CHARACTERIZATION OF PRIORS IN 
THE STEIN PROBLEM

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The so-called Stein problem is addressed in the estimation of a mean vector of a multivariate normal distribution with a known covariance matrix. For general prior distributions with sphericity, the paper derives conditions on priors under which the resulting generalized Bayes estimators are minimax relative to the usual quadratic loss. It is also shown that the conditions can be expressed based on the inverse Laplace transform of the general prior. Stein’s super-harmonic condition is derived from the general conditions. Finally, the priors are characterized for the admissibility.

Key words and phrases: Admissibility, decision theory, estimation, generalized Bayes estimator, inverse Laplace transform, James-Stein estimator, minimaxity, quadratic loss, risk function, shrinkage estimation, Stein problem, uniform domination.

1. Introduction

The Stein problem is one of the most attractive topics in theoretical statistics. In the estimation of a mean vector of a multivariate normal distribution, Stein (1956) and James and Stein (1961) discovered the inadmissibility of the maximum likelihood estimator (MLE) when the dimension of the mean vector is larger than or equal to three. A considerable amount of studies have been devoted to this topic for half a century. Of these, Baranchik (1970), Brown (1971), Strawderman (1971), Alam (1973) and Berger (1976) developed classes of generalized Bayes estimators with minimaxity and/or admissibility. The classes of generalized Bayes minimax and/or admissible estimators have been extended by Faith (1978), Stein (1981), Fourdrinier et al. (1998) and Maruyama (1998). These results imply a characterization of prior distributions such that the resulting generalized Bayes estimators are minimax and/or admissible. Such a characterization of prior distributions in hierarchical Bayes models has been studied by Berger and Robert (1990) and Kubokawa and Strawderman (2007) for minimaxity and by Berger and Strawderman (1996) for admissibility. Most of these studies treated the scale-mixture of normal distributions as prior distributions except for Stein (1981) who derived the super-harmonic condition of the general prior distributions to satisfy the minimaxity of the Bayes estimators. In this paper, we obtain a class of general prior distributions with sphericity which results in the generalized Bayes estimators with minimaxity and/or admissibility.

To explain the outlines of the paper, we describe the model and the estimation problem. Let \( \mathbf{X} = (X_1, \ldots, X_p)^t \) be a random vector distributed as
\( \mathbf{X} \sim \mathcal{N}_p(\mathbf{\theta}, \mathbf{I}_p) \) for \( \mathbf{\theta} = (\theta_1, \ldots, \theta_p)^t \) and the \( p \times p \) identity matrix \( \mathbf{I}_p \). The problem of estimating the mean vector \( \mathbf{\theta} \) by \( \hat{\mathbf{\theta}} \) is considered relative to the quadratic loss \( L(\mathbf{\theta}, \hat{\mathbf{\theta}}) = \| \mathbf{\theta} - \hat{\mathbf{\theta}} \|^2 \). An estimator \( \hat{\mathbf{\theta}} \) is evaluated in terms of the risk function \( R(\mathbf{\theta}, \hat{\mathbf{\theta}}) = E_{\mathbf{\theta}}[L(\mathbf{\theta}, \hat{\mathbf{\theta}})] \). The maximum likelihood estimator of \( \mathbf{\theta} \) is \( \hat{\mathbf{\theta}}_0 = \mathbf{X} \). Since it is minimax with a constant risk \( R(\omega, \hat{\mathbf{\theta}}_0) = p \), the improvement on \( \hat{\mathbf{\theta}}_0 \) is equivalent to deriving minimax estimators other than \( \hat{\mathbf{\theta}}_0 \). To find a minimax estimator, Stein (1956) considered a class of estimators

\[
(1.1) \quad \hat{\mathbf{\theta}}_\psi = (1 - \psi(W)/W) \mathbf{X} \quad \text{for} \quad W = \| \mathbf{X} \|^2,
\]

where \( \psi(w) \) is a function of \( w \). As stated in Stein (1956), this is a class of estimators equivariant under the transformation \( \mathbf{X} \rightarrow \mathbf{\Gamma} \mathbf{X} \) and \( \mathbf{\theta} \rightarrow \mathbf{\Gamma} \mathbf{\theta} \) for any \( p \times p \) orthogonal matrix \( \mathbf{\Gamma} \), namely, \( \hat{\mathbf{\theta}}(\mathbf{\Gamma} \mathbf{X}) = \mathbf{\Gamma} \hat{\mathbf{\theta}}(\mathbf{X}) \). Out of the class, James and Stein (1961) found the estimator \( \hat{\mathbf{\theta}}^{JS} = (1 - (p - 2)/W) \mathbf{X} \), and established that if \( p \geq 3 \), then the shrinkage estimator \( \hat{\mathbf{\theta}}^{JS} \) dominates \( \hat{\mathbf{\theta}}_0 \), namely, \( \hat{\mathbf{\theta}}^{JS} \) is minimax. The James-Stein estimator can be further dominated by the positive-part Stein estimator, which is still inadmissible. This fact is the primary motivation to derive generalized Bayes and minimax estimators, some of which may be admissible and minimax.

In this paper, we handle the general form of prior distributions with sphericity, given by \( h(\| \mathbf{\theta} \|^2) d\mathbf{\theta} \). As noted in Section 2, the generalized Bayes estimator against the prior \( h(\| \mathbf{\theta} \|^2) d\mathbf{\theta} \) belongs to the class (1.1). In a precise sense, the generalized Bayes estimator for \( h(\| \mathbf{\theta} \|^2) d\mathbf{\theta} \) is identical to the generalized Bayes estimator against the prior \( \pi(\lambda) d\lambda = \lambda^{p/2-1} h(\lambda) d\lambda \) within the equivariant class (1.1) for \( \lambda = \| \mathbf{\theta} \|^2 \). This is called the Bayes equivariant estimator in this paper and denoted by \( \hat{\mathbf{\theta}}^\pi \).

