Let \( \{Y_n\} \) be a sequence of nonnegative random variables (r.v.s), and \( S_n = \sum_{j=1}^{n} Y_j, n \geq 1 \). It is first shown that independence of \( S_{k-1} \) and \( Y_k \), for all \( 2 \leq k \leq n \), does not imply the independence of \( Y_1, Y_2, \ldots, Y_n \). When \( Y_j \)'s are identically distributed exponential \( \text{Exp}(\alpha) \) variables, we show that the independence of \( S_{k-1} \) and \( Y_k, 2 \leq k \leq n \), implies that the \( S_k \) follows a gamma \( G(\alpha, k) \) distribution for every \( 1 \leq k \leq n \). It is shown by a counterexample that the converse is not true. We show that if \( X \) is a non-negative integer valued r.v., then there exists, under certain conditions, a r.v. \( Y \geq 0 \) such that \( \mathcal{N}(Y) \equiv X \), where \( \{\mathcal{N}(t)\} \) is a standard (homogeneous) Poisson process, and obtain the Laplace-Stieltjes transform of \( Y \). This leads to a new characterization for the gamma distribution. It is also shown that a \( G(\alpha, k) \) distribution may arise as the distribution of \( S_k \), where the components are not necessarily exponential. Several typical examples are discussed.

Key words and phrases: Discrete stable laws, gamma distribution, Laplace-Stieltjes transforms, negative binomial distribution, Poisson mixture, Poisson process, Poisson-Lindley distribution, positive stable laws.

1. Introduction

Let \( R_+ = (0, \infty) \) and \( Z_+ = \{0, 1, 2, \ldots\} \). Let \( \{Y_j\} \) be a sequence of \( R_+ \)-valued random variables (r.v.s), and \( S_n = \sum_{j=1}^{n} Y_j, n \geq 1 \). The study of the distribution of \( S_n \), where the \( Y_j \) are possibly dependent, arises in various disciplines such as applied probability, statistics, reliability and insurance mathematics. Vellaisamy (1996) and Vellaisamy and Punnen (2001) have shown that when the \( Y_j \) are arbitrary Bernoulli r.v.s, the distribution of \( S_n \) may be approximated by a binomial \( B(n, p) \) distribution, under certain conditions. In particular, they showed that a \( B(n, p) \) distribution arises more often as the distribution of \( S_n \) of dependent and non identical Bernoulli r.v.s. Recently, Vellaisamy and Upadhye (2007) have established that the negative binomial \( NB(r, p) \) distribution also possesses similar properties. That is, a \( NB(r, p) \) distribution may arise as a distribution of the sum of dependent and non-geometric variables. The above studies were motivated by practical applications. Drezner and Farnum (1993) considered a generalized binomial model, where \( P(Y_k = 1) \) depends on \( S_{k-1} \) and observed that such a model fits better than the classical binomial distribution, for the analysis of major league baseball data. Also, such models arise in the study of shock models in reliability theory. Vellaisamy and Sankar (2005) used a rather general dependent model for monitoring a continuous production process, where
the classical assumption of iid quality characteristic is not appropriate.

The gamma model has been widely used in several disciplines, such as queueing theory, manufacturing processes, insurance mathematics and hydraulic analysis. In the above situations, the sequence \{Y_k\} of waiting times may not be iid exponential variables. For example, the waiting time of a customer in the queue depends on the sum of waiting times of the customers already in the queue. As another example, the waiting period of a customer booked for a new vehicle depends on the total waiting times of the customers booked before him. These situations motivate us to consider the study the sum of exponential variables where \(Y_k\) may possibly depend on \(S_{k-1}\), for \(k \geq 2\). One of the main goals of this paper is to investigate if the gamma distribution could serve as a reasonable model, under the above-mentioned circumstances. In the process, we discuss some connections between a discrete distribution and a mixed Poisson process, with a particular reference to a gamma model.

