maxwell to seemingly plausible, but largely false, concepts.

The standard textbooks of HD and MHD are generally correct in their derivation of the dynamical equations from Newton, Maxwell, and Lorentz, but they do not address the popular misunderstandings and the rejection of HD and MHD. In fact, the basic equations of HD and MHD are simply derived and cannot be avoided in the large-scale bulk dynamics of a collisionless gas, with or without a magnetic field. The purpose of this article is to show that Newton, Maxwell, and Lorentz constrain the theory to HD and MHD and exclude the aforementioned misconceptions.

2 Hydrodynamics

Consider a collisionless gas in swirling motion in the absence of any applied forces, e.g. gravity, magnetic stresses, etc. In this simple case each particle moves in a straight line with constant speed regardless of the presence or absence of the other particles, and it is not immediately obvious that the sum of all these free particle motions has the overall large-scale nature of HD. So we must proceed formally from the principles of conservation of particles, momentum and energy.

The first question is how many particles are necessary to justify the fluid concept that we call HD? It is evident there must be *enough particles to provide a statistically well defined local mean number density* N. For if there are not enough particles, then neither the mean density nor the local bulk velocity can be defined with precision, and the fluid concept described by HD is not useful. Consider, then, a system with characteristic scale L. That implies a statistically well defined local mean density on some much smaller scale l. Experience with numerical simulations on a grid with spacing l suggests that l as small as $10^{-3}L$ works very well, allowing spatial derivatives to be computed with adequate accuracy. Using this as our criterion, it follows that there are Nl^3 particles in the basic cell, with volume $V = l^3$. The statistical fluctuation ΔN in the number Nl^3 of particles in the cell is of the order of $(Nl^3)^{1/2}$, so that $\Delta N/N$ is of the order of $1/(Nl^3)^{1/2}$. This condition is quite generally satisfied in astrophysical settings. For instance, the flow of the solar wind around the terrestrial magnetosphere is on a scale of $L \geq 10^8$ cm, so we might need l as small as 10^5 cm. The solar wind density is typically 5 ions/cm^3, so that the number of particles is $Nl^3 = 5 \times 10^{15}$. The statistical fluctuations are hardly more than one part in 10^8, and quite sufficient for our purposes. Needless to say, the internal small-scale structure of the standing shock upstream from Earth cannot be treated with HD, and we content ourselves there with application of the Rankine-Hugoniot relations.

Now, if the local mean density N is statistically well defined, then so is the local mean bulk velocity $\mathbf{v}$, or v_i. Denoting the velocity of an individual particle by u_i, it follows that

$$N = \frac{1}{V}\Sigma, \; N v_i = \frac{1}{V}\Sigma u_i, \tag{1}$$

where Σ means the sum over all the particles in the volume $V = l^3$. Let w_i represent the thermal velocity of an individual particle relative to the bulk motion v_i, so that

$$u_i = v_i + w_i \tag{2}$$

and

$$\frac{1}{V}\Sigma w_i = 0. \tag{3}$$

It follows that the momentum density of the particles of mass M is

$$\frac{1}{V}\Sigma M u_i = N M v_i, \tag{4}$$

and the flux of momentum density is

$$\frac{1}{V}\Sigma M u_i u_j = N M v_i v_j + p_{ij}, \tag{5}$$

where

$$p_{ij} = \frac{1}{V}\Sigma M w_i w_j \tag{6}$$

is the pressure tensor, representing the flux of thermal momentum transported by the thermal motions. A momentum flux is equivalent to a force per unit area, i.e. pressure, regardless of the presence or absence of interparticle collisions. The first term on the right hand side of Eq. (5) is the Reynolds stress tensor,

$$R_{ij} = NMv_i v_j \tag{7}$$

representing the momentum of the bulk flow transported by the bulk flow.

Note that the diagonal terms of $\frac{1}{V}\Sigma M u_i u_j$ represent twice the kinetic energy density, so the flux of kinetic energy is contained in the tensor

$$\begin{aligned}\frac{1}{V}\Sigma M u_i u_j u_k &= \frac{1}{V}\Sigma M(v_i + w_i)(v_j + w_j)(v_k + w_k)\\ &= NMv_i v_j v_k + v_i p_{jk} + v_j p_{ki} + v_k p_{ij} + T_{ijk},\end{aligned} \tag{8}$$

where

$$T_{ijk} = \frac{1}{V}\Sigma M w_i w_j w_k \tag{9}$$

represents the thermal transport of thermal energy, i.e. the heat flow tensor.

Now particles, momentum, and kinetic energy are conserved quantities. Suppose, then, that Q represents the density of a conserved quantity. The time rate of change of that quantity in a fixed volume U is equal to the rate at which Q flows inward across the surface S of the volume U,

$$\begin{aligned}\frac{\partial}{\partial t}\int_U d^3r\, Q &= -\int_S dS_j (Q_{\text{flux}})_j\\ &= -\int_U d^3r \frac{\partial}{\partial x_j}(Q_{\text{flux}})_j,\end{aligned}$$

upon applying Gauss's theorem. The equality holds for every volume U, requiring that

$$\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial x_j}(Q_{\text{flux}})_j \tag{10}$$

at every point. So the time rate of change of the density Q of a conserved quantity is equal to the negative divergence of the flux of Q.

It follows that conservation of particles requires

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_j} N v_j = 0. \tag{11}$$

Conservation of momentum requires

$$\frac{\partial}{\partial t} NMv_i + \frac{\partial}{\partial x_j} NMv_i v_j = -\frac{\partial p_{ij}}{\partial x_j}. \tag{12}$$

Conservation of kinetic energy requires conservation of the diagonal terms in p_{ij}, requiring conservation of the tensor p_{ij}. So the time rate of change of $\Sigma M u_i u_j$ must be equal to the negative divergence of $\Sigma M u_i u_j u_k$, leading to

$$\frac{\partial}{\partial t} N M v_i v_j + \frac{\partial}{\partial x_k} N M v_i v_j v_k + \frac{\partial p_{ij}}{\partial t} = -\frac{\partial}{\partial x_k}(v_i p_{jk} + v_j p_{ki} + v_k p_{ij}) - \frac{\partial T_{ijk}}{\partial x_k}. \quad (13)$$

Multiply Eq. (11) by v_i and subtract from Eq. (12), providing the familiar Euler equation

$$N M \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p_{ij}}{\partial x_j}. \quad (14)$$

Equations (11) and (12) can be used to reduce Eq. (13) to

$$\frac{\partial p_{ij}}{\partial t} + v_k \frac{\partial p_{ij}}{\partial x_k} = -p_{jk} \frac{\partial v_i}{\partial x_k} - p_{ki} \frac{\partial v_j}{\partial x_k} - p_{ij} \frac{\partial v_k}{\partial x_k} - \frac{\partial T_{ijk}}{\partial x_k}. \quad (15)$$

Equations (11), (14), and (15) are the basic equations of HD. The pressure p_{ij} may be anisotropic in the absence of collisions, of course, but the principles are HD.

