ON MP TEST AND THE MVUEs IN A $N(\theta, c\theta)$ DISTRIBUTION WITH $\theta$ UNKNOWN: ILLUSTRATIONS AND APPLICATIONS

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Consider a sequence of independent observations $X_1, \ldots, X_n$ from a $N(\theta, c\theta)$ distribution with $0 < \theta < \infty$. We assume that $\theta$ is unknown, but $c(> 0)$ is known. We begin with the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ where $\theta_0, \theta_1 (\theta_0 \neq \theta_1)$ are specified values of $\theta$. The most powerful (MP) level $\alpha$ test depends upon $\sum_{i=1}^{n} X_i^2$, a complete and sufficient statistic for $\theta$, which has a multiple of a non-central chi-square distribution with its non-centrality parameter involving $n$ and the true parameter value $\theta$ under $H_0, H_1$. We first target type-I and type-II error probabilities $\alpha$ and $\beta$ respectively, with $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$. We set out to determine the required exact sample size which will control these error probabilities and provide two useful large-sample approximations for the sample size. The three methods provide nearly the same required sample size whether $n$ is small, moderate or large. We also show how one may derive the minimum variance unbiased estimators (MVUEs) for a number of interesting and useful functionals of $\theta$ by combining some previous work from Mukhopadhyay and Cicconetti (2004) and Mukhopadhyay and Bhattacharjee (2010). All methodologies are illustrated with both simulated data and real data.

Key words and phrases: Exact method, large-sample method, minimum variance unbiased estimation, monotone likelihood ratio, most powerful test, non-central chi-square distribution, one-parameter exponential family, required sample size determination, type-I error probability, type-II error probability.

1. Introduction

Suppose that we have a random sample $X_1, \ldots, X_n$ from a normal distribution whose variance is a known constant multiple of the mean. That is, each $X$ has a common probability density function (p.d.f.)

\begin{equation}
    g(x; \theta, c) = (2\pi c\theta)^{-1/2} \exp\{- (x - \theta)^2 / (2c\theta)\},
\end{equation}

with $-\infty < x < \infty$, $0 < \theta, c < \infty$. We assume that $\theta$ is unknown, but $c$ is known. Now, observe that if $X \sim N(\theta, c\theta)$, then $X/c \sim N(\theta/c, \theta/c)$, that is a normal distribution with the same mean and variance.

Hence, without any loss of generality, we consider ignoring the constant $c$ by assigning $c = 1$ so that we will treat $X_1, \ldots, X_n$ to be $n$ independently and identically distributed (i.i.d.) random variables with the following common p.d.f.

\begin{equation}
    f(x; \theta) \equiv g(x; \theta, 1), -\infty < x < \infty, \quad 0 < \theta < \infty.
\end{equation}
Note that the p.d.f. from (1.2) belongs to a regular one parameter exponential family (Barndorff-Nielson (1988), Barndorff-Nielson and Cox (1994)) since one can rewrite

\begin{equation}
    f(x; \theta) = a(\theta) h(x) \exp\{b(\theta)p(x)\}
\end{equation}

where, 
\begin{align*}
a(\theta) &= (2\pi\theta)^{-1/2} \exp\{-\frac{\theta^2}{2}\}, \\
h(x) &= \exp(x), \\
b(\theta) &= -\frac{1}{2} \theta \\
p(x) &= x^2 
\end{align*}

for all 0 < x, \theta < \infty. It becomes obvious that a complete and sufficient statistic for \theta is \( T \equiv \sum_{i=1}^{n} X_i^2 \).

From Neyman-Pearson lemma (Neyman and Pearson (1933)) it follows that for testing \( H_0 : \theta = \theta_0 \) against an alternative \( H_1 : \theta = \theta_1 (> \theta_0) \), the level \( \alpha \) most powerful (MP) test would correspond to the following test function with \( t > 0 \):

\begin{equation}
    \psi(t) = \begin{cases} 
        1 & \text{if } t > k \\
        0 & \text{if } t \leq k 
    \end{cases}
\end{equation}

where \( k \equiv k_\alpha (> 0) \) is determined in such a way that

\begin{equation}
    E_{\theta}\left[\psi(T)\right] = \alpha \text{ under } \theta = \theta_0.
\end{equation}

Next, the p.d.f. from (1.2) belongs to a monotone likelihood ratio (MLR increasing) family in \( T \). Thus, using the theorem of Karlin and Rubin (1956), the same test (1.4) would be the uniformly most powerful (UMP) level \( \alpha \) test for deciding between \( H_0 : \theta \leq \theta_0 \) against \( H_1 : \theta > \theta_0 \). One may also refer to Corollary 2 from Lehmann (1986, p. 80) or Rao (1973, Chapter 7) among other sources including Lehmann and Casella (1998), Mukhopadhyay (2000), and Lehmann and Romano (2005).

In this paper, we report on various exact and large-sample methods of determination of the cut-off point \( k \) and practical implementation of the level \( \alpha \) MP test described by (1.4)–(1.5) having additionally a preassigned type-II error probability \( \beta \), \( 0 < \beta < 1 \) when the sample size \( n \equiv n(\alpha, \beta) \) is chosen appropriately. After nearly 80 years since the foundational work of Neyman and Pearson (1933), our present task may superficially appear trivial to some readers. But, indeed \( N(\theta, \theta) \) happens to be a very special distribution from a regular one parameter exponential family with its own intricacies requiring tender loving care and attention to minute details.

Hence, we first motivate what we are doing and why. One finds numerous examples where an implemented form of a MP test function \( \psi \) involves a sufficient statistic, \( S \equiv S(X_1, \ldots, X_n) \), customarily having a central distribution under \( H_0 \), but having a non-central distribution under \( H_1 \). Throughout the literature of MP tests, we have not found one single explicit example of a MP test based on i.i.d. observations from a normal distribution involving a sufficient statistic \( S \) that has non-central distributions under both \( H_0 \) and \( H_1 \).

