The Lidov–Kozai Oscillation and Hugo von Zeipel

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Received February 3, 2018; Revised August 24, 2018; Accepted December 4, 2018; Online published November 8, 2019.


The following contents in this file are Supplementary Information attached to the article. They are in the format provided by the authors, and unedited by the editor or the publisher.
1. Complete table of contents

Due to MEEP’s editorial policy, the main body of this monograph does not include a table of contents. It is attached as a part of the online version, but there is no page number information in that table. The main body contains a large number of sections, subsections, subsubsections, and named paragraphs across many pages. Therefore, placing the table of contents here with specific page number information will allow the readers to see the entire monograph as a whole and make it easier to understand its structure.

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2. Complete bibliography

As we mentioned in p. 3 of the main body of the monograph, use of the Cyrillic alphabets and the Japanese characters is prohibited in the main body due to a technical limitation about font in the \LaTeX\ typesetting process by the publisher. Also, some hyperlinks to the URLs embedded in the References section do not properly function, although the URLs themselves are correct. This is due to another technical limitation in this monograph’s \LaTeX\ typesetting process. Therefore we have created a more complete, alternative bibliography for this monograph using the Cyrillic and Japanese characters with fully functional hyperlinks.

References


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Beust, H., and A. Duret (2006), Dynamics of the young multiple system GG Tauri. II. Relation between the stellar system and the circumbinary disk, Astronomy and Astrophysics, 446, 137–154, [https://doi.org/10.1051/0004-6361:20053163].

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3. URLs of the relevant websites

In the main body of the monograph, we limited the use of URLs (Uniform Resource Locators) minimum due to the following two reasons. First, there is a technical limitation about the use of hyperlink (embedded URLs in the text) in the LATEX typesetting process by the publisher. Also, we wanted to avoid clutter by having many complicated URLs that often become sources of distractions. Instead, for the readers’ convenience, here we made a list of the URLs of the websites that are mentioned or cited in the main body of the monograph.

Orbit databases

- The JPL Horizons web-interface cited in p. 10, 11, 24, 25, and 38
  https://ssd.jpl.nasa.gov/horizons.cgi
- The JPL Small-Body Database Search Engine cited in p. 58 and 98
  https://ssd.jpl.nasa.gov/sbdb_query.cgi
- The JPL Small-Body Database Browser cited in p. 84
  https://ssd.jpl.nasa.gov/sbdb.cgi
- The Asteroid Orbital Elements Database (astorb.dat) cited in p. 81 and 84
  https://asteroid.lowell.edu/main/astorb
- Minor Planet Circulars (MPC) 9770 (1985 July 2) cited in p. 23

Obituaries for Yoshihide Kozai (mentioned in p. 11)

Here we just picked several obituaries published in English. Note that there are many others published in Japanese (and probably in other languages) that we did not mention here.

- American Astronomical Society (AAS)
- International Astronomical Union (IAU)
  https://www.iau.org/administration/membership/individual/1985/
- The Planetary Society
- The Japan Academy
- National Astronomical Observatory of Japan (NAOJ)
- Asian Scientist
- The Asahi Newspaper
  http://www.asahi.com/ajw/articles/AJ201802150020.html
4. How we mention Hugo von Zeipel’s name

As we wrote in p. 39 of the monograph, Hugo von Zeipel’s name is sometimes mentioned just as “Zeipel.” This manner may reflect the fact that the German language surname von has been a part of names of the noble families in the Nordic countries such as Sweden, and that use of von is currently prohibited in new names. Regarding this subject, let us cite a sentence from a webpage in Svenska Wikipedia about von:

“In Sweden, von is strongly associated with the nobility, and since 1901 it cannot be used in new names.” (translated from https://sv.wikipedia.org/wiki/Von as of June 11, 2019. The original Swedish expression is “I Sverige är von starkt associerat med adeln och sedan 1901 kan det inte användas i nya namn.”)

On the other hand, we find a heading “Zeipel, von” in Nordisk familjebok, 33 (1922), 710–712 (http://runeberg.org/nfcm/0387.html and http://runeberg.org/nfcm/0388.html). As an additional complication for us, he wrote his own name as “H. v. Zeipel” in all of his publications that we are aware of: von appears just in the form of “v.”.