In Section 3, we obtain general conditions on \( h(\lambda) \) under which \( \hat{\mathbf{\theta}}^\pi \) is minimax. For the first derivative \( h'(\lambda) \), the function \( k(\lambda) \equiv -\{2h'(\lambda) + (p - 2)h(\lambda)\}/\{\int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}\} \) is assumed to be decomposed as \( k(\lambda) = k_1(\lambda) + k_2(\lambda) \), where \( k_1(\lambda) \) is a nondecreasing function of \( \lambda \) and \( k_2(\lambda) \) is a function. Then, the general conditions are described as

1. the first derivative of \( h(\lambda) \) is not positive,
2. \( h(\lambda) \) satisfies the inequality

\[
k_0 - 2 \inf_{\lambda>0} \left\{ \frac{2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda) \int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}]'}{\int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}} \right\} - \inf_{\lambda>0} k_2(\lambda) \leq \frac{p - 2}{2},
\]

for a constant \( k_0 \) defined by (3.29), where \( h''(\lambda) \) is the second derivative. Under these conditions, the Bayes equivariant estimator is minimax. Especially, in the case that \( k(\lambda) \) is nondecreasing, the condition (2) can be simplified as \( k_0 \leq p - 2 \). Examples using the general conditions are given in Section 3.

In Section 4, the general conditions can be expressed based on the inverse Laplace transform of \( h(\lambda) \). When \( h(\lambda) \) is written as \( h(\lambda) = \int_0^\infty H(t)e^{-\lambda t}dt \), the function \( H(t) \) is called the inverse Laplace transform of \( h(\lambda) \). The general conditions on \( h(\lambda) \) derived in Section 3 can be rewritten by simple conditions
based on the inverse Laplace transform $H(t)$. Especially, in the case that $h(\lambda)$ is completely monotone, it is known that $H(t)$ is a nonnegative function. For the detailed account of completely monotone functions, see Feller (1971). Then, the Bayes equivariant estimator is minimax if $H(t)$ satisfies the inequality

$$K_0 + 2 \sup_t K_2(t) - \inf_t K_2(t) \leq p - 3,$$

where $K(t) \equiv -(p-4)t + t(1 + 2t)H'(t)/H(t)$ is decomposed as $K(t) = K_1(t) + K_2(t)$ for a nonincreasing function $K_1(t)$ and a function $K_2(t)$, and $K_0$ is a constant defined by (4.5). This condition is similar to that of Fourdrinier et al. (1998), though more general conditions are provided in Section 4. When we check the conditions for the minimaxity for a given function $h(\lambda)$, the conditions given in Section 4 are not very useful, because, in general, it is hard to derive the inverse Laplace transform $H(t)$ of $h(\lambda)$. However, the conditions in Section 4 are useful for constructing prior distributions of the form $h(\lambda) = \int H(t)e^{-\lambda t}dt$ such that the resulting Bayes equivariant estimators are minimax.

Section 5 explains how Stein’s super-harmonic condition can be derived from the general conditions in Section 3. Examples are given where the conditions in Section 4 do not work, but Stein’s super-harmonic condition works well.

The admissibility of the Bayes equivariant estimators is studied in Section 6 based on Brown’s admissibility condition. The prior distributions for the admissibility are characterized, and some examples of admissible and minimax estimators are provided.

Finally, it is remarked that the idea of using the inverse Laplace transform appeared in Kubokawa (2006) who dealt with a linear regression model with an error term having a normal distribution with unknown variance. Since the generalized Bayes estimators are complicated in the case of unknown variance, the estimators treated in Kubokawa (2006) were focused on a class of estimators (1.1) with monotone nondecreasing functions $\psi(\cdot)$. This paper, however, handles more general classes without assuming the monotonicity of $\psi(\cdot)$.

### 2. Bayes equivariant estimators

In this section, we derive the Bayes equivariant estimator, namely, the Bayes estimator within the class of equivariant estimators.

We begin with providing the risk function of the equivariant estimator $\hat{\theta}_\psi$ given by (1.1). It is assumed that the function $\psi(w)$ is absolutely continuous with respect to the Lebesgue measure and satisfies that $E[\psi(W)^2/W] < \infty$. Using integration by parts called the Stein identity, Stein (1973, 1981) showed that the risk function of $\hat{\theta}_\psi$ is given by

$$R(\lambda, \hat{\theta}_\psi) = p + E[\psi^2(W) - 2(p - 2)\psi(W)]/W - 4\psi'(W),$$

which can be expressed as

$$R(\lambda, \hat{\theta}_\psi) = p + \int_0^\infty \{[\psi^2(w) - 2(p - 2)\psi(w)]/w - 4\psi'(w)\}g(w; \lambda)dw,$$
where \( g(w; \lambda) \) is a density of a noncentral chi-square distribution \( \chi_p^2(\lambda) \) with \( p \) degrees of freedom and the noncentrality \( \lambda = \|\theta\|^2 \), given by

\[
g(w; \lambda) = 2^{-p/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j! \Gamma(p/2 + j) 2^j} e^{-\lambda/2} w^{p/2+j-1} e^{-w/2}.
\]

Let \( \pi(\lambda) \) be a prior distribution of \( \lambda \), and the marginal density of \( W \) be given by \( g_\pi(w) = \int_0^\infty g(w; \lambda) \pi(\lambda) d\lambda \). Then the difference of the Bayes risks of the estimators \( \hat{\theta}_\psi \) and \( \hat{\theta}_0 \) is written as

\[
\Delta(\pi, \hat{\theta}_\psi) = \int_0^\infty \{ R(\lambda, \hat{\theta}_\psi) - R(\lambda, \hat{\theta}_0) \} \pi(\lambda) d\lambda = \int_0^\infty \{ \psi^2(w) - 2(p - 2)\psi(w) \} g_\pi(w) dw
\]

where this integral is assumed to be finite. Using integration by parts, it is noted that

\[
\int_0^\infty \psi'(w) g_\pi(w) dw = [\psi(w) g_\pi(w)]_w^\infty - \int_0^\infty \psi(w) g'_\pi(w) dw,
\]

where \( \lim_{w \to -\infty} \psi(w) g_\pi(w) = \lim_{w \to 0} \psi(w) g_\pi(w) = 0 \), because \( \int \{ \psi^2(w)/w \} g_\pi(w) dw \) is finite. Then,

\[
\Delta(\pi, \hat{\theta}_\psi) = \int_0^\infty \{ \psi^2(w) - 2(p - 2)\psi(w) \} g_\pi(w)/w + 4\psi(w) g'_\pi(w) dw,
\]

which is minimized at \( \psi_\pi(w) = p - 2 - 2wg'_\pi(w)/g_\pi(w) \), and we get the Bayes estimator

\[
(2.2) \quad \hat{\theta}_\pi = (1 - \psi_\pi(W)/W) X.
\]