Several comments are in order here. First of all, the complexity of the right hand side of Eq. (15) is nothing more than a representation of adiabatic heating and cooling in the presence of compression and expansion, respectively, as well as viscosity in the presence of velocity shear. To illustrate the effects consider a uniform plasma density and uniform temperature subject to the one dimension expansion $\partial v_1 / \partial x_1$ with all other elements of $\partial v_i / \partial x_j$ equal to zero. The left hand side of Eq. (15) is the Lagrangian derivative dp_{ij}/dt, so that

$$\frac{dp_{11}}{dt} = -3 p_{11} \frac{\partial v_1}{\partial x_1}, \ \frac{dp_{22}}{dt} = -p_{22} \frac{\partial v_1}{\partial x_1}, \ \frac{dp_{33}}{dt} = -p_{33} \frac{\partial v_1}{\partial x_1}, \quad (16)$$

$$\frac{dp_{12}}{dt} = -2 p_{12} \frac{\partial v_1}{\partial x_1}, \ \frac{dp_{13}}{dt} = -2 p_{13} \frac{\partial v_1}{\partial x_1}, \ \frac{dp_{23}}{dt} = -p_{23} \frac{\partial v_1}{\partial x_1}. \quad (17)$$

Noting that

$$\frac{1}{N} \frac{dN}{dt} = -\frac{\partial v_1}{\partial x_1}, \quad (18)$$

it follows that

$$p_{11} \propto N^3; \ p_{12}, p_{13} \propto N^2; \ p_{22}, p_{33}, p_{23} \propto N. \quad (19)$$

Anisotropic expansion in two and three dimensions provides somewhat more complicated results.

Consider, then, a uniform shear, with $\partial v_1 / \partial x_2$ as the only nonvanishing term in $\partial v_i / \partial x_j$. The result is

$$\frac{dp_{11}}{dt} = -2 p_{12} \frac{\partial v_1}{\partial x_2}, \ \frac{dp_{22}}{dt} = \frac{dp_{33}}{dt} = 0, \quad (20)$$

$$\frac{dp_{12}}{dt} = -p_{22} \frac{\partial v_1}{\partial x_2}, \ \frac{dp_{23}}{dt} = 0, \ \frac{dp_{31}}{dt} = -p_{23} \frac{\partial v_1}{\partial x_2}. \quad (21)$$

Noting that p_{22}, p_{23}, p_{33} do not vary with time, it follows that

$$p_{12}(t) = p_{12}(0) - p_{22}\frac{\partial v_1}{\partial x_2}t,\ p_{13}(t) = p_{13}(0) - p_{23}\frac{\partial v_1}{\partial x_2}t, \tag{22}$$

so that

$$p_{11}(t) = p_{11}(0) - 2p_{12}(0)\frac{\partial v_1}{\partial x_2}t + p_{22}\left(\frac{\partial v_1}{\partial x_2}\right)^2 t^2. \tag{23}$$

Note that $p_{12}(t)$ represents the transport of the $i = 1$ component of momentum in the $j = 2$ direction across the velocity gradient. It is the viscous stress across the velocity gradient. It increases linearly with time as particles arrive from farther away across the velocity gradient. The increase in the component p_{11} represents the associated viscous heating. Interparticle collisions would limit growth to the time between collisions. (See Parker, 2007, p. 86 for further discussion).

The essential point is that there is nothing mysterious about a collisionless fluid. The interpenetration of differently moving regions of fluid is rapid, as we would expect from their free passage. The effective viscosity increases rapidly with the passage of time as the interpenetration continues. But it all has the nature of HD with initial conditions and boundary conditions determining the evolution of the fluid.

In astrophysical settings a purely collisionless state is a rarity. The ubiquitous magnetic fields limit the particles to cyclotron gyrations in the two dimensions perpendicular to the magnetic field, so the two pressure components are isotropic. Plasma waves scatter the particles between the parallel and perpendicular directions, and any strong anisotropy is unstable to the generation of more plasmas waves. The thermal anisotropies observed in the solar wind at the orbit of Earth are muted by scattering of some sort after leaving the Sun. The principal expansion of the wind is transverse to the radial expansion, and the approximately radial magnetic field, once the wind is up to speed, so a wind without particle scattering would yield a very low transverse temperature, well below 10^4 K. But it is never so low, and sometimes actually exceeds the radial temperature component at time of solar activity. Obviously there is strong transverse wave heating. The scattering and heating is highly variable and cannot be predicted in any useful way, so the theorist usually can do no better than to pursue the HD with a simple scalar pressure p and a plausible estimate of the manner of variation of the temperature, looking for the basic HD behavior of the streaming fluid.

This brings us to our second comment on the HD equations (11), (14), and (15). They do not form a complete set, because of the unknown energy deposition and scattering just mentioned. Computing higher order tensors (computing higher velocity moments of the collisionless Boltzmann equation) does not help, because the physics of the plasma waves in the astrophysical setting is unknown, and no amount of mathematics can conjure it up. As a practical matter we write Eq. (15) as

$$\frac{dp_{ij}}{dt} = -p_{jk}\frac{\partial v_i}{\partial x_k} - p_{ki}\frac{\partial v_j}{\partial x_k} - p_{ij}\frac{\partial v_k}{\partial x_k} + S_{ij}. \tag{24}$$

Where S_{ij} is the total heat input, including thermal conduction, radiative heating, and plasma wave heating. One simply makes the best estimate of S_{ij} and proceeds from there in pursuit of the HD. (For further discussion see Parker, 2007, chap. 8 and 9.)

