ANALYSIS OF MODELS WITH COMPLEX ROOTS ON THE UNIT CIRCLE

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This paper deals with nonstationary autoregressive (AR) models with complex roots on the unit circle. We examine the asymptotic properties of the least squares estimators (LSEs) in the model. We also extend the model to the case where the error term follows a stationary linear process. We show that the limiting distribution of the LSE of the unit root parameter has a property comparable to that of the LSE in the standard nonstationary seasonal model with period two. Percent points and moments of the limiting distribution are computed by numerical integration.

Key words and phrases: Complex B-N decomposition, complex roots, limiting distribution, numerical integration, unit root.

1. Introduction

This paper deals with nonstationary AR models with two complex conjugate roots on the unit circle. Thus the model has the AR filter of the form \((1-e^{i\theta}L)(1-e^{-i\theta}L)\), where \(L\) is the lag operator and \(\theta\) (\(0 < \theta < \pi\)) is a dominant frequency. The simplest model of this type is the AR(2). This was dealt with by Ahtola and Tiao (1987), where the asymptotic distributions of the LSEs were derived. Chan and Wei (1988) obtained the asymptotic distributions in a much simpler way.

The purpose of the present paper is to extend the above AR(2) model by allowing the error term to follow a stationary process, and to discuss the asymptotic properties of the LSEs. It seems that there is no theoretical result for that extension. One of the advantages of the present model is that it can deal with any frequency that is dominant in the nonstationary series. In practice, we need to identify an appropriate model for explaining the frequency contribution. For that purpose Akaike’s information criterion (Akaike (1973)) may be useful.

In Section 2 we present a basic model, and the model is extended in Section 3. In both sections we derive the asymptotic distributions of the LSEs to find a similarity to the LSE in the standard seasonal model. Section 4 is concerned with numerical computation of the limiting distributions, and Section 5 concludes this paper.

2. Simple case

In this section we consider a nonstationary AR(2) model with two complex conjugate roots on the unit circle, which is given by

\[
(1 - e^{i\theta}L)(1 - e^{-i\theta}L)y_j = \varepsilon_j 
\Leftrightarrow y_j = \phi_1 y_{j-1} + \phi_2 y_{j-2} + \varepsilon_j,
\]

(2.1)
where \( \phi_1 = 2 \cos \theta \) and \( \phi_2 = -1 \). The parameter \( \theta \) is restricted to be \( 0 < \theta < \pi \), and gives a frequency contribution to the model with period \( 2\pi/\theta \). To generate the process we assume that \( y_{-1} = y_0 = 0 \). It is also assumed in this section that \( \{\varepsilon_j\} \sim \text{i.i.d.}(0, \sigma^2) \).

The model in (2.1) was dealt with earlier in Ahtola and Tiao (1987) and Chan and Wei (1988), where the asymptotic properties of the LSEs of \( \phi_1 \) and \( \phi_2 \) were studied. In particular, the limiting expressions for the normalized estimators in the sense of weak convergence were obtained. The former also gives percent points of the limiting distributions as well as those of finite sample distributions by simulations, while the latter gives limiting expressions which are much neater than the former.

Since, in the next section, we extend the model to the case where the error process \( \{\varepsilon_j\} \) follows a general stationary process by employing the basic asymptotic arguments associated with the present model, we briefly summarize the arguments here.

For this purpose we first obtain

\[
y_j = \frac{\varepsilon_j}{(1 - e^{i\theta}L)(1 - e^{-i\theta}L)} = \frac{1}{2i \sin \theta} \left[ \frac{e^{i\theta} - e^{-i\theta}}{1 - e^{i\theta}L - 1 - e^{-i\theta}L} \right] \varepsilon_j
\]

\[
= \frac{1}{2i \sin \theta} \left[ e^{i\theta} \{e^{i(j-1)\theta} \varepsilon_1 + e^{i(j-2)\theta} \varepsilon_2 + \cdots + e^{i\theta} \varepsilon_{j-1} + \varepsilon_j\} - e^{-i\theta} \{e^{-i(j-1)\theta} \varepsilon_1 + e^{-i(j-2)\theta} \varepsilon_2 + \cdots + e^{-i\theta} \varepsilon_{j-1} + \varepsilon_j\} \right]
\]

\[
= \frac{1}{\sin \theta} \sum_{k=1}^{j} \frac{e^{i(j-k+1)\theta} - e^{-i(j-k+1)\theta}}{2i} \varepsilon_k = \frac{1}{\sin \theta} \sum_{k=1}^{j} \varepsilon_k \sin(j - k + 1)\theta
\]