As such, speakers of the non-European languages like us are not familiar with von, and it is difficult to understand the usage of the word correctly. We wondered which way is more appropriate to refer to him, “Zeipel” or “von Zeipel”, and eventually sent a query about it to Uppsala University where he served as a professor in astronomy for a long time. Then we received an answer as follows:

“The correct way to write is ‘von Zeipel’. It is no longer possible to be honored in Sweden. However, it does not change the fact that noble families still use ‘von’.” (The Communications Department of Uppsala University, April 19, 2019)

We believe the above answer is more authentic and reliable than any other information as to how we should mention Hugo von Zeipel’s name. Therefore, by following the above instruction, we decided to use “von Zeipel” throughout the monograph.
5. Orbit diagrams of the objects in Table 4

In Table 4 of the main body of the monograph (p. 84), we made a list of osculating orbital elements $a$, $e$, $I$ and the parameter $k^2$ of several small objects in the actual solar system that can match von Zeipel’s description in his Section Z28: “Not very eccentric, with an inclination close to 90°, and located somewhat outside the orbit of perturbing planet.” For better facilitating an understanding of what kind of orbits these objects have, we generated orbital diagrams of all the 16 objects in this table as supplementary figures (Figs. S1 through S7).

First, similar to Fig. 12 of the main body of the monograph, we made secular orbital configuration diagrams on the $(e \cos g, e \sin g)$ plane (Figs. S1). The meanings of the color circles (black, red, and blue) remain the same as in the main body: The red and blue circles represent the conditions where the orbits of the perturbed body and the perturbing planet intersect each other at the ascending node (red) and at the descending node (blue) of the perturbed body. The black circles represent the theoretically largest eccentricity of the small body ($k' = \sqrt{1-k^2}$; see Eq. (Z43-224)). The black points denote the current orbital locations of each object.

As is evident in Fig. S1, seven out of the ten objects in the “small $k^2$” category in Table 4 are in the “rings in a chain” state (see the locations of the black points). In other words, these objects are located in the region of possible orbit intersection with the planets: Four objects (2007 VW$_{266}$, 2010 CR$_{140}$, 2012 YO$_{6}$, 2008 KV$_{42}$) stay inside the domain $B'$ defined by von Zeipel in Fig. 11 of the monograph, and three objects (2008 YB$_{3}$, 2011 MM$_{4}$, 2011 KT$_{19}$) stay inside the domain $B$. The remaining three objects are marginally located on boarders: 2015 KG$_{157}$ is between the domains $A$ and $B'$, 2014 JJ$_{57}$ is between $C$ and $B$, and 2007 BP$_{102}$ is between $C$ and $B'$. This circumstance implies that the objects in the “small $k^2$” category are likely dynamically unstable due to strong perturbation from the perturbing planets. On the other hand, all the six objects in the “small $e$ and large $I$” category (2016 LN$_{8}$, 2014 XZ$_{40}$, 2013 SA$_{87}$, 2004 DF$_{77}$, 2006 HU$_{122}$, and 2016 FM$_{59}$) stay in the middle of the domain $C$ without a possibility of orbit intersection. This is realized by the very small eccentricity of these objects, which indicates a possibility of their dynamical stability.

Next, we made a set of orbit diagrams of each of the objects in the actual $(x, y, z)$ space. Figs. S2–S5 are for the objects in the “small $k^2$” category, and Figs. S6–S7 are for those in the “small $e$ and large $I$” category. The unit of axes is au. The epoch and reference frame are based on J2000.0. For all the objects, we made three kinds of orbital diagrams projected onto three planes: the $(x, y)$, $(x, z)$, and $(y, z)$ planes. The orbits of the perturbed bodies are drawn in red in all the panels. The orbits of the major perturbing planet (which each of the perturbed bodies is located “somewhat outside”) are indicated in blue. We easily see the difference of orbital shapes between the objects in the “small $k^2$” category (Figs. S2–S5) and those in the “small $e$ and large $I$” category (Figs. S6–S7). In particular, the circular orbital shape of the latter objects is clear at a glance.
Fig. S1. Secular orbital configurations of the 16 objects in Table 4 visualized on the \((e \cos g, e \sin g)\) plane. See the description in p. S18 for the meanings of the black, red, and blue circles. The black points denote the current orbital location of each object.
Fig. S2. Orbital diagrams of Jupiter (blue) and five objects (red) in the (x, y), (x, z), and (y, z) planes: 2015 KG$_{157}$, 2007 VW$_{266}$, 2010 CR$_{140}$, 2012 YO$_6$, and 2014 JJ$_{57}$. 

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Fig. S3. Orbital diagrams of Saturn (blue) and 2008 YB₃ (red).

Fig. S4. Orbital diagrams of Uranus (blue) and two objects (red): 2011 MM₄ and 2007 BP₁₀₂.
Fig. S5. Orbital diagrams of Neptune (blue) and two objects (red): 2011 KT$_{19}$ and 2008 KV$_{42}$.

Fig. S6. Orbital diagrams of Jupiter (blue) and 2016 LN$_{8}$ (red).
Fig. S7. Orbital diagrams of Neptune (blue) and five objects (red): 2014 XZ40, 2013 SA87, 2004 DF77, 2006 HU122, and 2016 FM59.
6. Supplementary figures for Appendix B

In Appendix B of the main body (p. 113), after placing Eqs. (B.3) and (B.4) we stated “It is also easy to confirm that both the second terms are positive in the entire region of $0 \leq k^2 \leq 1$”. Here we plot the actual values of the second terms (Fig. S8a) together with their derivatives by $k$ (Fig. S8b). We clearly see that the second terms remain positive throughout the entire range of $k^2$, although their derivatives remain negative.