So far as the momentum equation is concerned, there are generally applied forces F_i, e.g. gravity, magnetic stresses, etc., so that the momentum equation (14) becomes

$$NM\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\frac{\partial p_{ij}}{\partial x_j} + F_i. \tag{25}$$

In terms of the magnetic stress tensor,

$$M_{ij} = -\delta_{ij}\frac{B^2}{8\pi} + \frac{B_i B_j}{4\pi},$$

representing the isotropic pressure $B^2/8\pi$ and the tension $B^2/4\pi$ along the field, the force per unit volume exerted on the plasma is

$$F_i = \frac{\partial M_{ij}}{\partial x_j}. \tag{26}$$

3 Magnetohydrodynamics

MHD treats the large-scale dynamics of an ionized gas, i.e. an electrically conducting fluid, with a magnetic field **B** throughout. So it is an extension of HD to include the magnetic stresses. We need, then, to write down the appropriate induction equation for the time variation of **B**. The essential feature of an electrically conducting fluid is its inability to sustain a significant electric field in its own moving frame of reference, resulting in the magnetic field being carried along bodily with the moving fluid. The electric current density **j** is determined by the form of **B** and described by Maxwell's equation,

$$4\pi\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B}$$

in cgs electrostatic units. The development is limited to the non relativistic case ($v \ll c$), from which it is evident that $\partial/\partial t$ is small compared to c curl by $O(v/c)$. It follows from Eq. (32) that **E** is small compared to **B** to the same order, so that is small compared to $c\nabla \times \mathbf{B}$ to second order in v/c. That is the same order of smallness as Lorentz-Fitzgerald contraction and time dilatation, which we also neglect. So Maxwell's equation reduces to Ampere's law,

$$4\pi\mathbf{j} = c\nabla \times \mathbf{B}. \tag{27}$$

Thus a field **B** with a characteristic scale L has a current density of the order of magnitude

$$j = O\left(\frac{c}{4\pi}\frac{B}{L}\right) \tag{28}$$

associated with it. The electrical resistivity of an ionized gas, or plasma, is so small that, over the large scales L appropriate for the astrophysical universe, only a very weak electric field $\mathbf{E}'$ in the moving frame of reference of the plasma is required to drive $\mathbf{j}$. To provide a simple illustrative example, suppose that the plasma is so dense as to be collision dominated, so that the scalar Ohm's law is a useful approximation. Denoting the electrical conductivity by σ it follows that $\mathbf{j} = \sigma\mathbf{E}'$. Then, in order of magnitude,

$$\frac{E'}{B} = O\left(\frac{\eta}{cL}\right), \tag{29}$$

where $\eta = c^2/4\pi\sigma$ is the resistive diffusion coefficient. In ionized hydrogen

$$\eta \approx \frac{4 \times 10^{12}}{T^{\frac{3}{2}}},$$

so that

$$\frac{E'}{B} = O\left[\left(\frac{10^{-4}}{L}\right)\left(\frac{10^4}{T}\right)^{\frac{3}{2}}\right]. \tag{30}$$

As an example, let $T = 10^4$ K and L be as small as 10^7 cm, comparable to the photospheric diameter of a magnetic fibril in the Sun. It follows that $E'/B \approx 10^{-11}$. The electric stresses, $E'^2/8\pi$, are only the fraction 10^{-22} of the magnetic stress, $B^2/8\pi$, and quite negligible. So the electric field in the moving frame of the local plasma plays no role in the large-scale dynamics of the plasma. The dynamics is described by HD in the presence of the magnetic force $F_i = \partial M_{ij}/\partial x_j$ on the right hand side of Eq. (25).

Now in the astrophysical world, where most velocities $\mathbf{v}$ are small compared to the speed of light c, the electric field $\mathbf{E}'$ in the moving frame of the plasma is related to the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ in the coordinate system by the nonrelativistic Lorentz transformation

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}, \tag{31}$$

with $\mathbf{B}' = \mathbf{B}$, of course. It is evident from Eq. (30) that $\mathbf{E}'$ can be neglected, with the result that there is an electric field

$$\mathbf{E} = -\frac{\mathbf{v} \times \mathbf{B}}{c} \tag{32}$$

in the frame of the coordinates. This electric field exists *because* there is no electric field in the moving frame of the plasma. The stresses in the electric field $\mathbf{E}$ are much larger than $\mathbf{E}'$ in the moving frame of the plasma, but still quite negligible in the nonrelativistic approximation, being small $O(v^2/c^2)$ compared to the magnetic stresses. So $\mathbf{E}$ plays no dynamical role when $v \ll c$, in spite of an incorrect popular notion that $\mathbf{E}$ drives motions in the terrestrial ionosphere and magnetosphere.

It is unfortunate that there is sometimes confusion as to the appropriate electric field for particle acceleration or for driving an electric current in a moving system.

So we digress briefly to make clear how the problem should be addressed. Consider the question whether there is an electric field pervading the laboratory or office where you, the reader, are working. There is a magnetic field, of course, of 0.3–0.5 Gauss associated with Earth. There may be an atmospheric electric field of the order of 1 Volt/cm as a consequence of the ongoing tropical thunderstorms charging Earth negative by some 3×10^5 Volts relative to the ionosphere. So enclose the laboratory with a fine grounded copper screen to eliminate this natural nuisance. The answer to the question would seem to be that there is no electric field ($\mathbf{E} = 0$) in the laboratory. However, if you rise from your chair and walk across the laboratory with velocity $\mathbf{v}_1$, Eq. (31) indicates that you observe an electric field $\mathbf{E}'(\mathbf{v}_1) = \dfrac{\mathbf{v}_1 \times \mathbf{B}}{c}$. Walking in another direction with velocity $\mathbf{v}_2$ provides an electric field $\mathbf{E}'(\mathbf{v}_2) = \dfrac{\mathbf{v}_2 \times \mathbf{B}}{c}$, etc. It is obvious, then, that a more general answer to the question would be that there are infinitely many distinct electric fields $\mathbf{E}(\mathbf{v}_\mathrm{n})$ in the laboratory, each in its own moving reference frame. This emphasizes that the physically relevant electric field is the field in the reference frame of the particle or of the plasma in which the electric current flows. The electric field is different in different moving frames so the outcome of a theoretical calculation depends on picking the correct electric field out of the infinitely many different fields in the laboratory. This is one of the reasons that attempts to formulate the dynamics in terms of $\mathbf{E}$ and $\mathbf{j}$ become intractable in general time dependent situations.