This is called the Bayes equivariant estimator, for it minimizes the Bayes risk within the class of equivariant estimators. The above expression of \( \psi_\pi(w) \) was derived by Haff (1991) through the variational method. When the prior \( \pi(\lambda) \) is improper, we can handle estimator (2.2) as a generalized Bayes estimator if \( \psi_\pi(w) \) is finite. In this paper, we thus treat the Bayes equivariant estimator \( \hat{\theta}_\pi \) regardless of the finiteness of the Bayes risk and \( \int \pi(\lambda) d\lambda \). Another expression of \( \psi_\pi(w) \) given below will be useful for deriving conditions for the minimaxity of \( \hat{\theta}_\pi \). Carrying out the differentiation \( g'_\pi(w) \), we may write \( \psi_\pi(w) \) as

\[
(2.3) \quad \psi_\pi(w) = w - 2w \sum_{j=1}^{\infty} d_j j w^{j-1}/\sum_{j=0}^{\infty} d_j w^j
\]

where \( d_j = \int_0^\infty [j! \Gamma(p/2 + j) 2^{2j} - 1] \exp\{-\lambda/2\} \lambda^j \pi(\lambda) d\lambda \). We thus get the form

\[
(2.4) \quad \psi_\pi(w) = w - 2w \sum_{j=0}^{\infty} \left\{ j! \Gamma(p/2 + j + 1) 2^{2j+2} - 1 \right\} w^j f_0^\infty \lambda^j \exp\{-\lambda/2\} \pi(\lambda) d\lambda
\]

\[
\sum_{j=0}^{\infty} \left\{ j! \Gamma(p/2 + j) 2^{2j} - 1 \right\} w^j f_0^\infty \lambda^j \exp\{-\lambda/2\} \pi(\lambda) d\lambda.
\]
It may be interesting to note that the Bayes equivariant estimator \( \hat{\theta}^\pi \) can be derived as the generalized Bayes estimator against a spherically symmetric prior distribution of \( \theta \), given by
\[
\theta \sim h(\|\theta\|^2) d\theta.
\]
In fact, the generalized Bayes estimator against prior (2.5) is given by
\[
\hat{\theta}_{GB} = \frac{\int \int \theta \exp\{-\|X - \theta\|^2/2\} h(\|\theta\|^2) d\theta}{\int \int \exp\{-\|X - \theta\|^2/2\} h(\|\theta\|^2) d\theta}.
\]
Using the same arguments as in Kubokawa (2006), we can show that \( \hat{\theta}_{GB} \) is identical to the Bayes equivariant estimator \( \hat{\theta}^\pi \), given by (2.2), against the prior \( \pi(\lambda) = \lambda^{p/2-1} h(\lambda) \). Hereafter, the prior distribution of \( \lambda \) is supposed to be of the form
\[
\pi(\lambda) = \lambda^{p/2-1} h(\lambda).
\]
Therefore, \( d_j \) given above (2.4) is written as
\[
d_j = \int_0^\infty [j!\Gamma(\frac{p}{2}+j)2^{2j}]^{-1} \exp\{-\lambda/2\} \lambda^{p/2+j-1} h(\lambda) d\lambda,
\]
and investigate the minimaxity of the Bayes equivariant estimator \( \hat{\theta}^\pi \).

3. General characterization of priors for minimaxity

We now address the problem of showing the minimaxity of the Bayes equivariant estimator \( \hat{\theta}^\pi \) against the prior \( \pi(\lambda) = \lambda^{p/2-1} h(\lambda) \). In this section, we derive general sufficient conditions on \( h(\lambda) \) for the minimaxity. To this end, assume the following condition as well as \( p \geq 3 \).

(A.1) The function \( h(\lambda) \) is twice differentiable, and the first and second derivatives of \( h(\lambda) \) are denoted by \( h'(\lambda) \) and \( h''(\lambda) \). The functions \( h(\lambda) \) and \( h'(\lambda) \) are absolutely continuous and satisfy that
\[
\sum_{j=0}^{\infty} c_j^{(p)} \int \lambda^{p/2+j-1} \{h(\lambda) + j|h'(\lambda)|\} e^{-\lambda/2} d\lambda < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} \{|h'(\lambda)| + j|h''(\lambda)|\} e^{-\lambda/2} d\lambda < \infty
\]
for
\[
c_j^{(a)} = w_j /[j!\Gamma(a/2+j)2^{2j}].
\]