\[
= \frac{1}{\sin \theta} \left[ x_j^{(\theta)} \sin(j + 1)\theta - y_j^{(\theta)} \cos(j + 1)\theta \right] = \frac{1}{\sin \theta} (a_j^{(\theta)})^T z_j^{(\theta)},
\]

where

\[
z_j^{(\theta)} = \begin{pmatrix} x_j^{(\theta)} \\ y_j^{(\theta)} \end{pmatrix} = \sum_{k=1}^{j} \begin{pmatrix} \cos k\theta \\ \sin k\theta \end{pmatrix} \varepsilon_k, \quad a_j^{(\theta)} = \begin{pmatrix} \sin(j + 1)\theta \\ -\cos(j + 1)\theta \end{pmatrix}.
\]

It is noticed that \( \{z_j^{(\theta)}\} \) is a two-dimensional random walk expressed by

\[
(2.2) \quad z_j^{(\theta)} = z_{j-1}^{(\theta)} + v_j^{(\theta)}, \quad v_j^{(\theta)} = \begin{pmatrix} \cos j\theta \\ \sin j\theta \end{pmatrix} \varepsilon_j, \quad z_0^{(\theta)} = 0,
\]

where the error term \( \{v_j^{(\theta)}\} \) is a sequence of martingale differences that are independent with heterogeneous variances.

On the basis of the random walk \( \{z_j^{(\theta)}\} \), let us construct, for \( 0 \leq t \leq 1 \), the two-dimensional partial sum process

\[
(2.3) \quad Z_T^{(\theta)}(t) = \sqrt{2} \left[ \frac{1}{\sqrt{T\sigma}} z_{[Tt]}^{(\theta)} + (Tt - [Tt]) \frac{v_{[Tt]+1}^{(\theta)}}{\sqrt{T\sigma}} \right],
\]
where \([x]\) is the greatest integer not exceeding \(x\). Note that \(Z_T^{(\theta)} = \{Z_T^{(\theta)}(t)\}\) is a stochastic process in a function space \(C^2 = C[0,1] \times C[0,1]\), where \(C[0,1]\) is the space of all real-valued continuous functions defined on \([0,1]\). Then the following functional central limit theorem (FCLT) holds (Helland (1982), Chan and Wei (1988)).

**Theorem 1.** For the two-dimensional partial sum process \(\{Z_T^{(\theta)}(t)\}\) in (2.3) constructed from the random walk \(\{z_j^{(\theta)}\}\) in (2.2), the following FCLT holds:

\[
Z_T^{(\theta)} \Rightarrow W,
\]

where \(W = \{W(t)\}\) is the two-dimensional standard Brownian motion. Moreover the following joint weak convergence of basic quantities holds:

\[
(2.4) \quad (Q_T, R_T) \Rightarrow \left( \int_0^1 W(t) W'(t) dt, \int_0^1 W(t) dW'(t) \right),
\]

where

\[
Q_T = \frac{2}{T^2 \sigma^2} \sum_{j=1}^{T} z_j^{(\theta)}(z_j^{(\theta)})' = \frac{1}{T} \sum_{j=1}^{T} Z_T^{(\theta)} \left( \frac{j}{T} \right) \left\{ Z_T^{(\theta)} \left( \frac{j}{T} \right) \right\}',
\]

\[
R_T = \frac{2}{\sigma^2} \sum_{j=1}^{T} z_j^{(\theta)}(z_j^{(\theta)})' = \sum_{j=1}^{T} Z_T^{(\theta)} \left( \frac{j-1}{T} \right) \left\{ \Delta Z_T^{(\theta)} \left( \frac{j}{T} \right) \right\}',
\]

\[
\]

Let us return to the model (2.1) and put \(\phi = (\phi_1, \phi_2)' = (2 \cos \theta, -1)'\). Let us denote the LSE of \(\phi\) based on a sample of size \(T\) by \(\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2)'\). Then we have \(T(\hat{\phi} - \phi) = A_T^{-1} b_T\), where

\[
(2.5) \quad A_T = \frac{1}{T^2 \sigma^2} \sum_{j=3}^{T} \left( \begin{array}{cc} y_{j-1}^2 & y_{j-1}y_{j-2} \\ y_{j-1}y_{j-2} & y_{j-2}^2 \end{array} \right), \quad b_T = \frac{1}{T \sigma^2} \sum_{j=3}^{T} \left( \begin{array}{c} y_{j-1} \epsilon_j \\ y_{j-2} \epsilon_j \end{array} \right).
\]