It is well known that the gamma \(G(\alpha, k)\) distribution, \(k \in \mathbb{Z}_+ \setminus \{0\}\), arises as the distribution of \(S_k\), where \(Y_j\)s are iid exponential, Exp(\(\alpha\)), variates (see Feller (1971)). In Section 2, we first show that the independence of \(S_{k-1}\) and \(Y_k\), for all \(2 \leq k \leq n\), does not imply the independence of \(Y_1, Y_2, \ldots, Y_n\). However, when \(Y_j\)s are identical Exp(\(\alpha\)) variates, independence of \(S_{k-1}\) and \(Y_k\), for all \(2 \leq k \leq n\), implies \(S_k \sim G(\alpha, k)\) for all \(1 \leq k \leq n\). We show by a counterexample that the converse is not true. Note that for the \(B(n, p)\) model the converse is also true (Vellaisamy (1996)), while it is not true for the \(NB(r, p)\) model (Vellaisamy and Upadhye (2007)). In Section 3, we explore the connections between a discrete distribution and a Poisson process. Let \(X\) be a \(Z_+\)-valued rv and \(\{N(t)\}\) be a standard (parameter is unity) homogeneous Poisson process. We show that, under certain conditions, there exists a unique rv \(Y \geq 0\) such that \(N(Y) \leq X\).

In Section 4, we derive a new characterization of a \(G(\alpha, k)\) distribution, using the connections between the Poisson process and the \(NB(r, p)\) distribution. It is also shown that a \(G(\alpha, 2)\) distribution arises as the distribution of \(S_2 = Y_1 + Y_2\), where the components \(Y_1\) and \(Y_2\) are independent and one of them is non-exponential.

### 2. The \(G(\alpha, k)\) based on dependent exponential variables

Let \(\{Y_j\}\) be a sequence of \(R_+\)-valued rvs, and \(S_n = \sum_{j=1}^n Y_j\), for \(n \geq 1\). We first show that the independence of \(S_{k-1}\) and \(Y_k\), for all \(2 \leq k \leq n\), does not imply the independence of \(Y_1, \ldots, Y_n\) and also discuss its implications.

**Example 1.** Let \(Y_1\) and \(Y_2\) be independent nonnegative continuous rvs with common distribution function \(F\), with \(F(1) < 1\). Then \(H(x) = F^{*2}(x)\), the two-fold convolution of \(F\), is the distribution function of \(S_2\). Define

- \(A_2 = \{(y_1, y_2) \mid 0 \leq y_2 \leq y_1; y_1 + y_2 \leq z\}\),
- \(B_2 = \{(y_1, y_2) \mid 0 \leq y_1 \leq y_2; y_1 + y_2 \leq z\}\),
- \(C_2 = \{(y_1, y_2) \mid y_1 + y_2 > z\}\),
where $z > 0$. Let $\alpha(y) = e^{F(y)}$ and $\beta(y) = (2 - c)F(y)$ for $0 \leq y \leq 1$, where $\max(0, 2 - \frac{1}{e^{1/c}}) < c < 1$. Let $Y_3$ be a nonnegative rv with conditional d.f.

$$F_3(y \mid (y_1, y_2) \in A_2) = \begin{cases} \alpha(y), & \text{for } 0 \leq y \leq 1 \\ 1 - (1 - \eta)\frac{1 - F(y)}{1 - F(1)}, & \text{for } y \geq 1; \end{cases}$$

$$F_3(y \mid (y_1, y_2) \in B_2) = \begin{cases} \beta(y), & \text{for } 0 \leq y \leq 1 \\ 1 - (1 - \delta)\frac{1 - F(y)}{1 - F(1)}, & \text{for } y \geq 1, \end{cases}$$

for any $0 \leq z \leq 1$, and

$$F_3(y \mid (y_1, y_2) \in C_1 \setminus C_2) = F(y), \quad \forall y \geq 0,$$

where $z > 1$, $\eta = \alpha(1)$ and $\delta = \beta(1)$. Then, it follows that $Y_3$ and $S_2$ are independent, though $Y_1, Y_2$ and $Y_3$ are not independent. Also, it can be seen that $Y_3 \tilde{\in} Y_1$.

**Remark 1.** Note that $\alpha(y)$ and $\beta(y)$ may be chosen more generally. In fact, we may take $\alpha(y)$ and $\beta(y)$ to be nonnegative, nondecreasing differentiable functions such that $\alpha(y) + \beta(y) = 2F(y)$, for $0 \leq y \leq 1$ and $\alpha(y) \neq \beta(y)$ for some $y \in [0, 1]$.