We work with a fixed number \( n \) of i.i.d. observations from a normal distribution belonging to a regular one-parameter exponential family having the MLR increasing property in \( T \). We also note that \( \theta^{-1} T \) has a non-central chi-square
distribution under both null and alternative hypotheses. This clearly makes the implementation of our proposed MP test (1.4) challenging. We will have more to say later. By the way, one may review non-central chi-square distributions from Johnson et al. (1995, Chapter 29).

Section 2 briefly summarizes how the cut-off point \( k \equiv k_\alpha(> 0) \) is determined exactly. Section 3 develops the exact and large-sample determinations of the required sample size \( n \equiv n(\alpha, \beta) \) associated with our MP test (1.4)–(1.5) having preassigned type-I and type-II error probabilities \( \alpha \) and \( \beta \). Section 4 gives a glimpse of data analysis with simulated as well as real data.

In Section 5, we briefly show how one may derive the minimum variance unbiased estimators (MVUEs) for a number of interesting and useful functionals of \( \theta \) by exploiting and combining some previous work from Mukhopadhyay and Cicconetti (2004) and Mukhopadhyay and Bhattacharjee (2010).

2. Determination of the cut-off point

We begin with a random sample \( X_1, \ldots, X_n \) from a \( N(\theta, \theta) \) distribution. The joint distribution of the random sample \( X_1, \ldots, X_n \) will be \( n \)-dimensional multivariate normal, \( N_n(\theta, I_\theta) \), where \( 1' = (1, \ldots, 1) \) is the \( 1 \times n \) unit vector and \( I \) is the \( n \times n \) identity matrix. The joint density is given by,

\[
\frac{1}{(2\pi\theta)^{n/2}} \exp \left\{ -\frac{1}{2\theta} (x - 1\theta)'(x - 1\theta) \right\}, \quad x \in \mathbb{R}^n, \ \theta \in \mathbb{R}^+.
\]

Though we have a normal distribution on hand, the test statistic for mean \( \theta \) is \( \sum_{i=1}^n X_i^2 \) instead of the more customary \( \sum_{i=1}^n X_i \) or \( \overline{X} \). A complete and sufficient statistic \( T = \sum_{i=1}^n X_i^2 \) is clearly a quadratic form and hence we see that \( \theta^{-1}T \) would have a non-central chi-square distribution with \( n \) degrees of freedom with an underlying non-centrality parameter, \( \lambda = \theta^{-1}(1\theta)'I(1\theta) = n\theta \) when \( \theta \) is the true parameter value. We write \( \sum_{i=1}^n X_i^2 \sim \chi^2_n[\lambda] \) with \( \lambda = n\theta \) when \( \theta \) is the true parameter value.

The fact that this distribution happens to be non-central highlights non-triviality of the testing problem associated with the \( N(\theta, \theta) \) distribution. A non-central chi-square distribution under both \( H_0 \) and \( H_1 \) is not a very easy situation to work with. We must address the problem of inverting the distribution function of \( \chi^2_n[n\theta] \) in order to come up with the appropriate cut-off point \( k \equiv k_\alpha(> 0) \) satisfying (1.5) when \( n \) is small, moderate or large.

The statistic \( \sum_{i=1}^n X_i^2 \sim \theta_i\chi^2_i[\lambda_i] \) where \( \lambda_i = n\theta_i \) under \( H_i, \ i = 0, 1 \). Without any loss of generality, let us assume that \( \theta_1 > \theta_0 \). Then, for a fixed sample size \( n \), the level \( \alpha \) MP test will reject \( H_0 \) in favor of \( H_1 \) if and only if

\[
\sum_{i=1}^n X_i^2 > \theta_0\chi^2_{n,\alpha}[\lambda_0] \quad \text{where} \quad \chi^2_{n,\alpha}[\lambda_0] \quad \text{is the upper 100}\alpha\% \text{ point of } \chi^2_{n,\alpha}[\lambda_0] \text{ distribution.}
\]

Formally, we may now rewrite our test function from (1.4) as follows with \( t > 0 \):

\[
psi(t) = \begin{cases} 
1 & \text{if } \ t > \theta_0\chi^2_{n,\alpha}[\lambda_0] \\
0 & \text{if } \ t \leq \theta_0\chi^2_{n,\alpha}[\lambda_0].
\end{cases}
\]
Obviously, we must have:

\[(2.4)\]

\[E_{\theta}[\psi(T)] = P_{\theta}(T > \theta_0 \chi^2_{n,\alpha}[n\theta_0]) = \alpha.\]

3. Controlling the type-II error probability: Sample size determination

In practice, one may want to determine the required sample size \(n\) so that the type-II error probability associated with (2.3), namely,

\[P_{\theta_1}\{\Sigma_{i=1}^{n}X_i^2 < \theta_0 \chi^2_{n,\alpha}[\lambda_0]\}\]

does not exceed some preassigned \(\beta, 0 < \beta < 1\), that is the power would be at least \(1 - \beta\).