**Theorem 3.1.** Assume that \( h(\lambda) \) satisfies (A.1). Then the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax if the function \( h(\lambda) \) satisfies the inequality
\[
\frac{\sum_{j=0}^{\infty} 4j c_j^{(p)} \int \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} d\lambda}{\sum_{j=0}^{\infty} c_j^{(p)} \int \lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda} - 2 \frac{\sum_{j=0}^{\infty} 4j c_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda}{\sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} h'(\lambda) e^{-\lambda/2} d\lambda} \leq 2p.
\]
Proof. From expression (2.3), the function \( \psi_\pi(w) \) is written as \( \psi_\pi(w) = \sum_{j=1}^{\infty} D_j w^j / \sum_{j=0}^{\infty} d_j w^j \), where \( D_j = d_{j-1} - 2jd_j \). Since

\[
\psi'_{\pi}(w) = \frac{\sum_{j=1}^{\infty} jD_j w^{j-1}}{\sum_{j=0}^{\infty} d_j w^j} - \frac{\sum_{j=1}^{\infty} D_j w^j \sum_{j=1}^{\infty} jD_j w^{j-1}}{(\sum_{j=0}^{\infty} d_j w^j)^2},
\]

it is observed that

\[
\tilde{\Delta} \equiv (\psi_{\pi}(w)^2 - 2(p - 2)\psi_{\pi}(w)) / w - 4\psi'_{\pi}(w) = w \left( \frac{\sum_{j=1}^{\infty} D_j w^{j-1}}{\sum_{j=0}^{\infty} d_j w^j} \right)^2 - 2(p - 2) \frac{\sum_{j=1}^{\infty} D_j w^{j-1}}{\sum_{j=0}^{\infty} d_j w^j} - 4 \frac{\sum_{j=1}^{\infty} jD_j w^{j-1}}{\sum_{j=0}^{\infty} d_j w^j} + 4 \frac{\sum_{j=1}^{\infty} jD_j w^j}{(\sum_{j=0}^{\infty} d_j w^j)^2}.
\]

From (2.1), it follows that the Bayes equivariant estimator \( \tilde{\theta}^\pi \) is minimax if \( \tilde{\Delta}^* \) is not positive for any \( w \), where \( \tilde{\Delta}^* \) is defined by

\[
\tilde{\Delta}^* \equiv \tilde{\Delta} \sum_{j=0}^{\infty} d_j w^j / \sum_{j=1}^{\infty} D_j w^j - 4 \frac{\sum_{j=1}^{\infty} jD_j w^{j-1}}{\sum_{j=0}^{\infty} d_j w^j} + 4 \frac{\sum_{j=1}^{\infty} jD_j w^j}{(\sum_{j=0}^{\infty} d_j w^j)^2} - 2(p - 2).
\]

Since \( 2jd_j = d_{j-1} - D_j \), it is seen that

\[
2 \frac{\sum_{j=1}^{\infty} jD_j w^j}{\sum_{j=0}^{\infty} d_j w^j} = \sum_{j=1}^{\infty} D_j w^j / \sum_{j=0}^{\infty} d_j w^j = w - \sum_{j=1}^{\infty} D_j w^j / \sum_{j=0}^{\infty} d_j w^j,
\]

so that \( \tilde{\Delta}^* \) may be rewritten as

\[
(3.2) \quad \tilde{\Delta}^* = -\frac{\sum_{j=1}^{\infty} D_j w^j}{\sum_{j=0}^{\infty} d_j w^j} + 2 \frac{\sum_{j=1}^{\infty} (D_j - 2jd_j) w^j}{\sum_{j=1}^{\infty} D_j w^j} - 2(p - 2),
\]

where \( D_0 = 0 \).

To evaluate the term \( D_j \) for \( j \geq 1 \), from definition (2.7) of \( d_j \) it is noted that \( D_j / d_j = d_{j-1} / d_j - 2j \) is expressed as

\[
D_j / d_j = 4j(p/2 + j - 1) \int_0^\infty \frac{\lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda}{\lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda} - 2j.
\]

By integration by parts under assumption (A.1), it is noted that

\[
(3.3) \quad (p/2 + j - 1) \int_0^\infty \lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda
= [\lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2}]_{\lambda=0}^\infty + \int_0^\infty \lambda^{p/2+j-1} \{ h(\lambda) / 2 - h'(\lambda) \} e^{-\lambda/2} d\lambda
= \frac{1}{2} \int_0^\infty \lambda^{p/2+j-1} \{ h(\lambda) - 2h'(\lambda) \} e^{-\lambda/2} d\lambda,
\]
which yields that

\[(3.4) \quad D_j/d_j = -4j \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda / \int \lambda^{p/2+j-1} \lambda e^{-\lambda/2} d\lambda.\]

Hence, \(D_j\) is expressed as

\[(3.5) \quad D_j = \frac{4j}{j! \Gamma(p/2 + j/2^j) 2^j} \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda.\]

To evaluate the term \(D_{j-1} - 2jD_j\) for \(j \geq 1\), from (3.4) and (2.7), it is noted that the term may be written as

\[(3.6) \quad D_{j-1} - 2jD_j\]

\[= - \left( \frac{D_{j-1}}{D_j} - 2j \right) 4j \int \lambda^{p/2+j-2} \lambda' e^{-\lambda/2} d\lambda.\]

Using (3.4) and (2.7) again, we observe that

\[
\frac{D_{j-1}}{D_j} = \frac{(j-1) \int \lambda^{p/2+j-2} \lambda' e^{-\lambda/2} d\lambda / \int \lambda^{p/2+j-2} \lambda e^{-\lambda/2} d\lambda d_{j-1}}{j \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda / \int \lambda^{p/2+j-1} \lambda e^{-\lambda/2} d\lambda d_j}
\]

\[= 4(p/2 + j - 1)(j-1) \int \lambda^{p/2+j-2} \lambda' e^{-\lambda/2} d\lambda / \int \lambda^{p/2+j-1} \lambda e^{-\lambda/2} d\lambda.\]

since \(d_{j-1}/d_j = 4j(p/2 + j - 1) \int \lambda^{p/2+j-2} \lambda e^{-\lambda/2} d\lambda / \int \lambda^{p/2+j-1} \lambda e^{-\lambda/2} d\lambda.