We now discuss some implications of the above example. Let $S_n = \sum_{j=1}^n Y_j$, and $\text{Poi}(\lambda)$ denote the Poisson distribution with mean $\lambda$.

**Theorem 1.** Let $Y_1, Y_2, \ldots, Y_n$ be real-valued random variables with common d.f $F(y)$. If $Y_k$ is independent of $S_{k-1}$, for every $2 \leq k \leq n$, then $S_k \sim F^{*}(k)$, for all $2 \leq k \leq n$.

**Proof.** Let $\phi(t)$ be the characteristic function of $F$. Then the characteristic function of $S_k$ is

$$\phi_{S_k}(t) = E(e^{itS_k}) = E(e^{itS_{k-1} + itY_k}) = \phi_{S_{k-1}}(t)\phi(t),$$

which leads to $\phi_{S_k}(t) = \phi^k(t)$, for all $t \in \mathbb{R}$. Thus, $S_k \sim F^{*}(k)$, for all $2 \leq k \leq n$.

**Corollary 1.** Let $\{Y_n\}_{n \geq 1}$ be identically distributed $\text{Exp}(\alpha)$ variables. If $Y_k$ is independent of $S_{k-1}$, for all $k \geq 2$, then $S_k \sim G(\alpha, k)$ distribution, for all $k > 1$. Further, the associated counting process $N(t) = \max\{n \mid S_n \leq t\}$, $t > 0$, has the marginal distribution of $N(t) \sim \text{Poi}(\lambda t)$, for each $t > 0$.

**Proof.** The first part of the corollary follows from Theorem 1 and the second part follows from the well-known fact that $N(t) \sim \text{Poi}(\lambda t)$, for $t > 0$, if $S_n \sim G(\lambda, n)$, for all $n \geq 1$.

Note that Corollary 1 shows, in particular, that $G(\alpha, n)$ and $\text{Poi}(\lambda)$ distributions may arise out of identical, but dependent, $\text{Exp}(\alpha)$ variables. The converse of Corollary 1 is not true, as can be seen in the following counterexample.
Example 2. Let $X$ be an $\text{Exp}(\alpha)$ variable, and the density of $Y$ given $X = x$ be given by

$$f(y \mid x) = \alpha e^{-\alpha y} + \theta(x, y),$$

where

$$\theta(x, y) = \begin{cases} 
\theta_1, & \text{if } 0 < x < 1 \land 1 < y < 2 \\
-\theta_1, & \text{if } 0 < x < 1 \land 2 < y < 3 \\
\theta_2, & \text{if } 0 < y < 1 \land x < 2 \\
-\theta_2, & \text{if } 0 < y < 1 \land 2 < x < 3 \\
\theta_3, & \text{if } 1 < y < 2 \land x < 3 \\
-\theta_3, & \text{if } 1 < y < 2 \land 3 < x < 3 \\
0, & \text{elsewhere}
\end{cases}$$

with $\theta_1(x, y) = \theta e^{2\alpha x}$, $\theta_2(x, y) = -\theta e^{\alpha(x - 1)}$, and $\theta_3(x, y) = \theta e^{\alpha(2x - 2)}$, for some $\theta > 0$.

Note that $\theta(x, y) > -\alpha e^{-\alpha y}$, for all $x > 0$ and appropriate choice of $\theta$. Also, it can be shown that

(i) $\int_0^\infty \theta(x, y) dy = 0$, $\forall x > 0$;
(ii) $\int_0^\infty \theta(x, y) e^{-\alpha x} dx = 0$, $\forall y > 0$; and
(iii) $\int_{x+y<z} \theta(x, y) e^{-\alpha x} dx dy = 0$, $\forall z > 0$.