3.1. Exact small-sample method

Now, the type-II error probability will not exceed \(\beta\) provided that the probability of \(\Sigma_{i=1}^{n}X_i^2 < \theta_0 \chi^2_{n,\alpha}[\lambda_0]\) under \(\theta = \theta_1\), or equivalently,

\[(3.1)\]

\[P_{\theta_1}\{\theta_1^{-1}\Sigma_{i=1}^{n}X_i^2 < \theta_1^{-1}\theta_0 \chi^2_{n,\alpha}[\lambda_0]\} \leq \beta.\]

But, we know that \(\beta = P_{\theta_1}\{\theta_1^{-1}\Sigma_{i=1}^{n}X_i^2 < \chi^2_{n,1-\beta}[\lambda_1]\}\), and hence the required sample size must be determined as follows:

\[(3.2)\]

\[n \equiv n_{01}\text{ where } n_{01}\text{ is the smallest positive integer } n\]

\[\text{such that } \theta_1^{-1}\theta_0 \chi^2_{n,\alpha}[\lambda_0] \leq \chi^2_{n,1-\beta}[\lambda_1].\]

However, tables for \(\chi^2_{n,\gamma}[\lambda]\) are not readily available for a wide range of fixed values of \(\lambda, n, \gamma\) corresponding to our arbitrary choices of \(\theta_0, \theta_1\) and \(\alpha, \beta\) under consideration. Thus, we opted to determine \(n\) numerically using a R-program which is freely accessible. But, due to limited accuracy of computation through this R-program, we necessarily had to avoid considering large values of \(\theta_0\) and \(\theta_1\). Large values of \(\theta_0, \theta_1\) invariably led to large non-centrality parameter values under \(H_0, H_1\) which in turn made the determination of minimum required \(n\) subject to limited accuracy within the R-environment or another source.

One may be tempted to circumvent the added complications arising primarily from a non-central chi-square distribution as follows: Simply propose an appropriate level \(\alpha\) testing methodology of rejecting \(H_0\) in favor of \(H_1\) if and only if \((n - 1)S^2/\theta_0 > \chi^2_{n,\alpha}\) where \(S^2\) is the sample variance. But, why should one even consider such a text instead of the MP level \(\alpha\) test (1.4), or equivalently (2.2), based upon a complete sufficient statistic, \(T\)? One will encounter a significant loss of information about \(\theta\) when \(S^2\) is used compared with the case when \(T\) is used. Mukhopadhyay and Cicconetti (2004) evaluated Fisher-information about \(\theta\) in \(S^2\) and \(T\).

Given the excellent computing resources available today, the significant issue mainly arising due to the non-centrality of the distribution of \(T\) still continues to haunt us. For this reason, in what follows, we present two large-sample methods
of approximately determining the minimum required sample size for controlling both probabilities of type-I and type-II errors. In Table 1, Section 4, we will report comparisons of $n_{01}$ with the sample sizes obtained via large-sample approximations.

3.2. Large-sample method-I: Invoking central limit theorem

We first observe that

\[
\mu(\theta) \equiv E(X_2^2) = \theta + \theta^2, \quad E(X_4^1) = 3\theta^2 + 6\theta^3 + \theta^4
\]

so that $\sigma^2(\theta) \equiv \text{Var}(X_2^2) = 2\theta^2(1 + 2\theta).

Hence, by the central limit theorem, for a fixed but large sample size $n$, one has:

\[
\frac{\sqrt{n}(W_n - \mu(\theta))}{\sigma(\theta)} \xrightarrow{d} N(0, 1) \quad \text{as} \quad n \to \infty \text{ for true } \theta,
\]

where $W_n = n^{-1} \sum_{i=1}^{n} X_i^2$.

By (3.4), we may express a large-sample approximate level $\alpha$ MP test for $H_0$ vs. $H_1$ as follows: We will reject $H_0$ in favor of $H_1$ if and only if

\[
\frac{\sqrt{n}(W_n - \mu(\theta_0))}{\sigma(\theta_0)} > z_{\alpha} \quad \text{where } z_{\alpha} \text{ is the upper } 100\alpha\% \text{ point of a } N(0, 1) \text{ distribution.}
\]

Now, the probability of type-II error associated with the approximate level $\alpha$ MP test (3.5) will not exceed $\beta$ provided that we have:

\[
P_{\theta_1}\left\{ \frac{\sqrt{n}(W_n - (\theta_0 + \theta_1^2))}{\theta_0\sqrt{2(1 + 2\theta_0)}} \leq z_{\alpha} \right\} \leq \beta
\]

\[
\Rightarrow \Phi\left( z_{\alpha} \frac{\theta_0\sqrt{1 + 2\theta_1}}{\theta_1\sqrt{1 + 2\theta_0}} \right) \leq \beta = \Phi(-z_\beta),
\]

where we denote $\Phi(u) = \int_{-\infty}^{u} \left( \frac{1}{\sqrt{2\pi}} \right) e^{-x^2/2} \, dx, u \in R$. Thus, the required approximate sample size must be determined as follows:

\[
n \equiv n_{02} \text{ where } n_{02} \text{ is the smallest positive integer}
\]

\[
\geq \frac{2\theta_1^2(1 + 2\theta_1)}{(\theta_1 - \theta_0)^2(1 + \theta_0 + \theta_1)^2} \left[ \frac{\theta_0}{\theta_1} \sqrt{\frac{1 + 2\theta_0}{1 + 2\theta_1}} + z_{\beta} \right]^2.
\]

3.3. Large-sample method-II: Variance stabilization

Since the expression of $\sigma^2(\theta)$ from (3.3) involves $\theta$, we feel tempted to provide another large-sample approximation for the required sample size after we come up with the associated variance stabilizing transformation (Mukhopadhyay (2000, Section 12.4)). Now, for a non-zero continuously differentiable real valued
function \( g(.) \), the variance of the asymptotic distribution of \( \sqrt{n}(g(W_n) - g(\mu(\theta))) \) would be given by the expression of

\[
g'(\mu(\theta)) \sigma^2(\theta).
\]

We want to determine the function \( g(.) \) such that \( g'(\mu(\theta)) \sigma^2(\theta) \) becomes free from \( \theta \). That is, we must have:

\[
(3.8) \quad g(\mu(\theta)) = \int g'(\mu(\theta))d\mu(\theta) = \int \{2\theta^2(1 + 2\theta)\}^{-1/2}d\mu(\theta).
\]

In (3.8), we may make the following substitution: Let \( \sqrt{1 + 2\theta} = u \) so that we obtain:

\[
(3.9) \quad g(\mu(\theta)) = 2^{1/2} \left[ (1 + 2\theta)^{1/2} + \frac{1}{2} \log \left( \frac{(1 + 2\theta)^{1/2} - 1}{(1 + 2\theta)^{1/2} + 1} \right) \right], \quad \text{and}
\]

\[
g(W_n) = 2^{1/2} \left[ (1 + 4W_n)^{1/4} + \frac{1}{2} \log \left( \frac{(1 + 4W_n)^{1/4} - 1}{(1 + 4W_n)^{1/4} + 1} \right) \right].
\]

In view of (3.9), we can claim that \( \sqrt{n}(g(W_n) - g(\mu(\theta))) \xrightarrow{L} N(0, 1) \) as \( n \to \infty \).