We have also made comparisons of magnitudes of the first and the second terms of Eqs. (B.3) and (B.4). More specifically, we plotted the dependence of the first and the second terms (multiplied by the factor $\alpha^2$) on $k^2$ at different values of $\alpha'$ ($0.05, 0.30, 0.60, 0.95$) in the panels c, d, e, and f. The factor $\alpha^2$ accounts for the order difference between the first and the second terms of Eqs. (B.3) and (B.4). We see both the terms remain positive in the range of $0 \leq k^2 \leq 1/3$ as we wrote in Appendix B. In addition, we find the magnitude difference between the first and the second terms become smaller as $\alpha'$ gets larger. When $\alpha'$ is as large as 0.95, we see that the second terms are larger than the first term over almost entire range of $k^2$ (Fig. S8f). This is another exemplification that shows the importance to incorporate the $\alpha^2$ terms in the discussion here.

![Fig. S8. Panel a: (red) the second term in Eq. (B.3), (blue) that in Eq. (B.4), putting $\alpha' = 1$. This panel is equivalent to Fig. 31 of the main body (p. 113). Panel b: (red) the derivative of the second term in Eq. (B.3), (blue) that in Eq. (B.4), putting $\alpha' = 1$. Note that in b we plotted the quantities obtained by inverting the sign of the derivatives, because the derivatives remain negative. Panels c–f: Comparison between the first term (black) and the second terms (red and blue) when $\alpha' = 0.05$ (c), $\alpha' = 0.30$ (d), $\alpha' = 0.60$ (e), and $\alpha' = 0.95$ (f).]
7. Other relevant information

7.1 Lidov–Kozai oscillation in hydrodynamical disks

Theory of the Lidov–Kozai oscillation, or what should be called the von Zeipel–Lidov–Kozai oscillation from now on, is not just confined to the point mass systems such as the classical three-body problem. For the past decade it has also been applied to dynamical studies in hydrodynamical disks (e.g. Xiang-Gruess and Papaloizou, 2013; Martin et al., 2014; Trani et al., 2016; Lubow and Ogilvie, 2017; Franchini et al., 2019). Objectives of these studies extend from accretion processes in protoplanetary disks to dynamics of stellar disks in the galactic center. This is yet another exemplification of the substantial and ubiquitous importance of this mechanism in collective astronomy and planetary science.

7.2 More about the $c_2$-like parameter in later studies

In Section 6.2.2 of the main body (p. 88), we mentioned several examples of later “discovery” of Lidov’s $c_2$-like parameters. In the proofread stage of this monograph, we found that Froeschlé (1970) had brought in a pair of parameters equivalent to Lidov’s $c_1$ and $c_2$. After introducing a set of variational equations for orbital elements in the doubly averaged three-body problem (equivalent to Lidov’s Eq. (L54-147)), he states as follows:

“This system has the following first integrals:

\[
\begin{align*}
  a &= Ct, \\
  (1 - e^2) \cos^2 i &= A^2, \\
  e^2 (5 \sin^2 i \sin^2 \omega - 2) &= K,
\end{align*}
\]

(9)

where $A^2, K$ are constants which depends on the initial conditions.” (Froeschlé, 1970, p. 121)

Here $A^2 = c_1$ and $K = 5c_2$ are obvious, but there is no citation to Lidov’s work in Froeschlé (1970). Therefore we do not know if Froeschlé devised the function form of $K$ by himself or borrowed it from previous literature. Note also that Innanen et al. (1997, p. 1916) developed a slightly similar discussion after placing a set of variational equations (their Eq. (5)) that are equivalent to Lidov’s Eq. (L54-147).

7.3 No asteroids named after Moiseev or Lindstedt yet

On the course of preparing this monograph we noticed that, in addition to the asteroid (3040) Kozai that is named after Yoshihide Kozai, there are some more asteroids that are named after the great celestial mechanists and dynamical astronomers that we mentioned in this monograph: (2021) Poincare, (3663) Tisserand, (4236) Lidov, (8688) Delaunay, and (8870) von Zeipel. However, it seems that there are no asteroids named after Nikolay Dmitriyevich Moiseev or Anders Lindstedt yet. Considering their substantial contributions to the development of celestial mechanics and dynamical astronomy, this fact is rather unexpected. We presume that some of the readers of this monograph own the privilege to name asteroids. We hope those readers propose “Moiseev” and “Lindstedt” for asteroid names along with the standard procedure of International Astronomical Union for commemorating these two great pioneers.
8. **Supplementary bibliography**


(This supplement was last updated November 7, 2019)