Condition (i) shows that $f(y \mid x)$ is a valid conditional density and Condition (ii) shows that $f(x, y) = \alpha e^{-\alpha x} f(y \mid x)$ is a valid joint density. Condition (iii) shows that $P(X + Y \leq z) = G(z \mid \alpha, 2)$, the df of a $G(\alpha, 2)$ variable. It can be easily seen that $Y \sim \text{Exp}(\alpha)$ distribution. The verifications of the three conditions (i)–(iii) are routine, but tedious. We provide the details for Condition (iii) and for the case $3 < z \leq 4$, as an illustration. When $3 < z \leq 4$,

$$\int_{3 < x + y \leq z} \theta(x, y) e^{-\alpha x} dx dy$$

$$= \int_0^{z-3} \left( \int_{3-x}^3 -\theta e^{-\alpha x} dy \right) dx + \int_{z-3}^1 \left( \int_{3-x}^z -\theta e^{-\alpha y} dy \right) dx$$

$$+ \int_{1}^{z-2} \left( \int_2^{z-x} \theta e^{\alpha(y-1)} dy \right) dx + \int_{z-2}^1 \left( \int_{z-x}^z +\theta e^{\alpha(y-2)} dy \right) dx$$

$$+ \int_2^{z-1} \left( \int_{3-x}^1 \theta e^{\alpha(y-2)} dy \right) dx + \int_{z-1}^3 \left( \int_{z-x}^{z-2} \theta e^{\alpha(y-2)} dy \right) dx$$

$$= 0,$$

which follows by substituting the appropriate values of $\theta(x, y)$ and by routine calculations.

Thus, $S_2 = X + Y \sim G(\alpha, 2)$, but $X$ and $Y$ are not independent.

Remark 2. In the above example, we have explicitly shown that $S_2 = X + Y \sim G(\alpha, 2)$, where $X$ and $Y$ are dependent $\text{Exp}(\alpha)$ variables. A question to raise at this point is if either $X$ or $Y$ or both can be non-exponential, and
still $S_2 \sim G(\alpha, 2)$ distribution, for some $\alpha > 0$. Note also that the earlier constructive approach is quite difficult to pursue further. We, instead, follow a different approach where we exploit the connections between negative binomial and gamma distributions via the standard Poisson process (see Engel and Zijlstra (1980) or Grandell (1997)).

3. Discrete distributions and a mixed Poisson process

Let $\{N(t)\}$ be a homogeneous (suppressed henceforth) Poisson process with parameter $\lambda = 1$, called the standard Poisson process. Hereafter, whenever we consider $N(Z)$, with or without a suffix, it is assumed that $Z$ is a nonnegative rv independent of the standard Poisson process $\{N(t)\}$.

It is easily seen that the Laplace-Stieltjes transform (LST) $\phi_Z(t) = E(e^{-tZ})$ of $Z$ and the LST $\phi_{N(Z)}(t)$ of $N(Z)$ are related by the identity

$$\phi_{N(Z)}(t) = \phi_Z(1 - e^{-t}), \quad t \geq 0.$$  

(3.1)

Since $0 \leq 1 - e^{-t} < 1$, the function $\phi_{N(Z)}(t)$ determines the function $\phi_Z(t)$ only up to 1, while $\phi_{N(Z)}(t)$ is completely determined by the values of $\phi_Z(t)$ in the interval $[0, 1)$.

Let $X$ be a $Z_+^*$-valued rv with $P(X = j) = \alpha_j$ for $j \in Z_+$. The pgf $G_X(s)$ of $X$ satisfies

$$g(s) = G_X(1 - s) = \sum_{j=0}^{\infty} (1 - s)^j \alpha_j.$$  

(3.2)

According to Proposition 2.2 in Grandell (1997), for $X$ to be a mixed Poisson rv, that is, for $X \overset{d}{=} N(Y)$ to hold for some $Y \geq 0$, it is necessary and sufficient that $G_X(s)$ is absolutely monotone for all $s \in (-\infty, 1)$. However, even in simple situations where $G_X(s)$ is a polynomial, it will be very difficult to check this condition because of the arbitrary nature of $\alpha_j$.

Note that $g(s)$ is completely monotone in $(0, 1)$ and if $g(s)$ also happens to be completely monotone in $[1, \infty)$, it will indeed be an LST, since $g(0) = 1$, and we shall have a $Y$ such that $X \overset{d}{=} N(Y)$ with LST of $Y$ being $g(s)$. The following result may be useful as it eliminates tedious calculations in checking the absolute monotonicity of $G_X(s)$ for negative values of $s$ and in situations where $X$ is not a Poisson mixture (see Remark 3 (ii)).