In what follows, we first prove the monotonicity and concavity properties of the \( g(.) \) function found in (3.9).

**Lemma 1.** The function \( g(.) \) given by (3.9) is monotonically increasing and concave.

**Proof.** Let us denote

\[
f(x) = (1 + 4x)^{1/4} + \frac{1}{2} \log \left( \frac{(1 + 4x)^{1/4} - 1}{(1 + 4x)^{1/4} + 1} \right)
\]

\[
= (1 + 4x)^{1/4} + \frac{1}{2} \log \{(1 + 4x)^{1/4} - 1\} - \log \{(1 + 4x)^{1/4} + 1\},
\]

\( x > 0 \). Thus, we can express

\[
f'(x) = \frac{d}{dx} f(x)
\]

\[
= \frac{1}{4(1 + 4x)^{3/4}} + \frac{1}{8(1 + 4x)^{3/4}} \left[ \frac{1}{(1 + 4x)^{1/4} - 1} - \frac{1}{(1 + 4x)^{1/4} + 1} \right]
\]

\[
= \frac{1}{4(1 + 4x)^{1/4}((1 + 4x)^{1/2} - 1)},
\]

which is positive for all \( x > 0 \). Thus, \( f(x) \) is an increasing function of \( x \), and so is \( g(x) \).

We can also verify that \( f''(x) < 0 \) for all \( x > 0 \). Thus, \( f(x) \) is a concave function of \( x \), and so is \( g(x) \). \( \square \)
At this point, a second large-sample approximate level $\alpha$ MP test for $H_0$ vs. $H_1$ may be restated as follows: We will reject $H_0$ in favor of $H_1$ if and only if

$$\sqrt{n}(g(W_n) - g(\mu(\theta_0))) > z_\alpha,$$

with the $g(.)$ function found in (3.9). Later, we compare performances of this test with those of the other tests via computer simulations.

Now, the probability of type-II error associated with the approximate level $\alpha$ MP test (3.10) will not exceed $\beta$ provided that the required approximate sample size is determined as follows:

$$n = n_{03} \text{ where } n_{03} \text{ is the smallest positive integer } \geq \left[ \frac{z_\alpha + z_\beta}{\mu_1 - \mu_0} \right]^2 \left[(1 + 2\theta_i)^{1/2} + \frac{1}{2} \log \left(\frac{(1 + 2\theta_i)^{1/2} - 1}{(1 + 2\theta_i)^{1/2} + 1}\right)\right],$$

where $\mu_i = g(\mu(\theta_i)) = 2^{1/2} \left[(1 + 2\theta_i)^{1/2} + \frac{1}{2} \log \left(\frac{(1 + 2\theta_i)^{1/2} - 1}{(1 + 2\theta_i)^{1/2} + 1}\right)\right], i = 0, 1.$

A comparison of the required sample sizes, namely, $n_{01}$ given in (3.2), $n_{02}$ given in (3.7) and $n_{03}$ given in (3.11) is in order for different combinations of $\alpha, \beta, \theta_0$, and $\theta_1$. This is explored in the next section.

4. Illustrations: Simulations and real data

In this section, we will present some data analysis in order to illustrate practical merits of the theoretical findings from previous sections.

4.1. Simulated data from a $N(\theta, \theta)$ distribution

We begin our numerical illustration with the help of simulated data from a $N(\theta, \theta)$ distribution. For this data we calculated the MP as well as asymptotic sample sizes under fixed probabilities of type-I and type-II errors. The main goal, as stated earlier, was to look at the behavior of the three different sample size choices (proposed in Section 3) with change in error probabilities as well as change in $\theta$.

We fixed three combinations of type-I error and type-II error probabilities, namely, (i) $\alpha = 0.01$, $\beta = 0.01$, (ii) $\alpha = 0.05$, $\beta = 0.05$, and (iii) $\alpha = 0.10$, $\beta = 0.10$. We also fixed the following combinations of $\theta$-values: (a) $\theta_0 = 14$, $\theta_1 = 15$, (b) $\theta_0 = 5$, $\theta_1 = 5.36$, (c) $\theta_0 = 14$, $\theta_1 = 17$, (d) $\theta_0 = 5$, $\theta_1 = 6.07$, (e) $\theta_0 = 5$, $\theta_1 = 6$, and (f) $\theta_0 = 5$, $\theta_1 = 8$.

In each case, we then determined the exact sample size from (3.2) required to meet our specific goals for $\alpha, \beta$ associated with the MP test from (2.3)–(2.4). In column 1 of Table 1, we show these $n_{01}$ values. The columns 6 and 7 of Table 1 respectively show the associated large-sample approximations $n_{02}, n_{03}$ from (3.7) and (3.11) respectively.

Next, under each configuration, we generated independent random samples of size $n \equiv n_{01}$ from $N(\theta_0, \theta_0)$, that is under $H_0$, and replicated this 1000 times. From these 1000 replications we calculated the proportion ($\hat{\alpha}$) of rejecting $H_0$ and
its estimated standard error (s.e.). These are provided in column 4 of Table 1. Also, under each configuration, we generated independent random samples of size \( n \equiv n_{01} \) from \( N(\theta_1, \theta_1) \), that is under \( H_1 \), and replicated this 1000 times independently of those that ran under \( H_0 \). From these 1000 replications we calculated the proportion (\( \hat{\beta} \)) of accepting \( H_0 \) and its estimated standard error which are provided in column 5 of Table 1.