Then,

\[(3.7) \quad - \left( \frac{D_{j-1}}{D_j} - 2j \right) \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda \]

\[= -4(j-1)(p/2 + j - 1) \int \lambda^{p/2+j-2} \lambda' e^{-\lambda/2} d\lambda \]

\[+ 2j \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda.\]

Similarly to (3.3), for \(j \geq 2\), we can get the equality

\[(p/2 + j - 1) \int \lambda^{p/2+j-2} \lambda' e^{-\lambda/2} d\lambda = \frac{1}{2} \int \lambda^{p/2+j-1} \{h'(\lambda) - 2h''(\lambda)\} e^{-\lambda/2} d\lambda,\]

under assumption (A.1). Then, the r.h.s. of equality (3.7) is rewritten as

\[(3.8) \quad - \left( \frac{D_{j-1}}{D_j} - 2j \right) \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda \]

\[= 4(j-1) \int \lambda^{p/2+j-1} \lambda'' e^{-\lambda/2} d\lambda + 2 \int \lambda^{p/2+j-1} \lambda' e^{-\lambda/2} d\lambda,\]
for $j \geq 1$. Hence from (3.6) and (3.8),

\begin{equation}
D_{j-1} - 2jD_j = \frac{4j}{j!\Gamma(p/2 + j)2^{2j}} \left\{ 4(j - 1) \int \lambda^{p/2+j-1}h''(\lambda)e^{-\lambda/2}d\lambda + 2 \int \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}d\lambda \right\}.
\end{equation}

We now rewrite $\tilde{\Delta}^*$ given by (3.2) by using expressions (3.5) and (3.9). From (3.5),

\[ \frac{\sum_{j=1}^{\infty} D_j w^j}{\sum_{j=0}^{\infty} d_j w^j} = -\frac{\sum_{j=1}^{\infty} \lambda c_j^{(p)} \int \lambda^{p/2+j-1}h''(\lambda)e^{-\lambda/2}d\lambda}{\sum_{j=0}^{\infty} c_j^{(p)} \int \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}d\lambda}. \]

Also from (3.5) and (3.9), it can be observed that

\[ \frac{\sum_{j=1}^{\infty} (D_{j-1} - 2jD_j) w^j}{\sum_{j=1}^{\infty} D_j w^j} = -\frac{\sum_{j=1}^{\infty} 4^2 j(j - 1) c_j^{(p)} \int \lambda^{p/2+j-1}h''(\lambda)e^{-\lambda/2}d\lambda}{\sum_{j=1}^{\infty} 4j c_j^{(p)} \int \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}d\lambda} - 2. \]

Since $4j c_j^{(p)} = 4j c_j^{(p+2)} = 0$ for $j = 0$, $\tilde{\Delta}^*$ can be rewritten as

\[ \frac{\sum_{j=0}^{\infty} 4j c_j^{(p)} \int \lambda^{p/2+j-1}h''(\lambda)e^{-\lambda/2}d\lambda}{\sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j}h'\lambda)e^{-\lambda/2}d\lambda} - 2p, \]

which leads to condition (3.1). \(\square\)

Various conditions for minimaxity given in this paper can be obtained from the general condition (3.1) of Theorem 3.1. The following lemma is heavily utilized for deriving minimaxity conditions. For the proof of Lemma 3.1, see Theorem 2 in Wijsman (1985).

**Lemma 3.1.** Let $X$ be a random variable, and let $f(x)$, $g(x)$ and $u(x)$ be functions. If both $g(x)/f(x)$ and $u(x)$ are monotone in the same direction, then the inequality $E[g(X)u(X)] \cdot E[f(X)] \geq E[g(X)] \cdot E[f(X)u(X)]$ holds, where it is assumed that all the expectations exist and $E[f(X)] > 0$. The reversed inequality holds if $g(x)/f(x)$ and $u(x)$ are monotone in opposite directions.

We here provide a simple example using the condition of Theorem 3.1.

**Example 3.1** (Stein’s harmonic prior). Consider a prior distribution of the form $\pi(\theta) = ||\theta||^{-2c}$ for $0 < c \leq (p - 2)/2$. In this case, $h(\lambda) = \lambda^{c-1}$. Since $h'(\lambda) = -c\lambda^{c-1}$ and $h''(\lambda) = c(c + 1)\lambda^{c-2}$, the integrals in condition (3.1) can be easily calculated, and (3.1) can be expressed as

\begin{equation}
2(c + 1) \frac{\sum_{j=0}^{\infty} e_j A_j / [(p + 2j)(p - 2c - 2 + 2j)]}{\sum_{j=0}^{\infty} e_j / (p + 2j)} - c \frac{\sum_{j=0}^{\infty} e_j A_j / (p - 2c - 2 + 2j)}{\sum_{j=0}^{\infty} e_j} \leq 2p,
\end{equation}

where $A_j$ is the $j$th term of the series.
where \( e_j = c_j^{(p)} \Gamma(p/2 - c + j)2^{p/2-c+j} \). It is here shown that
\[
(3.11) \quad \frac{\sum_{j=0}^{\infty} e_j j/[(p + 2j)(p - 2c - 2 + 2j)]}{\sum_{j=0}^{\infty} e_j / (p + 2j)} \leq \frac{\sum_{j=0}^{\infty} e_j j / (p - 2c - 2 + 2j)}{\sum_{j=0}^{\infty} e_j}.
\]

In fact, let \( f(j) = e_j \), \( g(j) = 1/(p + 2j) \) and \( u(j) = j/(p - 2c - 2 + 2j) \) in Lemma 3.1. Since \( u(j) \) is nondecreasing in \( j \) for \( p - 2c - 2 \geq 0 \) while \( g(j) \) is decreasing, inequality (3.11) follows from Lemma 3.1. Using inequality (3.11), we can see that inequality (3.10) holds if
\[
(c + 2) \frac{\sum_{j=0}^{\infty} e_j j / (p - 2c - 2 + 2j)}{\sum_{j=0}^{\infty} e_j} \leq 2p,
\]
which is satisfied if \( 2(c + 2) \leq 2p \), or \( c \leq p - 2 \), since \( p - 2c - 2 \geq 0 \). Hence, the minimaxity can be established for \( 0 < c \leq (p - 2)/2 \). When \( c = (p - 2)/2 \), the prior distribution \( \|\theta\|^{-(p-2)} \) is a harmonic prior proposed by Stein (1973).