**Theorem 2.** Let $X$ be a $Z_+^*$-valued rv with $P(X = r) = \alpha_r$, for $r \in Z_+$. Consider the function $g$ on $[0, 1]$ defined by

$$g(t) = \sum_{r=0}^{\infty} (1 - t)^r \alpha_r.$$  

(3.3)

Let $g$ have an extension $g^*$ to $R_+$ as the LST of a nonnegative rv $Y$. Then, on a suitable probability space, there exists a standard Poisson process $\{N(t)\}$ and a rv
Y, independent of the process, with \( g^* \) as its LST such that \( P(N(Y) = r) = \alpha_r \), that is, \( N(Y) \overset{D}{=} X \). Conversely, whenever there exists a \( Y \) such that \( X \overset{D}{=} N(Y) \), the function \( g(t) \) is the LST of \( Y \).

**Proof.** For any rv \( Y \geq 0 \) with \( F_Y \) as its df and \( \phi_Y \) as LST, we have for \( k \geq 1 \),

\[
P(N(Y) = k) = \int_0^\infty e^{-y} \frac{y^k}{k!} dF_Y(y) = \frac{(-1)^k}{k!} \phi_Y^{(k)}(1),
\]

where \( \phi_Y^{(k)}(t) \) denotes the \( k \)-th derivative of \( \phi_Y(t) \). Since \( \phi_Y(t) \) is differentiable any number of times at all \( t > 0 \), we have, using Taylor’s expansion and (3.4)

\[
\phi_Y(1 + s) = \phi_Y(1) + \sum_{k=1}^{\infty} \frac{\phi_Y^{(k)}(1)}{k!} s^k = \sum_{k=0}^{\infty} (-s)^k P(N(Y) = k),
\]

for \( |s| \leq 1 \). Thus,

\[
\phi_Y(t) = \sum_{k=0}^{\infty} (1 - t)^k P(N(Y) = k)
\]

for \( 0 \leq t \leq 2 \). Set \( g^* = \phi_Y \). Then \( g \), given at (3.3), coincides with \( \phi_Y \) for some \( Y \geq 0 \), on a non degenerate interval and hence by uniqueness of power series expansions, \( P(N(Y) = r) = \alpha_r \), which proves the first part. The converse is obvious from (3.6) and Proposition 2.2 in Grandell (1997).

**Remark 3.** (i) If an LST has the form given at (3.3) when restricted to \([0, 1]\), then there exists a rv \( Y \) with LST \( \phi_Y(t) \) for which \( N(Y) \overset{D}{=} X \). In the event of existence of an extension \( g^* \) to \( \mathbb{R}_+ \) as an LST, the rv \( X \) is expressible as a Poisson mixture and by the fact that the mixing distribution of \( Y \) is identifiable in Poisson mixtures \( N(Y) \), the LST of \( Y \) is \( g^* \).

(ii) Note that \( g \) need not always have an extension \( g^* \) to \( \mathbb{R}_+ \) as an LST. For example, if \( X \) takes only a finite number of values, then \( g \) is a polynomial and \( g^* \), the unique analytic extension of \( g \) to \( \mathbb{R}_+ \), is also \( g \) which can not be an LST because polynomials are unbounded at \( \infty \). Thus, if \( X \sim \text{Be}(p) \), the Bernoulli distribution, then \( X \) does not admit the form \( X \overset{D}{=} N(Y) \) for any \( Y \geq 0 \).

(iii) Let \( X \) be a nonnegative integer valued rv. Then, we obtain a relation between the mgf of \( X \) and its factorial moments, namely,

\[
M_X(t) = E[(1 + t)^X] = \sum_{n=0}^{\infty} P(X = n) \sum_{r=0}^{n} t^r \binom{n}{r} = \sum_{r=0}^{\infty} t^r E\left(\frac{X}{r}\right),
\]

after the rearrangement of (nonnegative) terms and when the rhs of (3.7) converges for some \( t > 0 \).