With these preliminary remarks, consider the dynamics of the large-scale bulk motion of a plasma and magnetic field. With the electric field given by Eq. (32) the Faraday induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}, \tag{33}$$

becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{34}$$

recognizable as the MHD induction equation. The flux of electromagnetic energy is given by the Poynting vector

$$\begin{aligned} \mathbf{P} &= c\frac{\mathbf{E} \times \mathbf{B}}{4\pi}, \\ &= \mathbf{v}_\perp \frac{B^2}{4\pi}, \end{aligned} \tag{35}$$

where $v_\perp$ is the component of the plasma velocity $\mathbf{v}$ perpendicular to $\mathbf{B}$, and $B^2/4\pi$ is the magnetic enthalpy density. This shows directly that the magnetic field is carried bodily with the plasma, i.e. the magnetic field moves in the frame of reference in which there is no electric field. So MHD is inescapable in the large-scale swirling plasmas that make up the astrophysical universe.

An immediate consequence of the bodily transport of magnetic field in the plasma is the deformation of the magnetic field, producing the nonvanishing Lorentz force

$F_i = \partial M_{ij}/\partial x_j$ on the plasma, as already described by Eqs. (25) and (26). The Lorentz force can also be written

$$\mathbf{F} = -\nabla \frac{B^2}{8\pi} + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}, \tag{36}$$

$$= \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}, \tag{37}$$

$$= \frac{\mathbf{j}\times\mathbf{B}}{c}. \tag{38}$$

Equation (38) is written using Ampere's law, Eq. (27), to replace $\nabla\times\mathbf{B}$ by the current density $\mathbf{j}$. This simple mathematical substitution must not be interpreted as a dynamical role for $\mathbf{j}$, for the current contains no significant energy or stress in itself. The current represents only the slow conduction drift of the electrons relative to the ions, with negligible inertia. It is a passive quantity, driven by $\mathbf{E}'$ so as always to conform to Ampere's law, i.e. to the deformation $\nabla\times\mathbf{B}$ of the magnetic field.

Now the physical universe is not an ideal system and $\mathbf{E}'$ is small but not identically zero. In fact over smaller scales, and particularly in gases with only a slight ionization, $\mathbf{E}'$ may be non-negligible. Shock fronts and the intense current sheets associated with rapid reconnection represent small scales. Then the solar photosphere has but little ionization, so that $\mathbf{E}'$ may be non-negligible over dimensions of no more than 10^7 cm. As an example, consider the case again that the plasma density is sufficiently large that the familiar scalar Ohm's law is a useful approximation. Then with $\mathbf{E}' = \mathbf{j}/\sigma$ the induction equation takes the familiar form

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \nabla\times(\eta\nabla\times\mathbf{B}), \tag{39}$$

reducing to

$$\frac{\partial\mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}$$

when the resistive diffusion coefficient η is uniform. The reciprocal of the magnetic Reynolds number, defined as $R_\mathrm{m} = vL/\eta$ in terms of the characteristic velocity v and scale L, is a measure of the smallness of the resistive diffusion term compared to the first term on the right hand side of the induction equation. On astrophysical scales, R_m is very large compared to one. So the diffusion term is small and can be neglected so far as the gross dynamics of the system is concerned. It must be appreciated, however, that over time even a very small amount of diffusion may play an important role by reconnecting the field lines, as in the rapid reconnection phenomenon where the field spontaneously develops small scales. Indeed, in all but the simplest field line topologies the Lorentz force pushes the field toward the formation of surfaces of tangential discontinuity, i.e. intense current sheets, where resistive diffusion becomes important (Parker, 1972, 1994). It must be appreciated, too, that the MHD $\alpha\omega$-dynamo requires substantial diffusion if it is to function efficiently in stars and galaxies. It is usually assumed that turbulent diffusion takes care of the problem, but it is not clear how turbulent diffusion can be accomplished in the strong magnetic fields to be found in stars and galaxies (Parker, 1992; Vainshtein *et al.*, 1993). The turbulent motions would appear to be restricted to Alfven waves, whose passage provides little or no diffusion and dissipation of fields. But that is a problem for another time.

A plasma has internal microscopic structure, e.g. the cyclotron gyrations of the ions and electrons, and perhaps only partial ionization. It is sometimes stated that MHD does not apply to a partially ionized gas because MHD ignores the Hall effect and the Pedersen resistivity. As a matter of fact, these effects are the same order in $1/L$ as the resistive diffusion term, and there is no reason why they should not be included in MHD, along with the dominant MHD induction $\nabla \times (\mathbf{v} \times \mathbf{B})$ term, in circumstances where they play a significant role. What is more, the Hall effect exists whether the gas is wholly or only partly ionized and whether there are interparticle collisions or not. To illustrate the variety of effects that arise consider a partially ionized gas, consisting of N neutral atoms per unit volume and n electrons and n singly charged ions per unit volume. The ions, electrons, and neutral atoms all collide with each other, which we represent with linear scattering terms. The time over which an individual ion collides with a neutral atom is denoted by τ_{i}, with τ_{e} the time in which an electron does the same. The time over which an electron collides with an ion is τ. The mass of the neutral atom and the ion are both designated by M, and the electron mass is m. Then denote the bulk velocity of the neutral atoms by $\mathbf{v}$, the ions by $\mathbf{w}$, and the electrons by $\mathbf{u}$ (not to be confused with the $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ employed in Section 2 in the discussion of HD). The equation of motion for the neutral atoms becomes

$$NM\frac{d\mathbf{v}}{dt} = -\nabla p + \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_{\mathrm{i}}} + \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_{\mathrm{e}}} + N\mathbf{F}, \tag{40}$$

where $\mathbf{F}$ is the external force, e.g. gravity, exerted on each atom. Treating the ions as a fluid, the equation of motion is