In general, it is hard to evaluate condition (3.1) directly. To derive tractable conditions on \( h(\lambda) \) from Theorem 3.1, we need to assume the following condition:

(A.1') The function \( h(\lambda) \) satisfies not only (A.1), but also \( \int_0^{\lambda_{p/2-1}} h'(\lambda)e^{-\lambda/2}d\lambda < \infty \) and \( \int_0^{\lambda_{p/2}} |h''(\lambda)|e^{-\lambda/2}d\lambda < \infty \).

It is noted that (A.1) assumes that \( \lim_{\lambda \to 0} \lambda^{p/2+1} h'(\lambda) = \lim_{\lambda \to 0} \lambda^{p/2+2} \).
\( h''(\lambda) = 0 \) while (A.1') assumes that \( \lim_{\lambda \to 0} \lambda^{p/2} h'(\lambda) = \lim_{\lambda \to 0} \lambda^{p/2+1} h''(\lambda) = 0 \), which is slightly more restrictive than (A.1). Define \( C_a(\lambda, w) \) by
\[
C_a(\lambda, w) = \sum_{j=0}^{\infty} c_j^{(a)} \lambda^{a/2+j-1}e^{-\lambda/2}.
\]

**Theorem 3.2.** Assume that \( h(\lambda) \) satisfies (A.1'). Then the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax if the function \( h(\lambda) \) satisfies the inequality
\[
(3.12) \quad \frac{\int C_{p-2}(\lambda, w)(2\lambda h'(\lambda) + (p-2)h(\lambda))d\lambda}{\int C_{p-2}(\lambda, w) \int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} - 2\frac{\int C_p(\lambda, w)(2\lambda h''(\lambda) + ph'(\lambda))d\lambda}{\int C_p(\lambda, w) \int_\lambda^\infty h'(t)e^{-t/2}dte^{\lambda/2}d\lambda} \leq \frac{p - 2}{2}.
\]

To prove this theorem, we use the following lemma which can be derived by integration by parts.

**Lemma 3.2.** Assume that a function \( f(\lambda) \) satisfies that \( \int C_{a+2}(\lambda, w) \cdot |f(\lambda)|d\lambda < \infty \). Then, for \( a > 0 \),
\[
(3.13) \quad \int C_{a+2}(\lambda, w)f(\lambda)d\lambda = \int C_a(\lambda, w) \int_\lambda^\infty f(t)e^{-t/2}dte^{\lambda/2}d\lambda.
\]
In fact, it can be seen that the condition \( \int C_{a+2}(\lambda, w) f(\lambda) |d\lambda | < \infty \) implies that \( C_{a+2}(\lambda, w) \int_{\lambda}^{\infty} f(t) e^{-t/2} dt e^{\lambda/2} \) converges to zero as \( \lambda \) goes to zero or infinity.

**Proof of Theorem 3.2.** We shall show that the condition given in (3.1) leads to condition (3.12). Since \( \int C_p(\lambda, w) |h''(\lambda)| |d\lambda | < \infty \) and \( \int C_{p+2}(\lambda, w) |h''(\lambda)| |d\lambda | < \infty \) from assumption (A.1'), it is observed that

\[
\sum_{j=0}^{\infty} 4 j c_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda
= \sum_{j=0}^{\infty} 4(j + p/2) e_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda
- 2p \sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda
= 4 \int C_p(\lambda, w) \lambda h''(\lambda) d\lambda - 2p \int C_{p+2}(\lambda, w) h''(\lambda) d\lambda.
\]

Similarly, under assumption (A.1'),

\[
\sum_{j=0}^{\infty} 4 j c_j^{(p)} \int \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} d\lambda
= 4 \int C_{p-2}(\lambda, w) \lambda h'(\lambda) d\lambda - 2(p - 2) \int C_p(\lambda, w) h'(\lambda) d\lambda.
\]

Using identity (3.13) of Lemma 3.2 gives that

\[
\int C_{p+2}(\lambda, w) h''(\lambda) d\lambda = \int C_p(\lambda, w) \int_{\lambda}^{\infty} h''(t) e^{-t/2} dt e^{\lambda/2} d\lambda.
\]

Further, by integration by parts,

\[
\int_{\lambda}^{\infty} h''(t) e^{-t/2} dt = -h'(\lambda) e^{-\lambda/2} + \frac{1}{2} \int_{\lambda}^{\infty} h'(t) e^{-t/2} dt.
\]

Combining (3.16) and (3.17) yields that

\[
\int C_{p+2}(\lambda, w) h''(\lambda) d\lambda
= - \int C_p(\lambda, w) h'(\lambda) d\lambda + \frac{1}{2} \int C_p(\lambda, w) \int_{\lambda}^{\infty} h'(t) e^{-t/2} dt e^{\lambda/2} d\lambda
- \int C_p(\lambda, w) h'(\lambda) d\lambda + \frac{1}{2} \int C_{p+2}(\lambda, w) h'(\lambda) d\lambda.
\]

Note that \( \sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} h'(\lambda) e^{-\lambda/2} d\lambda = \int C_{p+2}(\lambda, w) h'(\lambda) d\lambda \), which is expressed as \( \int C_p(\lambda, w) \int_{\lambda}^{\infty} h'(t) e^{-t/2} dt e^{\lambda/2} d\lambda \). Then from (3.14) and (3.18), we can see that

\[
\sum_{j=0}^{\infty} 4 j c_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda
\]

\[
\sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} h'(\lambda) e^{-\lambda/2} d\lambda
= 2 \int C_p(\lambda, w) \{ 2 \lambda h''(\lambda) + ph'(\lambda) \} d\lambda
\]

\[
\int C_p(\lambda, w) \int_{\lambda}^{\infty} h'(t) e^{-t/2} dt e^{\lambda/2} d\lambda - p.
\]
Applying the same arguments to (3.15), we can show that

\[
\sum_{j=0}^{\infty} 4 j c_j^{(p)} \int \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} d\lambda \\
\sum_{j=0}^{\infty} j c_j^{(p)} \int \lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda \\
= 2 \int C_{p-2}(\lambda, w) \left\{ 2 \lambda h'(\lambda) + (p - 2) h(\lambda) \right\} d\lambda \\
\int C_{p-2}(\lambda, w) \int_{\lambda}^{\infty} h(t) e^{-t/2} dte^{\lambda/2} d\lambda - (p - 2).
\]

Substituting (3.19) and (3.20) into (3.1), we obtain condition (3.12), and the proof of Theorem 3.2 is complete. \( \square \)

To derive a sufficient condition on \( h(\cdot) \) from Theorem 3.2, we use the following lemmas.