We now present a condition on the binomial moments of the rv \( X \), which helps us to obtain the rv \( Y \), when it exists, such that \( N(Y) \overset{D}{=} X \).
Theorem 3. Suppose $X$ is a $\mathbb{Z}_+$-valued rv whose mgf $M_X(t)$ is finite for $0 \leq t < t_0$. Suppose the LST $\phi_Y(t)$ of a nonnegative rv $Y$ is of the form

$$(3.8) \quad \phi_Y(t) = \sum_{k=0}^{\infty} (1-t)^k P(X = k)$$

for $0 \leq t < t_0$, where $N(Y) \overset{?}{=} X$. Then $Y$ is completely determined and its LST is given by

$$(3.9) \quad \phi_Y(t) = \sum_{k=0}^{\infty} (-t)^k E \left( \frac{X}{k} \right), \quad t \geq 0.$$  

Proof. From (3.8), we have, for $r \geq 1$,

$$(3.10) \quad (-1)^r E(Y^r) = \lim_{t \to 0} \phi_Y^{(r)}(t) = (-1)^r r! E \left( \frac{X}{r} \right).$$

Then by Remark 3(iii), (3.7) and (3.10), we observe that $\phi_Y(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} EY^k$ converges absolutely for $0 \leq \lambda < \lambda_0$. From analytic function theory, it is known that, in this case, the above series determines $\phi_Y(\lambda)$ for all $\lambda \geq 0$. Thus, the result in (3.9) follows.

Remark 4. When the rhs of (3.7) converges for some $0 \leq s < s_0$ and $s_0 > 1$, the power series expressions at (3.8) and (3.9) coincide on a non-degenerate interval and lead to

$$P(X = r) = \sum_{k=r}^{\infty} (-1)^{k-r} \binom{k}{r} E \left( \frac{X}{r} \right) - \sum_{k=0}^{r-1} (-1)^{k-r} \binom{k}{r} E \left( \frac{X}{r} \right), \quad r \in \mathbb{Z}_+,$$

which was earlier obtained by Takačs (1967).

Next, we discuss two known examples where $X \overset{?}{=} N(Y)$ holds.

Example 3. (Engel and Zijlstra (1980)). A nonnegative rv $Y \sim G(\alpha, p)$ distribution iff $N(Y) \sim NB(p, \alpha^{-1})$ distribution.

Example 4. Let $X$ follow the Poisson-Lindley distribution (Johnson et al. (2005, p. 370)) with

$$(3.11) \quad P(X = n) = \alpha^2 \frac{\alpha + 2 + n}{(\alpha + 1)^{n+1}}$$

for $n \in \mathbb{Z}_+$ and $\alpha > 0$. Then the distribution of $Y$ for which $N(Y) \overset{?}{=} X$ has the pdf

$$(3.12) \quad g(y) = \frac{\alpha^2}{\alpha + 1} (1+y)e^{-\alpha y}, \quad y > 0, \ \alpha > 0,$$

which is called the Lindley distribution.
In the following example, both $X$ and $Y$ have infinite expectations and still $N(Y) \overset{d}{=} X$ holds.

Example 5. Suppose $X$ follows the discrete stable distribution (Steutel and van Harn (2003)) with probability generating function
\[
G_X(s) = \exp\{-c(1 - s)^\alpha\}, \quad 0 < \alpha < 1, \quad c > 0,
\]
and $Y$ follows the positive stable distribution with characteristic exponent $\alpha$, $0 < \alpha < 1$ (the LST of $Y$ being $\exp\{-ct^\alpha\}$). Then, it can be seen that $N(Y) \overset{d}{=} X$ holds.

Remark 5. The only other discrete stable distribution corresponds to the case $\alpha = 1$ (see Steutel and van Harn (1979, Theorem 4)) in which $X \sim \text{Poi}(c)$ and $Y$ is degenerate. Thus, we have a characterizing property between discrete and continuous stable laws (Devroye (1993)). Further, for $0 < \alpha < 1$, $EX = \infty$ and the mgf of $X$ does not exist.

4. The $G(\alpha, k)$ based on dependent and non-exponential variates

We address here the question raised in Remark 2 regarding the nature of $G(\alpha, k)$. First we need the following lemma whose proof easily follows from the definition of the Poisson process.

**Lemma 1.** Let \{N(t)\} be the standard Poisson Process, and $Y_1$ and $Y_2$ be two independent nonnegative continuous rvs. Then $N(Y_1 + Y_2) - N(Y_1) \overset{d}{=} N(Y_2)$.

The next result, which is of independent interest, presents some additional information on the connection between gamma and negative binomial distributions.