Let us first consider the second moment of \(\{y_j\}\), which is

\[
\frac{1}{T^2} \sum_{j=1}^{T} y_j^2 = \frac{1}{T^2 \sin^2 \theta} \sum_{j=1}^{T} \left[ x_j^{(\theta)} \sin(j+1) \theta - y_j^{(\theta)} \cos(j+1) \theta \right]^2
\]

\[
= \frac{\sigma^2}{4T \sin^2 \theta} \sum_{j=1}^{T} \left\{ Z_T^{(\theta)} \left( \frac{j}{T} \right) \right\}' Z_T^{(\theta)} \left( \frac{j}{T} \right) - \frac{1}{2T^2 \sin^2 \theta}
\]

\[
\times \sum_{j=1}^{T} \left[ \left( x_j^{(\theta)} \right)^2 - \left( y_j^{(\theta)} \right)^2 \right] \cos 2(j+1) \theta + 2x_j^{(\theta)} y_j^{(\theta)} \sin 2(j+1) \theta.
\]

Since it follows from Chan and Wei (1988) that

\[
(2.6) \quad \sup_{1 \leq j \leq T} \left| \sum_{k=1}^{j} e^{ik \theta} w_k \right| = o_p(T^2), \quad (\epsilon^{i \theta} \neq 1),
\]
where \( w_k \) is either \((x_k^{(\theta)})^2\) or \((y_k^{(\theta)})^2\) or \(x_k^{(\theta)} y_k^{(\theta)}\), it holds that
\[
\frac{1}{T^2 \sigma^2} \sum_{j=1}^{T} y_j^2 \Rightarrow \frac{1}{4 \sin^2 \theta} \int_0^1 W'(t) W(t) dt.
\]

Similarly, we have
\[
\sum_{j=2}^{T} y_{j-1} y_j = \frac{1}{\sin^2 \theta} \sum_{j=2}^{T} [x_{j-1}^{(\theta)} \sin j \theta - y_{j-1}^{(\theta)} \cos j \theta]
\times [x_j^{(\theta)} \sin (j + 1) \theta - y_j^{(\theta)} \cos (j + 1) \theta]
= \frac{1}{2 \sin^2 \theta} \sum_{j=2}^{T} [(x_{j-1}^{(\theta)} x_j^{(\theta)} + y_{j-1}^{(\theta)} y_j^{(\theta)}) \cos \theta + (x_{j-1}^{(\theta)} y_j^{(\theta)} - x_j^{(\theta)} y_{j-1}^{(\theta)}) \sin \theta]
- (x_{j-1}^{(\theta)} x_j^{(\theta)} - y_{j-1}^{(\theta)} y_j^{(\theta)}) \cos (2j + 1) \theta
- (x_{j-1}^{(\theta)} y_j^{(\theta)} + x_j^{(\theta)} y_{j-1}^{(\theta)}) \sin (2j + 1) \theta
= \frac{T \sigma^2 \cos \theta}{4 \sin^2 \theta} \sum_{j=1}^{T} \left\{ Z_T^{(\theta)} \left( \frac{j}{T} \right) \right\}' \frac{1}{T} Z_T^{(\theta)} \left( \frac{j}{T} \right) + o_p(T^2),
\]
which yields
\[
\frac{1}{T^2 \sigma^2} \sum_{j=2}^{T} y_{j-1} y_j \Rightarrow \frac{\cos \theta}{4 \sin^2 \theta} \int_0^1 W'(t) W(t) dt.
\]

Therefore we have
\[
(2.7) \quad A_T \Rightarrow \int_0^1 \frac{W'(t) W(t) dt}{4 \sin^2 \theta} \begin{pmatrix} 1 & \cos \theta \\ -1 & 1 \end{pmatrix}.
\]