**Lemma 3.3.** The ratio \( C_p(\lambda, w)/C_{p-2}(\lambda, w) \) is increasing in \( \lambda \) if \( p \geq 3 \).

**Proof.** Since \( C_p(\lambda, w)/C_{p-2}(\lambda, w) = \sum_{j=0}^{\infty} c_j^{(p)} \lambda^{j+1} / \sum_{j=0}^{\infty} c_j^{(p-2)} \lambda^j \), the derivative of \( C_p(\lambda, w)/C_{p-2}(\lambda, w) \) with respect to \( \lambda \) is proportional to

\[
\sum_{j=0}^{\infty} (j + 1) c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} c_j^{(p-2)} \lambda^j - \sum_{j=0}^{\infty} c_j^{(p)} \lambda^{j+1} \sum_{j=0}^{\infty} j c_j^{(p-2)} \lambda^j.
\]

This is equal to

\[
\sum_{j=0}^{\infty} (j + 1) c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} (p/2 - 1 + j) c_j^{(p)} \lambda^j - \sum_{j=0}^{\infty} c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} j (p/2 - 1 + j) c_j^{(p)} \lambda^j,
\]

which is further rewritten as

\[
(p/2 - 1) \left\{ \sum_{j=0}^{\infty} c_j^{(p)} \lambda^j \right\}^2 + \left\{ \sum_{j=0}^{\infty} j c_j^{(p)} \lambda^j \right\}^2 \\
- \sum_{j=0}^{\infty} c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} j (j - 1) c_j^{(p)} \lambda^j.
\]

Note that

\[
\sum_{j=0}^{\infty} j c_j^{(p)} \lambda^j = \frac{w \lambda}{4} \sum_{j=0}^{\infty} \frac{1}{p/2 + j} c_j^{(p)} \lambda^j,
\]

and

\[
\sum_{j=0}^{\infty} j (j - 1) c_j^{(p)} \lambda^j = \frac{w \lambda}{4} \sum_{j=0}^{\infty} \frac{j}{p/2 + j} c_j^{(p)} \lambda^j.
\]

Hence, (3.21) is expressed as

\[
(p/2 - 1) \left\{ \sum_{j=0}^{\infty} c_j^{(p)} \lambda^j \right\}^2 \\
+ \frac{w \lambda}{4} \left\{ \sum_{j=0}^{\infty} \frac{1}{p/2 + j} c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} j c_j^{(p)} \lambda^j - \sum_{j=0}^{\infty} c_j^{(p)} \lambda^j \sum_{j=0}^{\infty} \frac{j}{p/2 + j} c_j^{(p)} \lambda^j \right\}.
\]
Since $p \geq 3$, the first term in (3.22) is positive. Also the second term is nonnegative if

\begin{equation}
E[(p/2 + J)^{-1}]E[J] \geq E[(p/2 + J)^{-1}J],
\end{equation}

where $E[\cdot]$ is an expectation with respect to the probability $P[J = j] = c_j^{(p)} \lambda^j / \sum_{i=0}^{\infty} c_i^{(p)} \lambda^i$. Hence inequality (3.23) follows from Lemma 3.1. Therefore, the requested monotonicity is proved. \hfill \Box

**Lemma 3.4.** (i) Assume that $b(\lambda)$ and $h(\lambda)$ are functions such that $\int C_{p-2}(\lambda, w)|b(\lambda)|d\lambda < \infty$ and $\int C_p(\lambda, w)|h(\lambda)|d\lambda < \infty$ for $p \geq 3$. If $b(\lambda)e^{-\lambda/2} / \int_{\lambda}^{\infty} h(t)e^{-t/2}dt$ is nondecreasing (resp. nonincreasing) in $\lambda$, then the ratio of integrals

\[ \int C_{p-2}(\lambda, w)b(\lambda)d\lambda / \int C_p(\lambda, w) \int_{\lambda}^{\infty} h(t)e^{-t/2}dt e^{\lambda/2}d\lambda \]

is nondecreasing (resp. nonincreasing) in $w$.

(ii) Assume that $b(\lambda)$ is a differentiable function satisfying $\lim_{\lambda \to \infty} b(\lambda) \cdot e^{-\lambda/2} = 0$. Also assume that $\int_{\lambda}^{\infty} |h(t)|e^{-t/2}dt < \infty$. Then the ratio $b(\lambda)e^{-\lambda/2} / \int_{\lambda}^{\infty} h(t)e^{-t/2}dt$ is nondecreasing (resp. nonincreasing) in $\lambda$ if $\{b'(\lambda) - b(\lambda)/2\} / h(\lambda)$ is nonincreasing (resp. nondecreasing) in $\lambda$.