**Theorem 4.** Let $Y_2$ be a nonnegative continuous rv and \{N(t)\} be a standard Poisson process. If $Y_2 \sim G(\alpha, p)$, then $N(Y_1 + Y_2) - N(Y_1) \sim NB(p, \frac{\alpha}{1+\alpha})$, for all continuous $Y_1 \geq 0$, independent of $Y_2$.

Conversely, if for some continuous $Y_1 \geq 0$, independent of $Y_2$, $N(Y_1 + Y_2) - N(Y_1) \sim NB(p, \theta)$, for some $Y_1 \geq 0$. Again, by Lemma 1,
\[
N(Y_1 + Y_2) - N(Y_1) \overset{d}{=} N(Y_2) \sim NB(p, \theta)
\]
and hence $Y_2 \sim G(\frac{\theta}{1+\theta}, p)$. This proves the theorem.

**Remark 6.** An application of Theorem 4 is the following. Suppose that the time duration ($Y_i$) between the consecutive power failures, starting from
Suppose that a continuous production process produces defective items according to a Poisson process \( \{N(t)\} \) and only the power failure causes the stoppage of the production process, during a production run. Assume that it is physically possible to sort and find out the number of defective units produced only when the power fails and the production comes to a halt. Suppose \( U \) is a random time at which the first power failure occurs after installation/commissioning of the production process or a production run. It is reasonable to assume \( U \) and \( Y \) are mutually independent. Then, by Theorem 4, the number \( N(U + Y_1) - N(U) \) of defective units produced between first and second power failures follows a negative binomial. Indeed, the number \( N(U + \sum_{i=1}^{n} Y_i) - N(U + \sum_{i=1}^{n-1} Y_i) \) of defective units produced between successive failures also follows a negative binomial distribution.

We next establish a kind of additivity result for the mixed Poisson distributions, which is useful and needed in Example 6.

**THEOREM 5.** Suppose \( X_1 \) and \( X_2 \) are independent \( \mathbb{Z}_+ \)-valued rvs and let \( Y_1 \) and \( Y_2 \) be independent rvs such that \( X_1 \sim N_1(Y_1) \) and \( X_2 \sim N_2(Y_2) \), where \( N_1 \) and \( N_2 \) are independent standard Poisson processes. Then, there exists a standard Poisson process \( \{N(t)\} \) such that

\[
X_1 + X_2 \sim N_1(Y_1) + N_2(Y_2) \sim N(Y_1 + Y_2).
\]

**Proof.** The first distributional equality is obvious. To prove the second, recall from (3.6) that the LSTs of \( Y_1 \) and \( Y_2 \) are given by

\[
\phi_{Y_1}(t) = \sum_{r=0}^{\infty} (1 - t)^r P(X_1 = r),
\]

\[
\phi_{Y_2}(t) = \sum_{s=0}^{\infty} (1 - t)^s P(X_2 = s),
\]

for some \( 0 \leq t \leq 1 \). As \( Y_1 \) and \( Y_2 \) are independent, we have for \( 0 \leq t \leq 1 \),

\[
\phi_{Y_1 + Y_2}(t) = \phi_{Y_1}(t) \phi_{Y_2}(t) = \left( \sum_{r=0}^{\infty} (1 - t)^r P(X_1 = r) \right) \left( \sum_{s=0}^{\infty} (1 - t)^s P(X_2 = s) \right) = \sum_{n=0}^{\infty} (1 - t)^n \left( \sum_{j=0}^{n} P(X_1 = j) P(X_2 = n - j) \right) = \sum_{n=0}^{\infty} (1 - t)^n P(X_1 + X_2 = n),
\]

since \( X_1 \) and \( X_2 \) are independent.
As noted in Remark 3(i), there exists a rv $Y \geq 0$ such that $N(Y) \overset{L}{=} X_1 + X_2$ and $\phi_Y(t) = \phi_{Y_1+Y_2}(t)$. Thus, $Y_1 + Y_2 \overset{L}{=} Y$. Hence, by Theorem 2.1 (i) of Grandell (1997), $N(Y_1 + Y_2) \overset{L}{=} N(Y) \overset{L}{=} X_1 + X_2$, which completes the proof.