$$nM\frac{d\mathbf{w}}{dt} = -\nabla p_{\mathrm{i}} + ne\left(\mathbf{E} + \frac{\mathbf{w}\times\mathbf{B}}{c}\right) - \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_{\mathrm{i}}} - \frac{nm(\mathbf{w}-\mathbf{u})}{\tau} + n\mathbf{f}_{\mathrm{i}}, \tag{41}$$

where p_{i} is the ion pressure and $\mathbf{f}_{\mathrm{i}}$ is the external force on each ion. The equation of motion for the electrons in the same fluid approximation is

$$nm\frac{d\mathbf{u}}{dt} = -\nabla p_{\mathrm{e}} - ne\left(\mathbf{E} + \frac{\mathbf{u}\times\mathbf{B}}{\mathbf{c}}\right) - \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_{\mathrm{e}}} + \frac{nm(\mathbf{w}-\mathbf{u})}{\tau} + n\mathbf{f}_{\mathrm{e}}, \tag{42}$$

where p_{e} is the electron pressure and $\mathbf{f}_{\mathrm{e}}$ is the external force on each electron.

It is convenient to introduce the notation

$$n\mathbf{G} = -\nabla p_{\mathrm{i}} - nM\frac{d\mathbf{w}}{dt} + n\mathbf{f}_{\mathrm{i}}, \tag{43}$$

$$n\mathbf{H} = -\nabla p_{\mathrm{e}} - nm\frac{d\mathbf{u}}{dt} + n\mathbf{f}_{\mathrm{e}}, \tag{44}$$

so that Eqs. (41) and (42) become

$$0 = \mathbf{G} - \frac{M(\mathbf{w}-\mathbf{v})}{\tau_\mathrm{i}} - \frac{m(\mathbf{w}-\mathbf{u})}{\tau} + e\left(\mathbf{E} + \frac{\mathbf{w}\times\mathbf{B}}{c}\right), \tag{45}$$

$$0 = \mathbf{H} - \frac{m(\mathbf{u}-\mathbf{v})}{\tau_\mathrm{e}} + \frac{m(\mathbf{w}-\mathbf{u})}{\tau} - e\left(\mathbf{E} + \frac{\mathbf{u}\times\mathbf{B}}{c}\right). \tag{46}$$

The electric current density is

$$\mathbf{j} = ne(\mathbf{w}-\mathbf{u}), \tag{47}$$

and Ampere's law can be written

$$\mathbf{u} = \mathbf{w} - \frac{c\nabla\times\mathbf{B}}{4\pi ne}. \tag{48}$$

Add Eqs. (45) and (46), obtaining

$$n\mathbf{G} + n\mathbf{H} = \frac{nM(\mathbf{w}-\mathbf{v})}{\tau_\mathrm{i}} + \frac{nm(\mathbf{u}-\mathbf{v})}{\tau_\mathrm{e}} - \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}. \tag{49}$$

Then solve Eqs. (48) and (49) for **u** and **w**, obtaining

$$\mathbf{w} = \mathbf{v} + \frac{cm/\tau_\mathrm{e}}{4\pi neQ}\nabla\times\mathbf{B} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi nQ} + \frac{\mathbf{G}+\mathbf{H}}{Q}, \tag{50}$$

$$\mathbf{u} = \mathbf{v} - \frac{cM/\tau_\mathrm{i}}{4\pi neQ}\nabla\times\mathbf{B} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi nQ} + \frac{\mathbf{G}+\mathbf{H}}{Q}, \tag{51}$$

where

$$Q = \frac{M}{\tau_\mathrm{i}} + \frac{m}{\tau_\mathrm{e}}.$$

Solve Eq. (46) for **E** and use Eqs. (50) and (51) to eliminate **w** and **u**, with the result

$$\mathbf{E} = -\frac{\mathbf{v}\times\mathbf{B}}{c} + \frac{c}{4\pi ne^2}\left[\frac{m}{\tau} + \frac{\frac{m}{\tau_\mathrm{e}}\frac{M}{\tau_\mathrm{i}}}{Q}\right]\nabla\times\mathbf{B} + \frac{\frac{M}{\tau_\mathrm{i}} - \frac{m}{\tau_\mathrm{e}}}{4\pi neQ}(\nabla\times\mathbf{B})\times\mathbf{B}$$
$$-\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{4\pi ncQ} + \frac{\frac{\mathbf{H}M}{\tau_\mathrm{i}} - \frac{\mathbf{G}m}{\tau_\mathrm{e}}}{eQ} - \frac{(\mathbf{G}+\mathbf{H})\times\mathbf{B}}{cQ}. \tag{52}$$

The next step is to substitute this expression for **E** into the Faraday induction equation. Obviously the first term on the right hand side of Eq. (52) provides the familiar MHD induction term $\nabla\times(\mathbf{v}\times\mathbf{B})$. So MHD is the basic effect. The next three terms involve $\nabla\times\mathbf{B}$, which is smaller by $O(1/L)$ than the first term, where L is the characteristic large scale of variation of **B**. These three terms represent the resistive term, already discussed, the Pedersen effect, and the Hall effect, respectively. The fifth and sixth terms on the right hand side of Eq. (52) are also small $O(1/L)$ in the large scale bulk motion because **G** and **H** are small to that order. So there are these several effects appended to MHD, to be included wherever they may contribute significantly to the large-scale dynamics of the field and fluid.