We also have, for \( h > 0 \),
\[
\sum_{j=h+1}^{T} y_{j-h} \varepsilon_j = \frac{1}{\sin \theta} \sum_{j=2}^{T} [x_{j-h}^{(\theta)} \sin (j - h + 1) \theta - y_{j-h}^{(\theta)} \cos (j - h + 1) \theta] \varepsilon_j
= \frac{1}{\sin \theta} \sum_{j=h+1}^{T} (z_{j-h}^{(\theta)})' J_h(\theta) \Delta z_j^{(\theta)}
= \frac{1}{\sin \theta} \sum_{j=h+1}^{T} (z_{j-1}^{(\theta)})' J_h(\theta) \Delta z_j^{(\theta)} + o_p(T)
= \frac{T \sigma^2}{2 \sin \theta} \sum_{j=h+1}^{T} \left\{ Z_T^{(\theta)} \left( \frac{j-1}{T} \right) \right\}' J_h(\theta) \Delta Z_T^{(\theta)} \left( \frac{j}{T} \right) + o_p(T),
\]
where
\[
(2.8) \quad J_h(\theta) = \begin{pmatrix} -\sin(h-1)\theta & \cos(h-1)\theta \\ -\cos(h-1)\theta & -\sin(h-1)\theta \end{pmatrix}.
\]
Therefore we have

\( b_T \Rightarrow \frac{1}{2\sin \theta} \int_0^1 \left( \frac{W'(t)J_1(\theta)}{W'(t)J_2(\theta)} \right) dW(t). \) (2.9)

It follows from Theorem 1 and the continuous mapping theorem (CMT) that (2.7) and (2.9) lead us to

\( T(\hat{\phi} - \phi) = A_T^{-1} b_T \Rightarrow \left( \frac{Z_1}{Z_2} \right), \) (2.10)

where

\[
Z_1 = 2 \int_0^1 W'(t) \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} dW(t) / \int_0^1 W'(t) W(t) dt,
\]

\[
Z_2 = -2 \int_0^1 W'(t) dW(t) / \int_0^1 W'(t) W(t) dt.
\] (2.11) (2.12)

It is noticed that \( Z_2 \), which is the limit in distribution of \( T(\hat{\phi}_2 - \phi_2) \), does not depend on the seasonal frequency \( \theta \), although \( Z_1 \), which is the limit in distribution of \( T(\hat{\phi}_1 - \phi_1) \), does.

We note in passing that the limiting distribution of \( T(\hat{\phi}_2 - \phi_2) \) also appears when we deal with a standard nonstationary seasonal model with period \( m \):

\[
y_j = \rho y_{j-m} + \varepsilon_j, \quad (j = 1, \ldots, T),
\]

where the true value of \( \rho \) is unity, and the initial values of \( y_j (j = 1 - m, \ldots, 0) \) are all set at 0. Let \( \hat{\rho} \) be the LSE of \( \rho \). Then it holds (Tanaka (1996)) that

\[
N(\hat{\rho} - 1) \Rightarrow Z = \int_0^1 B'(t) dB(t) / \int_0^1 B'(t) B(t) dt,
\]

where \( N = [T/m] \) and \( \{B(t)\} \) is the \( m \)-dimensional standard Brownian motion. Then it is seen that the distribution of \(-2Z\) is the same as that of \( Z_2 \) when \( m = 2 \).

We shall show in the next section that this last property continues to hold when the present model is extended to the case where the error term \( \{\varepsilon_j\} \) becomes a stationary process. We shall also indicate in Section 4 how to compute numerically percent points of \( Z_1 \) and \( Z_2 \) together with their moments.

3. Extended case

In this section we extend the model (2.1) to the following case:

\[
y_j = \phi_1 y_{j-1} + \phi_2 y_{j-2} + u_j,
\]

\[
u_j = \alpha(L)\varepsilon_j = \sum_{l=0}^{\infty} \alpha_l \varepsilon_{j-l}, \quad \sum_{l=1}^{\infty} l|\alpha_l| < \infty,
\]

where \( \phi_1 = 2 \cos \theta \), \( \phi_2 = -1 \), and \( \{\varepsilon_j\} \sim \text{i.i.d.}(0, \sigma^2) \).
The error term \( \{u_j\} \) is now not independent, but follows a stationary linear process. To study the asymptotic properties of the LSEs of \( \phi_1 \) and \( \phi_2 \) in the present case, it is convenient to decompose \( \{u_j\} \) in the following way:

\[
(3.2) \quad u_j = \alpha(L)\varepsilon_j = [\alpha(e^{i\theta}) - (1 - e^{-i\theta}L)\tilde{\alpha}(\theta)(L)]\varepsilon_j = \alpha(e^{i\theta})\varepsilon_j + e^{-i\theta}\tilde{\varepsilon}_{j-1} - \tilde{\varepsilon}_{j},
\]

where

\[
\tilde{\varepsilon}_{j} = \tilde{\alpha}(\theta)(L)\varepsilon_j = \sum_{l=0}^{\infty} \tilde{\alpha}_l(\theta)\varepsilon_{j-l}, \quad \tilde{\alpha}_l(\theta) = \sum_{k=l+1}^{\infty} \alpha_k e^{i(k-l)\theta}.
\]