4.1.1. Some pertinent comments on Table 1

Some pertinent comments are in order. First, we note that under each configuration, the required sample size \( n_{01} \) decreased with the increase in error probabilities. Intuitively, this is expected. But now focus on section (a) \( \theta_0 = 14 \) and \( \theta_1 = 15 \) and section (e) \( \theta_0 = 5 \) and \( \theta_1 = 6 \). In both sections, the difference between \( \theta_0 \) and \( \theta_1 \) is 1 unit. But the sample sizes required to achieve the target \( \alpha, \beta \) are much less under (e). Indeed, for testing the hypotheses \( H_0 : \theta = 14 \) vs. \( H_1 : \theta = 15 \) the sample size required is nearly 2.7 times higher (uniformly for all error probabilities) than the sample size required for testing the hypotheses \( H_0 : \theta = 5 \) vs. \( H_1 : \theta = 6 \). A similar behavior is observed if one compares the sections (c) and (f). We observe similar situations when comparing section (a) with section (b) or section (c) with (d) from Table 1.

We recall that the specialty in a \( N(\theta, \theta) \) distribution is that the single unknown parameter \( \theta \) represents neither location nor scale and yet it belongs to a one-parameter exponential family. We are discovering that such specialty appears to translate into unconventional performances of a MP test in contrast with what one would normally expect from a MP test for \( H_0 : \mu = \mu_0 \) vs. \( H_1 : \mu = \mu_1 (> \mu_0) \) in a \( N(\mu, \sigma^2) \) distribution with known \( \sigma \). Obviously, the sample size required to meet a specified target \( \alpha, \beta \) would depend on \( \mu_1 - \mu_0 \) when \( \sigma \) is known and fixed.

We are seeing something that is completely different in the case of a \( N(\theta, \theta) \) distribution which makes our investigation worthwhile and interesting. Intuitively, a testing methodology ought to discriminate between the two distributions \( N(\theta_0, \theta_0) \) and \( N(\theta_1, \theta_1) \) with maximum power. This is clearly a more complex problem than discriminating between the two distributions \( N(\mu_0, \sigma^2) \) and \( N(\mu_1, \sigma^2) \) with \( \sigma^2 \) known!

Next, we focus on the comparing \( n_{01}, n_{02} \) and \( n_{03} \). We notice right away that large-sample approximations \( n_{02}, n_{03} \) are practically same as the exact required sample size \( n_{01} \) across the table whether \( n_{01} \) is small (under 30), medium (31 to 150) or large (151 and up).

This feature may not be entirely unexpected based on following explanations. From (3.5), we may claim that

\[
\chi^2_{n, \alpha}[\lambda_0] \approx n(1 + \theta_0) + z_\alpha \sqrt{2n(1 + 2\theta_0)}, \quad \text{and}
\]

\[
\chi^2_{n, 1-\beta}[\lambda_1] \approx n(1 + \theta_1) - z_\beta \sqrt{2n(1 + 2\theta_1)}.
\]

These approximations work well when \( \theta \) is large, even though \( n \) may be small. Indeed, the expression of \( n_{02} \) in (3.7) equivalently follows from these approximate expressions of \( \chi^2_{n, \alpha}[\lambda_0] \) and \( \chi^2_{n, 1-\beta}[\lambda_1] \).
Thus, we summarize our sentiment as follows: For hypotheses testing for \( \theta \) in a \( N(\theta, \theta) \) distribution, the sample sizes do not have to be very large for asymptotics to kick in. That is one may think of implementing the exact MP test procedure with the required sample size \( n \equiv n_{02} \) or \( n_{03} \) especially since it is not so simple to determine \( n_{01} \) in practice as it amounts to handling and inverting a non-central chi-square distribution. The large-sample approximations work just fine in practice for all values of \( \theta \).

One may be tempted to suggest that even better (than \( n \equiv n_{02} \) or \( n_{03} \)) approximations (to \( n \equiv n_{01} \)) may be possible by taking into account higher-order terms such as those considered by Anscombe (1948) in the cases of binomial, Poisson, and negative binomial data. From Table 1, however, we find that \( n \equiv n_{02}, n_{03} \) values are practically same as \( n \equiv n_{01} \) values even when \( n_{01} \) is small. Hence, in our view, any practical gain that may be possibly achieved from Anscombe-type analyses in this problem will be very marginal at best. So,
we do not pursue that route.

4.2. Illustrations with real data from a $N(\theta, c\theta)$ distribution

In this section, we illustrate our proposed MP test methodology using two sets of real data. In both cases, we will argue that the associated data may be reasonably assumed to have been generated from a $N(\theta, c\theta)$ distribution. We will introduce the two datasets separately in Subsections 4.2.1 and 4.2.2 respectively.