Consider, then, the relative magnitude of the fifth and sixth terms, involving **G** and **H**, given by Eqs. (43) and (44). In these expressions, the ion and electron pressures may be expressed in terms of the ion and electron temperatures, T_i and T_e, as nkT_i and nkT_e, respectively, so that, for instance,

$$\frac{\nabla p_\mathrm{i}}{n} = \frac{\nabla nkT_\mathrm{i}}{n} = O\left(\frac{kT_\mathrm{i}}{L}\right) = O\left(\frac{Mw_\mathrm{th}^2}{L}\right), \tag{53}$$

where $Mw_\mathrm{th}^2 = kT_\mathrm{i}$ so that w_th represents the ion thermal velocity. A similar order of magnitude expression can be constructed for the electrons. Then we estimate the acceleration terms $d\mathbf{w}/dt$ as

$$\begin{aligned}\frac{d\mathbf{w}}{dt} &= O\left[(\mathbf{w}\cdot\nabla)\mathbf{w}\right], \\ &= O\left(\frac{w^2}{L}\right).\end{aligned} \tag{54}$$

The bulk flow of the electrons is constrained to be comparable to the bulk flow of the ions, so

$$\frac{d\mathbf{u}}{dt} = O\left(\frac{w^2}{L}\right). \tag{55}$$

Assuming, then, that $\mathbf{f}_\mathrm{i}$ and $\mathbf{f}_\mathrm{e}$ are not greater in magnitude than the other terms, they can be ignored in the present estimate, and it follows from Eqs. (43) and (44) that the magnitudes of **G** and **H** are given by

$$G = O\left(M\frac{w_\mathrm{th}^2 + w^2}{L}\right), \tag{56}$$

$$H = O\left(m\frac{u_\mathrm{th}^2}{L}\right). \tag{57}$$

When the ion and electrons temperatures are of the same general order of magnitude, it is apparent that G and H are of comparable magnitude. Noting that the mean free paths for collisions with neutral atoms are comparable for ions and electrons, and that the electron thermal velocity exceeds the ion thermal velocity by a factor of the order of $(M/m)^{1/2}$, it follows that $\tau_\mathrm{i}/\tau_\mathrm{e}$ is of the order of $(M/m)^{1/2}$. Hence

$$Q \cong \frac{M}{\tau_\mathrm{i}}\left[1 + O\left(\frac{m}{M}\right)^{\frac{1}{2}}\right]. \tag{58}$$

It follows that the fifth term on the right hand side of Eq. (52) is of the order of

$$\frac{\dfrac{\mathbf{H}M}{\tau_\mathrm{i}} - \dfrac{\mathbf{G}m}{\tau_\mathrm{e}}}{eQ} \approx \frac{1}{e}\left(\mathbf{H} - \frac{m}{M}\frac{\tau_\mathrm{i}}{\tau_\mathrm{e}}\mathbf{G}\right), = \frac{1}{e}\left[O\left(\frac{mu_\mathrm{th}^2}{L}\right) - \left(\frac{m}{M}\right)^{\frac{1}{2}} O\left(\frac{M(w_\mathrm{th}^2 + w^2)}{L}\right)\right]. \tag{59}$$

With mu_th^2 and Mw_th^2 of comparable magnitude it is apparent that the second term is small $O(m/M)^{1/2}$ compared to the first term. That is to say,

$$\frac{\dfrac{\mathbf{H}M}{\tau_\mathrm{i}} - \dfrac{\mathbf{G}m}{\tau_\mathrm{e}}}{eQ} \cong -\frac{\nabla p_\mathrm{e}}{ne}\left[1 + O\left(\frac{m}{M}\right)^{\frac{1}{2}}\right]. \tag{60}$$

In a similar manner the sixth term on the right hand side of Eq. (52) can be shown to be of the order of

$$\frac{(\mathbf{G}+\mathbf{H})\times\mathbf{B}}{cQ} = O\left[\frac{B}{cQ}\left(\frac{\nabla p_\mathrm{i} + \nabla p_\mathrm{e}}{n} + M\frac{d\mathbf{w}}{dt}\right)\right], = \frac{\Omega_\mathrm{i}\tau_\mathrm{i}}{e} O\left(\frac{Mw_\mathrm{th}^2 + mu_\mathrm{th}^2 + Mw^2}{L}\right) \tag{61}$$

where Ω_i is the ion cyclotron frequency, eB/Mc, in the magnetic field $\mathbf{B}$. Comparing this with the dominant first term of the right hand side of Eq. (59), it may be seen that the sixth term is of the order of $\Omega_\mathrm{i}\tau_\mathrm{i}$ times the fifth term. The essential point is that both terms are small $O(1/L)$ compared to the first term $-\mathbf{v}\times\mathbf{B}/c$ on the right hand side of Eq. (52). It follows then that the second through the sixth terms are all small compared to the purely MHD term $-\mathbf{v}\times\mathbf{B}/c$. So the large-scale plasma dynamics is described by the MHD equations with whatever small additional effects are appropriate.

Consider, then, the physics of the second through the sixth terms of the right hand side of Eq. (52). The second term represents the familiar Ohmic dissipation, with the resistive diffusion coefficient

$$\eta = \frac{c^2}{4\pi ne^2}\left(\frac{m}{\tau} + \frac{\dfrac{m}{\tau_\mathrm{e}}\dfrac{M}{\tau_\mathrm{i}}}{Q}\right) \cong \frac{mc^2}{4\pi ne^2}\left(\frac{1}{\tau} + \frac{1}{\tau_\mathrm{e}}\right). \tag{62}$$

It contributes the term $-\nabla\times(\eta\nabla\times\mathbf{B})$ on the right hand side of the induction equation (39).

The third and fifth terms on the right hand side of Eq. (52) together provide the *Hall effect*, and the dominant terms are

$$\frac{\dfrac{M}{\tau_\mathrm{i}} - \dfrac{m}{\tau_\mathrm{e}}}{neQ} \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} + \frac{\dfrac{\mathbf{H}M}{\tau_\mathrm{i}} - \dfrac{\mathbf{G}m}{\tau_\mathrm{e}}}{eQ}$$
$$\cong \frac{1}{ne}\left[-\nabla p_\mathrm{e} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}\right]\left[1 + O\left(\frac{m}{M}\right)^{\frac{1}{2}}\right]. \tag{63}$$

In this approximation the Hall effect is independent of the collision times τ_i and τ_e between the ions and electrons and the neutral atoms. So the result applies to the collisionless plasma as well, and is readily derived from the fluid equations for a collisionless ion gas and electron gas,

$$nM\frac{d\mathbf{w}}{dt} = -\nabla p_\mathrm{i} + ne\left(\mathbf{E} + \frac{\mathbf{w} \times \mathbf{B}}{c}\right), \tag{64}$$

$$nm\frac{d\mathbf{u}}{dt} = -\nabla p_\mathrm{e} - ne\left(\mathbf{E} + \frac{\mathbf{u} \times \mathbf{B}}{c}\right). \tag{65}$$