Note here that it follows from the last condition in (3.1) that

\[
(3.3) \quad \sum_{l=0}^{\infty} |\tilde{\alpha}_l(\theta)| = \sum_{l=0}^{\infty} |\sum_{k=l+1}^{\infty} \alpha_k e^{i(k-l)\theta}| \leq \sum_{l=0}^{\infty} \sum_{k=l+1}^{\infty} |\alpha_k| = \sum_{l=0}^{\infty} l|\alpha_l| < \infty,
\]

which implies that the process \( \{\tilde{\varepsilon}_{j}\} \) is a complex-valued stationary process.

The expression in (3.2) for \( \{u_j\} \) is based on an expansion of \( \alpha(L) \) around \( L = e^{i\theta} \). This expansion is called the complex B-N decomposition (Phillips and Solo (1992)). Usefulness of this decomposition can be seen when dealing with the present model (3.1) with complex unit roots. To see this, consider

\[
y_j = \frac{u_j}{(1 - e^{i\theta}L)(1 - e^{-i\theta}L)} = \frac{1}{\sin \theta} [\tilde{x}_j(\theta) \sin(j + 1)\theta - \tilde{y}_j(\theta) \cos(j + 1)\theta]
\]

\[
= \frac{1}{\sin \theta} (a_j^{(\theta)})^\top \hat{z}_j^{(\theta)},
\]

where

\[
\hat{z}_j^{(\theta)} = \begin{pmatrix} \tilde{x}_j^{(\theta)} \\ \tilde{y}_j^{(\theta)} \end{pmatrix} = \sum_{l=1}^{j} \begin{pmatrix} \cos(l\theta) \\ \sin(l\theta) \end{pmatrix} u_l, \quad a_j^{(\theta)} = \begin{pmatrix} \sin(j + 1)\theta \\ -\cos(j + 1)\theta \end{pmatrix}.
\]

Applying the complex B-N decomposition to \( u_l \) in the expression for \( \hat{z}_j^{(\theta)} \), we obtain

\[
(3.4) \quad \hat{z}_j^{(\theta)} = \begin{pmatrix} \tilde{x}_j^{(\theta)} \\ \tilde{y}_j^{(\theta)} \end{pmatrix} = \begin{pmatrix} a(\theta) & -b(\theta) \\ b(\theta) & a(\theta) \end{pmatrix} \begin{pmatrix} x_j^{(\theta)} \\ y_j^{(\theta)} \end{pmatrix} + w_j^{(\theta)} = K(\theta)z_j^{(\theta)} + w_j^{(\theta)},
\]

where \( a(\theta) = \text{Re}[\alpha(e^{i\theta})] \), \( b(\theta) = \text{Im}[\alpha(e^{i\theta})] \) and

\[
K(\theta) = \begin{pmatrix} a(\theta) & -b(\theta) \\ b(\theta) & a(\theta) \end{pmatrix}, \quad w_j^{(\theta)} = \begin{pmatrix} \text{Re}[\varepsilon_0^{(\theta)} - e^{ij\theta}\tilde{\varepsilon}_j^{(\theta)}] \\ \text{Im}[\varepsilon_0^{(\theta)} - e^{ij\theta}\tilde{\varepsilon}_j^{(\theta)}] \end{pmatrix}.
\]

It is noticed that the quantity \( \hat{z}_j^{(\theta)} \) is made up of two terms: one is \( K(\theta)z_j^{(\theta)} \) which is \( O_p(\sqrt{j}) \), and the other is \( w_j^{(\theta)} \) which is \( O_p(1) \). Thus \( \hat{z}_j^{(\theta)} \) which reflects the stationarity assumption on the error term is approximately equal to \( K(\theta)z_j^{(\theta)} \),
where \( z_j^{(\theta)} \) is the quantity based on the independence assumption on the error term.