It is well known that if $X_1$ and $X_2$ are iid $Z_+\text{-valued rvs}$, and $X_1 + X_2 \sim NB$ distribution, then $X_1$ and $X_2$ must be geometric rvs. However, if $X_1$ and $X_2$ are independent (or dependent), but not identically distributed, then it is possible for either or both rvs to be nongeometric and still $X_1 + X_2$ is NB (see, Shishebor and Towhidi (2004), Vellaisamy and Upadhye (2007)).

As an application of Theorem 5, we next show that a sum of non-gamma variables may follow a gamma distribution.

**Example 6.** Consider three independent rvs $X_1, X_2$ and $X_3$, where $X_1 \sim Be \left(\frac{1}{8}\right)$,
\[
P(X_2 = r) = \frac{3}{4} \frac{2}{4^r} \left\{ \frac{1}{2} \left(1 - \frac{1}{2}\right)^r + \frac{1}{2} \right\}, \quad r \in Z_+.
\]
and
\[
P(X_3 = r) = \frac{2^{r-1}}{3^{r+1}} (8 + 5r), \quad r \in Z_+.
\]

Let $S_2 = X_1 + X_2$. Then it is known that (Example 2.3 of Vellaisamy and Upadhye (2007)) $S_2 \sim NB(1, \frac{1}{4})$ and $S_2 + X_3 \sim NB(2, \frac{1}{4})$. By Example 3, $S_2 \overset{L}{=} N_1(Y_1)$, where $Y_1 \sim Exp(3)$. Further, it can easily be seen that
\[
g(t) = \sum_{r=0}^{\infty} (1-t)^r P(X_3 = r) = \frac{1}{6(1+2t)} + \frac{5}{6(1+2t)^2}
\]
which (over $R_+$) is an LST (of a mixture of $Exp(\frac{1}{4})$ and $G(\frac{1}{4}, 2)$ distributions).

By Theorem 2, there exists a continuous $Y_2 \geq 0$ (with LST $g(t)$) such that $N_2(Y_2) \overset{L}{=} X_3$. Also, by Theorem 5,
\[
S_2 + X_3 \overset{L}{=} N_1(Y_1) + N_2(Y_2) \overset{L}{=} N(Y_1 + Y_2).
\]

Since $S_2 + X_3 \sim NB(2, \frac{1}{4})$, we have, by Example 3, $(Y_1 + Y_2) \sim G(\frac{1}{4}, 2)$, though $Y_2$ is not a gamma variable.

The above example motivated the following example.

**Example 7.** Consider three independent rvs $U, V$ and $J \in \{1, 2\}$, where $U \sim G(\alpha, 1)$, $V \sim G(\alpha, 2)$ and $P(J = 1) = p$ and $P(J = 2) = 1 - p$, $0 < p < 1$.

Let $Y_1 = UI(J = 1) + VI(J = 2)$ and $Y_2 = UI(J = 2) + VI(J = 1)$, where $I(A)$ denotes the indicator function of the set $A$. Then $Y_1 + Y_2 \sim G(\alpha, 3)$ and $Y_1$ and $Y_2$ are dependent non-gamma rvs.
5. Conclusions

It is shown that the gamma model arises also as the distribution of the sum of identical, but certain dependent, exponential sums. Some general relationships between a discrete distribution and a mixed Poisson process are investigated. In particular, some simpler and verifiable conditions are presented to check whether a discrete rv $X$ satisfies $X \sim N(Y)$ and when it holds, the LST of $Y$ is determined in terms of the factorial moments of $X$.

Often in practice, the count data exhibit overdispersion (variance is greater than mean) and in this case the NB model is more appropriate. In this paper, it is shown, using Theorem 4, that the NB model arises as the distribution of increments observed at random (gamma) times from a Poisson process. These results have practical applications as well. Also, a kind of additivity property for the mixed Poisson distributions is established and using this result it is shown that a gamma model can arise as the distribution of $Y_1 + Y_2$, where $Y_1$ is exponential, while $Y_2$ is a non-gamma rv. As another interesting application of Theorem 5, Vellaisamy and Upadhye (2009) recently obtained a random parameter representation for the convolution of gamma $G(\alpha_j, p_j)$ distributions. Finally, we hope that the results in this paper would lead to the consideration of generalized gamma models, where $Y_k$ strongly depends on $S_{k-1}$, for $k \geq 2$, that could be more flexible and useful for practical situations.

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