In fact we need only Eq. (65) for the electrons, because the Hall effect arises from the electric field required to hold the very fast mobile electrons in the company of the massive sluggish ions. Neglect the slight bulk inertia of the electrons and write Eq. (65) as

$$\mathbf{E} + \frac{\mathbf{w} \times \mathbf{B}}{c} = \frac{1}{ne}\left[-\nabla p_\mathrm{e} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}\right], \tag{66}$$

upon using Eq. (47) to express $\mathbf{u}$ in terms of $\mathbf{j}$ and Ampere's law to eliminate $\mathbf{j}$ in favor of $\nabla \times \mathbf{B}$. The left hand side of this equation represents the electric field $\mathbf{E}'$ in the moving frame of reference of the ions that is necessary to tie the electrons to the ions. It contributes to the MHD induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}) - c\nabla \times \left\{\frac{1}{ne}\left[-\nabla p_\mathrm{e} + \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}\right]\right\} \tag{67}$$

only insofar as $\mathbf{E}'$ provides a nonvanishing curl. This occurs when, for instance, $\nabla n \times \nabla T_\mathrm{e} \neq 0$. The Lorentz force,

$$\mathbf{L} = \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi}, \tag{68}$$

by itself provides a nondissipative energy circulation or Poynting vector

$$\mathbf{P} = c\frac{\mathbf{L}}{4\pi ne} \times \mathbf{B}. \tag{69}$$

Moving on to the second and sixth terms on the right hand side of Eq. (52), they represent the *Pedersen effect*, also known as *ambipolar diffusion*. Their contribution to the electric field $\mathbf{E}$ is

$$\mathbf{E}_\mathrm{P} = -\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{4\pi ncQ} - \frac{(\mathbf{G}+\mathbf{H})\times\mathbf{B}}{cQ}$$
$$= \frac{1}{cQ}\left\{\frac{\nabla p_\mathrm{i}+\nabla p_\mathrm{e}}{n} + M\frac{d\mathbf{w}}{dt} + m\frac{d\mathbf{u}}{dt} - \frac{1}{n}\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} - \mathbf{f}_\mathrm{i} - \mathbf{f}_\mathrm{e}\right\}\times\mathbf{B}. \quad (70)$$

The Pedersen effect arises because the pressure gradients, the Lorentz force, and other forces, exerted on the ions and electrons drive the ions and electrons through the neutral gas. The forced slippage of the ions and electrons relative to the neutral gas is opposed by the friction of the collisions, with characteristic collision times τ_i and τ_e.

The effect is dissipative, obviously, and allows the magnetic field to relax toward static equilibrium. The effect is sometimes referred to as the *Pedersen resistivity*, although it does not arise from collisions obstructing an electric current. *Ambipolar diffusion*, or just the *Pedersen effect*, is to be preferred.

Equation (70) is easily derived by noting that the slow drift of the ions and electrons, as they are pushed through the neutral gas, is balanced by the frictional forces. The balance of forces is described by adding Eqs. (41) and (42) and assuming that $\mathbf{u}$ and $\mathbf{w}$ are equal in the inertial terms, while their slight difference provides the current $\mathbf{j}$, to be replaced by $\nabla\times\mathbf{B}$ through Ampere's law. The result is

$$\mathbf{w}-\mathbf{v} = \frac{1}{nQ}\left[-\nabla p_\mathrm{i} - \nabla p_\mathrm{e} + \frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi} - n(M+m)\frac{d\mathbf{w}}{dt} + n\mathbf{f}_\mathrm{i} + n\mathbf{f}_\mathrm{e}\right]. \quad (71)$$

This extra velocity $\mathbf{w}-\mathbf{v}$ represents the drift of the ions relative to the neutral atoms, in addition to the bulk transport velocity $\mathbf{v}$ of the neutral atoms. So we would write

$$\mathbf{E} = -\frac{\mathbf{v}\times\mathbf{B}}{c} - \frac{(\mathbf{w}-\mathbf{v})\times\mathbf{B}}{c} \quad (72)$$

in Eq. (52) to account for the Pedersen effect. It is obvious by inspection of Eqs. (70), (71), and (72) that the term $(\mathbf{w}-\mathbf{v})\times\mathbf{B}/c$ represents the right hand side of Eq. (70) when the ions and electrons move together, $\mathbf{w}=\mathbf{u}$.

4 Popular Misconseptions

With the foregoing brief exposition of the dynamics of a swirling magnetized plasma, we review some of the more popular misconceptions encountered in the literature. For instance, it is claimed that the electric field $\mathbf{E} = -\mathbf{v}\times\mathbf{B}/c$ in the solar wind, as seen by an observer in the reference frame of Earth, actively penetrates into the geomagnetic field $\mathbf{B}_\mathrm{E}$ where it drives the plasma toward the electric drift velocity

$$\mathbf{v}_\mathrm{D} = c\frac{\mathbf{E}\times\mathbf{B}_\mathrm{E}}{B_\mathrm{E}^2}.$$

It is claimed that this effect is responsible for driving the convection of the magnetosphere. We have already pointed out that the electric stress density $E^2/8\pi$ is small

$O(v^2/c^2)$ compared to the magnetic stress density $B^2/8\pi$, and therefore negligible. So there cannot be any active dynamical effect to be attributed to the **E** in the Earth frame of reference. It should be appreciated, too, that there is no significant electric field $\mathbf{E}'$ in the moving frame of the solar wind, so if the calculation of the alleged dynamical effect were carried out in that frame of reference, the result would be quite different. But, of course, there is no effect at all in either reference frame.

The concern is sometimes expressed that the use of MHD overlooks the boundary conditions on the current (Melrose, 1995). In fact, the initial conditions and the boundary conditions together provide a *unique* solution for the magnetic field, which automatically provides the correct boundary conditions for the current. It has to be that way because there is no further freedom to adjust the solution if the current boundary conditions were not properly satisfied (Parker, 1996). It has to be remembered that Newton's equations and Maxwell's equations, i.e. Ampere's law, are fundamental laws of Nature. So they are mutually consistent. In particular, Newton's equations automatically provide the currents required by Ampere and Maxwell.