For later purposes we also define the spectrum of \( \{u_j\} \) by \( f(\omega) \), which may be expressed as

\[
(3.5) \quad f(\omega) = \frac{\sigma^2}{2\pi} \left| \sum_{l=0}^{\infty} \alpha_l e^{i\omega l} \right|^2 = \frac{\sigma^2}{2\pi} (a^2(\omega) + b^2(\omega)) = \frac{\sigma^2}{2\pi} |K(\omega)|.
\]

The above discussion leads us to derive

\[
(3.6) \quad \frac{2}{T^2 \sigma^2} \sum_{j=1}^{T} \tilde{z}_j^{(\theta)} (\tilde{z}_j^{(\theta)})' = \frac{2}{T^2 \sigma^2} K(\theta) \sum_{j=1}^{T} z_j^{(\theta)} (z_j^{(\theta)})' + \alpha_p(1)
\]

\[
= \frac{1}{T} K(\theta) \sum_{j=1}^{T} Z^{(\theta)} \left( \frac{j}{T} \right) \left\{ Z^{(\theta)} \left( \frac{j}{T} \right) \right\}' K'(\theta) + \alpha_p(1)
\]

\[
\Rightarrow K(\theta) \int_0^1 W(t) W'(t) dt K'(\theta),
\]

\[
(3.7) \quad \frac{2}{T^2 \sigma^2} \sum_{j=1}^{T} (\tilde{z}_j^{(\theta)})' \tilde{z}_j^{(\theta)} = \frac{2}{T^2 \sigma^2} \sum_{j=1}^{T} \left[ (\tilde{x}_j^{(\theta)})^2 + (\tilde{y}_j^{(\theta)})^2 \right]
\]

\[
\Rightarrow \text{tr} \left( K(\theta) \int_0^1 W(t) W'(t) dt K'(\theta) \right)
\]

\[
= \frac{2\pi f(\theta)}{\sigma^2} \int_0^1 W'(t) W(t) dt.
\]

Moreover let us consider \( R_T(h) = \sum_{j=h}^{T} y_j - h y_j / T^2 \) for a nonnegative integer \( h \). Then we have

\[
R_T(h) = \frac{1}{T^2 \sin^2 \theta} \sum_{j=1}^{T} \left[ \tilde{x}_j^{(\theta)} \sin(j - h + 1) \theta - \tilde{y}_j^{(\theta)} \cos(j - h + 1) \theta \right]
\]

\[
\times \left[ \tilde{x}_j^{(\theta)} \sin(j + 1) \theta - \tilde{y}_j^{(\theta)} \cos(j + 1) \theta \right]
\]

\[
= \frac{1}{2T^2 \sin^2 \theta} \sum_{j=h+1}^{T} \left[ (\tilde{x}_j^{(\theta)} + y_j^{(\theta)} \tilde{y}_j^{(\theta)}) \cos h \theta + (\tilde{x}_j^{(\theta)} - \tilde{y}_j^{(\theta)} \tilde{y}_j^{(\theta)}) \sin h \theta \right.
\]

\[
+ (\tilde{x}_j^{(\theta)} - \tilde{y}_j^{(\theta)} \tilde{y}_j^{(\theta)}) \sin h \theta + (\tilde{x}_j^{(\theta)} + \tilde{y}_j^{(\theta)} \tilde{y}_j^{(\theta)}) \cos(2j - h + 2) \theta 
\]

\[
= \frac{\cos h \theta}{2T^2 \sin^2 \theta} \sum_{j=h+1}^{T} \left[ (\tilde{x}_j^{(\theta)})^2 + (\tilde{y}_j^{(\theta)})^2 \right] + \alpha_p(1).
\]

Thus it follows from (3.7) that

\[
R_T(h) \Rightarrow R(h) = \frac{\pi f(\theta) \cos h \theta}{2 \sin^2 \theta} \int_0^1 W'(t) W(t) dt.
\]
As for a quantity related to the Ito integral, we consider

\[ S_T = \frac{1}{T} \sum_{j=2}^{T} \dot{z}_{j-1}(\Delta \hat{z}_j^{(\theta)})' = \frac{1}{T} \sum_{j=2}^{T} (K(\theta)z_j^{(\theta)} + w_j^{(\theta)})(K(\theta)\Delta z_j^{(\theta)} + \Delta w_j^{(\theta)})'. \]

After some algebra we arrive at

\[ S_T = \frac{1}{T} \left[ K(\theta) \sum_{j=2}^{T} z_j^{(\theta)}(\Delta z_j^{(\theta)})'K'(\theta) - K(\theta) \sum_{j=2}^{T} \Delta z_j^{(\theta)}(w_j^{(\theta)})' \right. \]

\[ + \sum_{j=2}^{T} w_j^{(\theta)}(\Delta z_j^{(\theta)})'K'(\theta) + \sum_{j=2}^{T} w_j^{(\theta)}(\Delta w_j^{(\theta)})' \right] + o_p(1), \]

which yields

\[ S_T \Rightarrow S = \frac{1}{2} \left[ \sigma^2 K(\theta) \int_0^1 W(t) dW'(t)K'(\theta) + \sum_{j=1}^{\infty} \gamma(j) P_j(\theta) \right], \]

where \( \gamma(j) = E(u_0 u_j) \), and \( P_j(\theta) \) is the orthogonal matrix defined by

\[ P_j(\theta) = \begin{pmatrix} \cos j\theta & \sin j\theta \\ -\sin j\theta & \cos j\theta \end{pmatrix}. \]