4.2.1. Illustration-I: Real data from a sequential fixed-width confidence interval procedure

Here, we first briefly summarize a problem of constructing a fixed-width confidence interval for a normal mean using the purely sequential procedure of Anscombe (1952), Ray (1957), and Chow and Robbins (1965). Suppose that we have i.i.d. random variables $Y_1, \ldots, Y_n$ from a $N(\mu, \sigma^2)$ distribution with both parameters unknown and $0 < \sigma < \infty$. One aims at constructing a fixed-width confidence interval,

$$J_n = [\bar{Y}_n - d, \bar{Y}_n + d]$$

for the unknown mean $\mu$ where $\bar{Y}_n = n^{-1} \sum_{i=1}^n Y_i$. Clearly $J_n$ has the fixed width of $2d$ where $d(>0)$ was preassigned. This confidence interval will have a confidence coefficient of at least $1 - q$, $0 < q < 1$, if and only if

$$n \text{ is the smallest integer } \geq n^* \equiv \frac{z_{q/2}^2 \sigma^2}{d^2}. \quad (4.1)$$

However, in practice, one may not have the true value of $\sigma$ and hence determine the required sample size purely sequentially by means of the following stopping rule:

$$L \equiv L(d) \text{ is the smallest integer } n(\geq m \geq 2) \text{ for which we observe } n \geq \frac{z_{q/2}^2 S_n^2}{d^2} \quad (4.2)$$

where $S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y}_n)^2$. Ghosh and Mukhopadhyay (1975) explored the asymptotic distribution of $L$ and showed:

$$\frac{L - n^*}{\sqrt{2n^*}} \xrightarrow{d} N(0,1) \quad \text{as } d \to 0. \quad (4.3)$$

Additionally, one may refer to Mukhopadhyay and Solanky (1994), Ghosh et al. (1997), and Mukhopadhyay and de Silva (2009) among other sources.

Thus, for small $d$, we may expect that the distribution of $\frac{1}{2}L$ would be approximated by $N(\frac{1}{2}n^*, \frac{1}{2}n^*)$. We used the built-in program for constructing fixed width confidence interval procedure accompanying the textbook of Mukhopadhyay and de Silva (2009) to generate 10000 values of $L$ defined by (4.2) after fixing $\mu = 10$, $\sigma^2 = 9.37$, $q = 0.95$, $d = 0.5$ so that $n^* = 143.99$ from (4.1). Starting with an initial sample of size 30(= $m$), we ran (4.2) 10000 times.
Table 2. Comparing the three tests for $H_0 : \theta = 70$ vs. $H_1 : \theta = 73$ when $n = 25$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>MP exact</th>
<th>Normal approx.</th>
<th>Variance stabilizing</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>reject $H_0$</td>
<td>reject $H_0$</td>
<td>accept $H_0$</td>
</tr>
<tr>
<td>0.05</td>
<td>accept $H_0$</td>
<td>accept $H_0$</td>
<td>accept $H_0$</td>
</tr>
</tbody>
</table>

Table 3. Comparing the three tests for $H_0 : \theta = 70$ vs. $H_1 : \theta = 73$ when $n = 50$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>MP exact</th>
<th>Normal approx.</th>
<th>Variance stabilizing</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>reject $H_0$</td>
<td>reject $H_0$</td>
<td>reject $H_0$</td>
</tr>
<tr>
<td>0.05</td>
<td>reject $H_0$</td>
<td>reject $H_0$</td>
<td>reject $H_0$</td>
</tr>
<tr>
<td>0.01</td>
<td>accept $H_0$</td>
<td>accept $H_0$</td>
<td>accept $H_0$</td>
</tr>
</tbody>
</table>

We saved a series of 10000 observed values on $L$. Then by (4.3), we may reasonably assume that we have 10000 approximate random samples from a $N(n^*, 2n^*)$ distribution with $n^* \approx 144$. Note that we may view $n^*$ as our parameter $\theta$ while the distribution under consideration is reasonably assumed as in (1.1) with known $c = 2$. Hence, we transformed our original data on $L$ to an equivalent set of data on $X \equiv \frac{1}{2}L$ so that we may reasonably postulate that the $X$-observations came from a $N(72, 72)$ distribution. From this “large” pseudo population of size 10000, we sampled $n$ observations without replacement and performed the exact MP test as well as other two large-sample tests with $n = 25, 50$. The population under consideration is so large that sampling without replacement may be treated practically same as sampling with replacement giving rise to nearly independent observations. We show our results in Tables 2 and 3.

From Tables 2 and 3, one can see that the performances of the three tests did not differ except in one case. When the sample size was $n = 25$, the large-sample test (3.10) using variance stabilization accepted the null hypothesis at 10% level of significance while the other two procedures had rejected $H_0$. But all three methods accepted null hypothesis at 5% level of significance when $n = 25$. On the other hand, when the sample size was $n = 50$, the decisions from all three tests were same for $\alpha = 0.10, 0.05, 0.01$. Hence for the exact procedure or the asymptotic procedures decisions do not differ by much. This gives a hint to the reader that one may be able to bypass the complicated exact test and go for the easier large-sample procedure while making inferences about the mean in a $N(\theta, \theta)$ distribution.

4.2.2. Illustration-II: Real data from 911 calls

Public emergency has always been an issue of immense importance. When one faces an emergency on the road or at home or elsewhere, the first thing that comes to mind is “Call 911”. So, it is very important to track 911 calls
or dispatches so that the citizens may get quick and efficient service when it is called for.

This illustration is based on the record of “Real-time 911 Dispatch” in Seattle, Washington. We have chosen the dispatches that occurred during August 25th to August 30th, 2009. The data is publicly available from the following website of Seattle Government: http://www2.seattle.gov/fire/realTime911/getDatePubTab.asp. Here, the amount of time taken to dispatch help due to each 911 call is recorded every day. We use the record on the number of 911 calls received during a period of 144 hours.

Before starting our analysis, we empirically examined this data for normality to hold approximately. In particular, in the context of our distribution of interest, we proceeded to check that the given data mean is proportional to the variance. The normality test on 144 values was satisfactory. The P-value for the Anderson-Darling test was 0.12. Next, under the assumption of normality, we looked at the value for the proportionality constant between the variance and the mean for this data.

For the entire data, this constant was 2.49. So, our guess for $c$ was 2.5, however, we empirically validated this assumption. We divided the data into 12 groups of 12 observations each and calculated the following ratios:

$$
\hat{c}_i = \frac{\text{sample variance of the } i\text{th group}}{\text{mean of the } i\text{th group}}, \quad i = 1, \ldots, 12.
$$

Then with these 12 estimated $c$ values, we performed both parametric (1-sample $t$-test) and non-parametric (sign test) tests for validity of our presumed value $c = 2.5$. For both parametric and non-parametric tests, the associated P-values were respectively 0.89 and 0.77, practically reassuring us that $c$ could be assumed 2.5.