A particularly popular notion concerns the behavior of the electric current associated through Ampere's law with the magnetic field in a swirling plasma. The current flows through a slightly resistive medium and has magnetic fields with large energy associated with them. So it is proposed that the flow of the current can be described by the familiar electric circuit equations for an electric circuit containing resistors, inductances, and perhaps capacitors (Alfven and Carlquist, 1967). The inductance L is deduced by equating the magnetic energy to $LI^2/2$ where I is the total current flowing around the circuit. The current I in the circuit then controls the associated **B** through Ampere's law. This idea is not derived from Newton and Maxwell, being considered so self evident as to need no derivation. The idea is appealing because it reduces the solution of the partial differential equations of MHD to the solution of the ordinary differential equations of laboratory circuit theory.

In fact the electric current in a swirling magnetized plasma flows under quite different conditions from the current in the laboratory circuit. Thus, the laboratory circuit is fixed in the laboratory so the topology (connectivity) of the current loops is fixed, and the currents experience the electric field in the frame of reference of the laboratory. In contrast, the electric current in the swirling plasma is carried with the plasma velocity **v**, and in that moving frame there is only the insignificant electric field $\mathbf{E}'$, given by Eqs. (30) and (31). So there can be no inductive effects on the current **I**. What is more, the topology of the electric current in the swirling plasma is determined from **B** through Ampere's law and may change as the magnetic field is carried along in the plasma and deformed with the passage of time. Thus, for instance, the abrupt current interruption that might be caused by the onset of plasma turbulence and anomalous resistivity, or an electric double layer, does not provide the large *emf* sometimes imagined to accelerate ions and electrons to high velocities. The sudden appearance of an obstacle in the current path causes the current to deviate around the obstacle, so there is no current interruption, only an immediate rerouting.

It is interesting to explore a simple example of a current interruption to see how the plasma side steps the electric field associated with the change in current flow pattern. Imagine an infinite space filled with an incompressible, infinitely conducting, inviscid fluid and a uniform magnetic field **B**. Orient the z-axis of the coordinate system along the direction of the field, and denote radial distance from the z-axis by $\omega = (x^2 + y^2)^{1/2}$. There is an azimuthal magnetic field $B_\phi(\omega)$ circling the z-axis from $z = 0$ out to $z = a$, and vanishing beyond $\omega \geq a$. That is to say, consider a twisted flux bundle of radius a lying along the z-axis. The field outside the bundle is uniform and parallel to the bundle.

Ampere's law dictates that there is an electric current $j_z(\omega)$ in $\omega < a$, given by

$$j_z = \frac{c}{4\pi}\frac{1}{\omega}\frac{d}{d\omega}(\omega B_\phi).$$

The total current carried in the bundle is identically zero, of course, for if it were not, then $B_\phi(\omega)$ would not vanish beyond $\omega = a$. In the simplest case, there is a net current **I** flowing one way along the flux bundle in the neighborhood of the axis and in the opposite direction at larger ω. We can imagine that the flux bundle extends along the z-axis between distant infinitely conducting end plates $z = \pm h$, where $h >>> a$. So we have an electric circuit with current I, and, according to the electric circuit analog, the inductance per unit length in the z-direction is related to $B_\phi(\omega)$ by the energy relation

$$\frac{1}{2}LI^2 = 2\pi \int_0^a d\omega\omega \frac{B_\phi^2(\omega)}{8\pi}.$$

The total inductance $2hL$ in the circuit is large for large h.

At time $t = 0$ slice across the flux bundle with a sheet of nonconducting material of thickness 2ϵ, so that $j_z(\omega) = 0$ throughout $-\epsilon < z < +\epsilon$, $0 \leq \omega < a$, thereby blocking the flow of I at $z = 0$. The electric circuit theory predicts that the sudden blocking of the current I flowing through an inductance $2hL$ provides a potential difference V given by

$$V = 2hL\frac{dl}{dt},$$

which can be very large if the interruption of the current is sudden.

In fact what happens is quite different. Within the insulating sheet, $-\epsilon < z < +\epsilon$, the field $B_\phi(\omega)$ is decoupled from the plasma, becoming a ring of magnetic field in a dielectric, which propagates away as an electromagnetic wave at the speed of light in the dielectric. This leaves the longitudinal field $z = \pm\epsilon$, which connects at $z = \pm\epsilon$ into the severed ends of the twisted flux bundle. According to Ampere's law the kink in the field lines at $z = \pm\epsilon$ causes a radial current to flow that connects between the I in the central region of the bundle and the opposite I in the outer region, thereby reconnecting the flow of the current I at $z = \pm\epsilon$. So I is not interrupted by the insertion of the insulating sheet. It is rerouted.

Then it must be realized that the torque $T(\omega)d\omega$ about the z-axis in the annulus $(\omega, \omega + d\omega)$ in the twisted flux bundle is

$$T(\omega)d\omega = \frac{B_\phi(\omega)B_z(\omega)}{4\pi}2\pi\omega^2 d\omega.$$

This torque is interrupted at $z = \pm\epsilon$ and is zero throughout $-\epsilon < z < +\epsilon$ where $B_\phi(\omega)$ vanishes. So the torque at the severed ends of the twisted bundle at $z = \pm\epsilon$ sets the plasma in motion, providing torsional Alfven waves which propagate away in both directions along the twisted flux bundle. The torsional waves unwind the twisted bundle, so that

$$\frac{1}{2}\rho v_\phi^2(\omega) = \frac{B_\phi^2(\omega)}{8\pi},$$

and

$$v_\phi(\omega) = \frac{B_\phi(\omega)}{(4\pi\rho)^{\frac{1}{2}}}.$$

The magnetic energy associated with the current I, then, is converted directly into the kinetic energy of rotation of the plasma about the axis of the bundle as the torsional wave fronts sweep away in both directions. Note then that no electric potential differences are created in the moving plasma. The plasma in the wave motion avoids even the modest electric field $\mathbf{E} = -\mathbf{v} \times \mathbf{B}/c$ in the laboratory frame. Electric circuit theory simply does not apply. It is MHD alone that describes the large-scale dynamical behavior of the magnetized plasma.

The reader is referred to Parker (2007), Vasyliunas (1999, 2001, 2005a, b), and Vasyliunas and Song (2005) for further and more detailed exploration of the basic physics.

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