Using the above result we can discuss the weak convergence of the following quantity for \( h > 0 \):

\[ S_{hT} = \frac{1}{T} \sum_{j=h+1}^{T} y_{j-h} u_j = \frac{1}{T \sin \theta} \sum_{j=h+1}^{T} \left( \tilde{x}_{j-h}^{(\theta)} \tilde{y}_{j-h}^{(\theta)} \right) \begin{pmatrix} \sin(j - h + 1)\theta \\ -\cos(j - h + 1)\theta \end{pmatrix} u_j. \]

After some algebra we obtain

\[ S_{hT} = \frac{1}{T \sin \theta} \sum_{j=h+1}^{T} \left( \tilde{x}_{j-h}^{(\theta)} \tilde{y}_{j-h}^{(\theta)} \left( \begin{array}{c} \Delta \tilde{y}_j^{(\theta)} \cos(h-1)\theta - \Delta \tilde{x}_j^{(\theta)} \sin(h-1)\theta \\ -\Delta \tilde{x}_j^{(\theta)} \cos(h-1)\theta - \Delta \tilde{y}_j^{(\theta)} \sin(h-1)\theta \end{array} \right) \right. \]

\[ + \sum_{j=h+1}^{T} \left( \tilde{x}_{j-h}^{(\theta)} \tilde{y}_{j-h}^{(\theta)} \Delta \tilde{x}_j^{(\theta)} \cos(h-1)\theta \right. \]

\[ \left. - \tilde{x}_{j-h}^{(\theta)} \tilde{y}_{j-h}^{(\theta)} \Delta \tilde{y}_j^{(\theta)} \sin(h-1)\theta \right) 
\]

\[ + u_j \sum_{l=1}^{h-2} u_{j+l+1-h} \sin l \theta \]

\[ \Rightarrow \frac{1}{\sin \theta} \left[ \{S(1, 2) - S(2, 1)\} \cos(h-1)\theta - \{S(1, 1) + S(2, 2)\} \sin(h-1)\theta \right. \]

\[ + \sum_{l=1}^{h-2} \gamma(h-l-1) \sin l \theta \]

\[ = S_h = \frac{1}{\sin \theta} \left[ \pi f(\theta) \int_0^1 W'(t)J_h(\theta) dW(t) + \sum_{j=h}^{\infty} \gamma(j) \sin(j - h + 1)\theta \right], \]
where $S(i, j)$ is the $(i, j)$-th element of $S$ defined in (3.8), whereas $J_h(\theta)$ is defined in (2.8).

Denoting by $\tilde{\phi}$ the LSE of $\phi$ in the present model, we can now establish that

$$T(\tilde{\phi} - \phi) = \left[ \frac{1}{T^2 \sigma^2} \sum_{j=3}^{T} \left( \begin{array}{cc} y_{j-1}^2 & y_{j-1}y_{j-2} \\ y_{j-1}y_{j-2} & y_{j-2}^2 \end{array} \right) \right]^{-1} \left[ \frac{1}{T^2 \sigma^2} \sum_{j=3}^{T} \left( y_{j-1}u_j - y_{j-2}u_j \right) \right]$$

$$\Rightarrow \left( \begin{array}{cc} R(0) & R(1) \\ R(1) & R(0) \end{array} \right)^{-1} \left( \begin{array}{c} S_1 \\ S_2 \end{array} \right) = \left( \begin{array}{c} \tilde{Z}_1 \\ \tilde{Z}_2 \end{array} \right),$$

where

$$\tilde{Z}_1 = \frac{2[\pi f(\theta) \int_0^1 W'(t) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) dW(t) + \sin \theta \sum_{j=1}^{\infty} \gamma(j) \cos(j-1)\theta]}{\pi f(\theta) \int_0^1 W'(t) W(t) dt},$$

$$\tilde{Z}_2 = \frac{-2[\int_0^1 W'(t) dW(t) + 1 - \gamma(0)/(2\pi f(\theta))] \int_0^1 W'(t) W(t) dt}{\int_0^1 W'(t) W(t) dt}.$$