From this point onward, for practical purposes, we pretend that our known $c$ is 2.5 even though this was truly an “estimated” value. Now we proceeded to implement our MP test on this data by drawing samples from it randomly without replacement. Here, again, simple random sampling without replacement may be treated practically similar to sampling with replacement giving rise to nearly independent observations. Note that the population size 144 is not too small.

We were interested in the average number ($\mu$) of dispatches per hour treating the number of dispatches per hour as a random sample from $N(\mu, 2.5\mu)$. But, in order to implement our hypotheses testing procedure, we divided each observation by 2.5 so that our scaled observations could be assumed to come from a $N(\theta, \theta)$ distribution with $\theta = \mu/2.5$. In the original scale, one could be interested in testing $H_0 : \mu = 8$ vs. $H_1 : \mu = 10$ so that we may equivalently postulate to test our hypotheses of interest $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ where $\theta_0 = 8/2.5 = 3.2$ and $\theta_1 = 10/2.5 = 4$. Our findings are summarized in Tables 4 and 5 when $n = 25, 50$ respectively.

When we compare the three tests, Tables 4 and 5 point in the direction of results similar in spirit to what we had reported in Tables 2 and 3. But, the
decisions appear to change greatly with the increase in the sample size \( n \) thereby leading all three tests to reject the null hypotheses at all levels of significance under consideration. But there is a more important feature associated with Tables 4 and 5. One may recall our comments made on the results from Tables 2 and 3. The observations made there hold true for Tables 4 and 5 also. That is, regardless of whether we have small sample or large sample, the results remain same for exact and asymptotic test procedures.

5. MVUE of some functionals

This section consists of ideas on estimation of some interesting functionals associated with \( N(\theta, \theta) \). First, one should note that the derivation of MVUE for \( \theta \) itself is not a very trivial exercise. This has been discussed in Mukhopadhyay and Cicconetti (2004) and Mukhopadhyay (2006). Along their lines, we find an expression for the MVUE of the parametric function \( \theta^k \) with arbitrary \( k(>0) \). According to Mukhopadhyay and Bhattacharjee’s (2010) methodology, the MVUEs of powers of \( \theta \) would be needed if one is interested in finding, for example, the MVUEs of the following functionals:

(i) the coefficient of variation, \( \text{cv}(\theta) = \frac{\theta}{\sqrt{\theta}} = \sqrt{\theta} \);

(ii) the probability, \( P_{\theta}(X_1 > 0) \).

We recall that we have \( T = \sum_{i=1}^{n} X_i^2 \). Then, following the steps shown in Mukhopadhyay and Cicconetti (2004), we will first derive the MVUE for the parametric function \( g(\theta) \equiv \theta^k, k > 0 \). In particular, if we fix \( k = \frac{1}{2} \) then we will have the MVUE for (i) \( \text{cv}(\theta) \).

5.1. Case (i)

Given a sample \( X_1, \ldots, X_n \) from a \( N(\theta, \theta) \) distribution, let us begin by finding the MVUE for \( \theta^k, k > 0 \). Our basic approach is based on classical Rao-Blackwellization. We take the sample variance, \( S^2 \), as a preliminary unbiased
estimator for the variance $\theta$ and then evaluate the conditional expectation of $S^2$ given the complete-sufficient statistic, $T = \Sigma_{i=1}^n X_i^2$. Next, we will find the expression of the following conditional expectation:

$$ h(T) = E[S^2 \mid T], $$

which, by the Lehmann-Scheffé theorem, will become the MVUE of $\theta$.

Let us construct the independent variables $Y_1, \ldots, Y_n$ by usual Helmert’s (orthogonal) transformation on $X_1, \ldots, X_n$. See Mukhopadhyay (2000, pp. 197–201). Then, we have $n^{1/2} \overline{X}_n = Y_1$ distributed as $N(n^{1/2}\theta, \theta)$, $Y_2, \ldots, Y_n$ are i.i.d. $N(0, \theta)$, $\Sigma_{i=1}^n X_i^2 = \Sigma_{i=1}^n Y_i^2$, and $S^2 = \frac{1}{n-1} \Sigma_{i=2}^n Y_i^2 = \frac{U}{n-1}$ with $U = \Sigma_{i=2}^n Y_i^2$. Hence, we have:

$$ h(t) = E[S^2 \mid \Sigma_{i=1}^n X_i^2 = t] = E \left[ \frac{1}{n-1} \Sigma_{i=2}^n Y_i^2 \mid \Sigma_{i=1}^n Y_i^2 = t \right] $$

$$ = \frac{1}{n-1} \int_0^t ua(u)du, $$

where the conditional p.d.f. of $\Sigma_{i=2}^n Y_i^2$ given $T$ is written as

$$ a(u) = a \left( \Sigma_{i=2}^n Y_i^2 = u \mid \Sigma_{i=1}^n Y_i^2 = t \right) $$

$$ = \frac{f_{1|U}(\Sigma_{i=1}^n Y_i^2 = t \mid \Sigma_{i=2}^n Y_i^2 = u) f_{U}(u)}{g(t)} $$

$$ = \frac{f_{Y_1^2}(Y_1^2 = t - u) f_{U}(u)}{g(t)}. $$

Here, $g(t)$ is the density of $T$. One can show that,

$$ f_{Y_1^2}(Y_1^2 = t - u) $$

$$ = \frac{1}{\sqrt{2\pi} \theta} \exp \left( -\frac{t - u - n\theta}{2\theta} \right) \sum_{j=0}^{\infty} \frac{n^j(t - u)^{j-1/2}}{2^{2j}j!\Gamma \left( \frac{1}{2} + j \right)}, \quad t \geq u \geq 0. $$