It is seen that the limiting distribution of $T(\tilde{\phi}_1 - \phi_1)$ depends on the frequency $\theta$, as in the independent error case. Moreover, that of $T(\tilde{\phi}_2 - \phi_2)$ now depends on $\theta$. This is quite comparable to the situation where we deal with the LSE in the following seasonal model with period $m$:

$$y_j = \rho y_{j-m} + u_j, \quad u_j = \sum_{l=0}^{\infty} \alpha_{lm} \varepsilon_{j-lm}, \quad \sum_{l=1}^{\infty} |l| \alpha_{lm} < \infty,$$

where the true value of $\rho$ is unity. Let $\tilde{\rho}$ be the LSE of $\rho$ based on a sample of size $T$. Then it is shown (Tanaka (1996)) that

$$N(\tilde{\rho} - 1) \Rightarrow \tilde{Z} = \frac{\int_0^1 B'(t)dB(t) + m/2 \{1 - \gamma(0)/(2\pi f(0))\}}{\int_0^1 B'(t)B(t) dt},$$

where $N = [T/m]$ and $\{B(t)\}$ is the $m$-dimensional standard Brownian motion. It is seen that the distribution of $-2\tilde{Z}$ reduces to that of $\tilde{Z}_2$ when $m = 2$ and $\theta = 0$.

4. Percent points and moments

The limiting distribution can be computed by numerical integration in the following way. Let us consider $\tilde{Z}$ in (3.10), which is of the form

$$X(a, m) = \frac{\int_0^1 B'(t)dB(t) + am/2}{\int_0^1 B'(t)B(t) dt} = \frac{U}{V}.$$

Then we compute

$$P(X(a, m) \leq x) = P(xV - U \geq 0) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{1}{u} \text{Im}[\phi(u; x)] du,$$
where $\phi(u; x)$ is the characteristic function of $xV - U$ given by

$$
\phi(u; x) = \left[ e^{-iu(1-a)} \left\{ \cos \sqrt{2iu}x + iu \frac{\sin \sqrt{2iu}x}{\sqrt{2iu}} \right\} \right]^{-m/2}.
$$

Care needs to be taken when computing square roots of complex-valued quantities, as is described in Tanaka (1996).

Moments of $X(a, m)$ can also be computed. We have

$$
E(X^j(a, m)) = \frac{1}{\Gamma(j)} \int_0^\infty u_2^{j-1} \left. \frac{\partial^j \psi(u_1, -u_2)}{\partial u_1^j} \right|_{u_1=0} du_2,
$$

where $\psi(u_1, u_2)$ is the joint moment generating function of $U$ and $V$ with

$$
\psi(u_1, u_2) = \left[ e^{u_1(1-a)} \left\{ \cosh \sqrt{2u_2} - u_1 \frac{\sinh \sqrt{2u_2}}{\sqrt{2u_2}} \right\} \right]^{-m/2}.
$$

On the basis of these results, we can compute percent points and moments of the limiting distributions of the LSEs discussed in this paper. Table 1 gives those of $Z_2 = -2X(0, 2)$ defined in (2.12).

<table>
<thead>
<tr>
<th></th>
<th>0.01</th>
<th>0.05</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
<th>0.95</th>
<th>0.99</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_2$</td>
<td>-2.956</td>
<td>-2.028</td>
<td>-1.539</td>
<td>0.775</td>
<td>6.010</td>
<td>8.389</td>
<td>14.115</td>
<td>1.664</td>
<td>3.460</td>
</tr>
</tbody>
</table>

5. Concluding remarks

We have discussed the asymptotic properties of the LSEs in extended AR(2) models with two complex conjugate unit roots $e^{\pm i\theta}$. It was shown that the limiting distribution associated with the complex unit roots is closely related to that of the LSE in standard nonstationary seasonal models with period two. The analysis may also be extended to cases of multiple complex unit roots $\exp\{\pm i\theta_j\}$ ($j = 1, \ldots, k$), and to cases where the error term follows a stationary long-memory process.

Testing for the existence of complex unit roots has been discussed in Hylleberg et al. (1990) for standard nonstationary seasonal models. Their approach rewrites the AR model representation in a convenient way, assuming the seasonal unit roots are known. In the present case, however, the frequency $\theta$ is supposed to be unknown. Thus it must be estimated before testing, but their idea may be useful.

References