Also, we have:

$$ f_{U}(u) = \frac{u^{(n-1)/2-1} e^{-u/(2\theta)}}{\Gamma \left( \frac{n-1}{2} \right) (2\theta)^{(n-1)/2}}, \quad u \geq 0 $$

and

$$ g(t) = (2\theta)^{-n/2} \exp \left\{ -\frac{1}{\theta} \left( \frac{t}{n\theta} \right) \right\} r(t, n; 0), \quad t \geq 0, $$

where we write

$$ r(t, n; k) = t^{n/2-1+k} \sum_{j=0}^{\infty} \frac{n^j t^j}{2^{2j}j!\Gamma \left( \frac{n}{2} + j + k \right)}. $$
Combining (5.2) through (5.5), it follows that
\[
a(u) = \sum_{j=0}^{\infty} \frac{n^j (t-u)^j}{2^j j! \Gamma \left( \frac{1}{2} + j \right) \Gamma \left( \frac{n-1}{2} \right)} u^{(n-1)/2-1}, \quad t \geq u \geq 0.
\]

Hence, (5.1) reduces to
\[
E_{\theta} [S^2 | \Sigma_{i=1}^{n} X_i^2 = t] = E_{\theta} \left[ \frac{U}{n-1} \right| \Sigma_{i=1}^{n} X_i^2 = t] = \frac{1}{2} \frac{r(t, n; 1)}{r(t, n; 0)}
\]
and in particular, we have:
\[
E_{\theta} [U^k | \Sigma_{i=1}^{n} X_i^2 = t] = \frac{\Gamma \left( \frac{n-1}{2} + k \right)}{\Gamma \left( \frac{n-1}{2} \right)} \frac{r(t, n; k)}{r(t, n; 0)}.
\]

Now, from the fact that \( \frac{U}{\theta} \sim \chi^2_{n-1} \), we have:
\[
E_{\theta} [U^k] = (2\theta)^k \Gamma \left( \frac{n-1}{2} + k \right) \left\{ \Gamma \left( \frac{n-1}{2} \right) \right\}^{-1}, \quad n > 1 - 2k.
\]

Hence, for \( k > 0 \), we have \( \{2^k \Gamma \left( \frac{n-1}{2} + k \right)\}^{-1} U^k \Gamma \left( \frac{n-1}{2} \right) \) as an unbiased estimator of \( \theta^k \), and consequently,
\[
r(T, n; k) = 2^kr(T, n; 0) \text{ is the MVUE for } \theta^k.
\]

Thus we have:
\[
\text{the MVUE for } \text{cv}(\theta) \text{ is } \frac{r(T, n; 1/2)}{2^{1/2}r(T, n; 0)}.
\]

Mukhopadhyay and Cicconetti’s (2004) expression for the MVUE of \( \theta \) coincides with the expression from (5.7) when \( k = 1 \).

5.2. Case (ii)

For estimating the functional \( P_{\theta}(X_1 > 0) = \Phi(\theta^{1/2}) \), we would capitalize on Mukhopadhyay and Bhattacharjee’s (2010) methodology to come up with its MVUE. Upon close inspection, one will realize that this kind of example is not immediately available in Ferguson (1967), Rao (1973), Lehmann (1986), or Lehmann and Casella (1998).

First we express \( \Phi(\theta^{1/2}) \) as:
\[
\Phi(\sqrt{\theta}) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{0}^{\sqrt{\theta}} \exp \left( -\frac{t^2}{2} \right) dt
\]
\[
= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!} \int_{0}^{\sqrt{\theta}} t^{2i} dt
\]
\[
= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!2^i} \frac{\theta^{i+1/2}}{2i + 1}.
\]
We also have:

\[ \sum_{i=0}^{\infty} \frac{1}{i!2^i(2i+1)} E_{\theta} \left[ \frac{r(T, n; i + \frac{1}{2})}{2^{i+1/2} r(T, n; 0)} \right] \]

\[ < \sum_{i=0}^{\infty} \frac{1}{i!} E_{\theta} \left[ \frac{r(T, n; i + \frac{1}{2})}{2^{i+1/2} r(T, n; 0)} \right] \]

\[ = \sum_{i=0}^{\infty} \frac{\theta^{i+1/2}}{i!}, \quad \text{for all fixed } \theta(>0). \]

Thus, the infinite sum on the left-hand side of (5.9) is finite. Now, combining (5.7)–(5.8), the required MVUE is given by:

\[ \Phi(\sqrt{\theta}) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{1}{i!2^i} \frac{r(T, n; i + \frac{1}{2})}{(2i + 1)2^{i+1/2} r(T, n; 0)}. \]

5.3. Brief numerical illustrations

In this section we present brief numerical illustrations. Table 6 shows the MVUEs for the two functionals mentioned in the beginning of Section 5. We have also given the Jackknife standard errors of the estimates.

Table 6. Illustrations of MVUEs using 911 data.

<table>
<thead>
<tr>
<th>n</th>
<th>cv(\theta)</th>
<th>( P_\theta(X_1 &gt; 0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.941618</td>
<td>0.9746329</td>
</tr>
<tr>
<td>Est. std. error</td>
<td>0.1149</td>
<td>0.0371</td>
</tr>
<tr>
<td>30</td>
<td>1.828608</td>
<td>0.9709182</td>
</tr>
<tr>
<td>Est. std. error</td>
<td>0.0989</td>
<td>0.0378</td>
</tr>
</tbody>
</table>

Acknowledgements

We are grateful to the Editor, an Associate Editor, and the two referees for their encouraging feedback. We thank them for sharing with us a number of critical remarks which have helped us immensely to come up with this improved version.

REFERENCES

INFERENCE IN A $N(\theta, c\theta)$ DISTRIBUTION