**Proof.** We prove only the case that $b(\lambda)e^{-\lambda/2} / \int_{\lambda}^{\infty} h(t)e^{-t/2}dt$ is nondecreasing. Differentiating $\int C_{p-2}(\lambda, w)b(\lambda)d\lambda / \int C_p(\lambda, w) \int_{\lambda}^{\infty} h(t)e^{-t/2}dt e^{\lambda/2}d\lambda$ with respect to $w$, we can see that the monotonicity is guaranteed if

\[ \sum_{j=0}^{\infty} j c_j^{(p-2)} \int \lambda^{p/2 + j - 2} e^{-\lambda/2} b(\lambda)d\lambda \sum_{j=0}^{\infty} c_j^{(p-2)} \int \lambda^{p/2 + j - 2} \int_{\lambda}^{\infty} h(t)e^{-t/2}dt d\lambda \]

\begin{align*}
&\geq \sum_{j=0}^{\infty} c_j^{(p-2)} \int \lambda^{p/2 + j - 2} e^{-\lambda/2} b(\lambda)d\lambda \\
&\quad \times \sum_{j=0}^{\infty} j c_j^{(p-2)} \int \lambda^{p/2 + j - 2} \int_{\lambda}^{\infty} h(t)e^{-t/2}dt d\lambda,
\end{align*}

which is expressed as

\begin{equation}
\int C_p(\lambda, w)b(\lambda)d\lambda \int C_{p-2}(\lambda, w) \int_{\lambda}^{\infty} h(t)e^{-t/2}dt e^{\lambda/2}d\lambda \\
\geq \int C_{p-2}(\lambda, w)b(\lambda)d\lambda \int C_p(\lambda, w) \int_{\lambda}^{\infty} h(t)e^{-t/2}dt e^{\lambda/2}d\lambda.
\end{equation}

From Lemma 3.3, $C_p(\lambda, w)/C_{p-2}(\lambda, w)$ is increasing in $\lambda$. Since $b(\lambda)e^{-\lambda/2} / \int_{\lambda}^{\infty} h(t)e^{-t/2}dt$ is nondecreasing, inequality (3.24) follows from Lemma 3.1, and part (i) is proved.

For part (ii), we consider only the case that $\{b'(\lambda) - b(\lambda)/2\}/h(\lambda)$ is nonincreasing. For the proof, it is sufficient to show that

\begin{equation}
\{b'(\lambda) - b(\lambda)/2\} \int_{\lambda}^{\infty} h(t)e^{-t/2}dt + b(\lambda)e^{-\lambda/2}h(\lambda)e^{-\lambda/2} \geq 0,
\end{equation}

\[ E[(p/2 + J)^{-1}]E[J] \geq E[(p/2 + J)^{-1}J], \]
which is derived by differentiating \( b(\lambda)e^{-\lambda/2}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dt \) with respect to \( \lambda \). Note that

\[
(3.26) \quad b(\lambda)e^{-\lambda/2} = -\int_{\lambda}^{\infty} \frac{d}{dt}\{b(t)e^{-t/2}\}dt = -\int_{\lambda}^{\infty} \{b'(t) - b(t)/2\}e^{-t/2}dt.
\]

Then, inequality (3.25) is written as

\[
\int_{\lambda}^{\infty} h(t)e^{-t/2} \left\{ \frac{b'(\lambda) - b(\lambda)/2}{h(\lambda)} - \frac{b'(t) - b(t)/2}{h(t)} \right\} dt \geq 0,
\]

which is guaranteed by monotonicity of \( \{b'(\lambda) - b(\lambda)/2\}/h(\lambda) \), and part (ii) is proved. \( \square \)

**Lemma 3.5.** Assume (A.1') and that

\begin{equation}
(A.2) \quad h(\lambda) \text{ is nonincreasing.}
\end{equation}

Assume that \( d(\lambda) \) is a differentiable function such that \( \int C_p(\lambda, w)|d'(\lambda)|d\lambda < \infty \) and \( \int C_{p-2}(\lambda, w)|d(\lambda)|d\lambda < \infty \). If \( d(\lambda)e^{-\lambda/2}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dt \) is nondecreasing in \( \lambda \), then

\[
(3.27) \quad \frac{\int C_{p-2}(\lambda, w)d(\lambda)d\lambda}{\int C_{p-2}(\lambda, w)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda} \geq \frac{\int C_p(\lambda, w)d'(\lambda)d\lambda}{\int C_p(\lambda, w)\int_{\lambda}^{\infty} h'(t)e^{-t/2}dte^{\lambda/2}d\lambda}.
\]

The reversed inequality holds if \( d(\lambda)e^{-\lambda/2}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dt \) is nonincreasing.

**Proof.** We here prove only the case that \( d(\lambda)e^{-\lambda/2}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dt \) is nondecreasing. From identity (3.13), it is noted that

\[
\int C_p(\lambda, w)d'(\lambda)d\lambda = \int C_{p-2}(\lambda, w)\int_{\lambda}^{\infty} d'(t)e^{-t/2}dte^{\lambda/2}d\lambda
\]

\[
= \int C_{p-2}(\lambda, w)\left\{-d(\lambda) + \frac{1}{2} \int_{\lambda}^{\infty} d(t)e^{-t/2}dte^{\lambda/2}\right\}d\lambda
\]

\[
= -\int C_{p-2}(\lambda, w)d(\lambda)d\lambda + \frac{1}{2} \int C_p(\lambda, w)d(\lambda)d\lambda.
\]

Also note that \( \int_{\lambda}^{\infty} h'(t)e^{-t/2}dt = -h(\lambda)e^{-\lambda/2} + (1/2) \int_{\lambda}^{\infty} h(t)e^{-t/2}dt \) and that

\[
\int C_{p-2}(\lambda, w)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda = \int C_p(\lambda, w)h(\lambda)d\lambda.
\]

Then inequality (3.27) is rewritten as

\[
\int C_{p-2}(\lambda, w)d(\lambda)d\lambda \int C_p(\lambda, w)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda
\]

\[
\leq \int C_{p-2}(\lambda, w)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda \int C_p(\lambda, w)d(\lambda)d\lambda.
\]
This inequality follows from Lemma 3.1 since $C_p(\lambda, w)/C_{p-2}(\lambda, w)$ is increasing in $\lambda$ from Lemma 3.3, and $d(\lambda)e^{-\lambda/2}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dt$ is nondecreasing. Therefore, inequality (3.27) is established. \[ \square \]

Define $k(\lambda)$ by

$$
(3.28) \quad k(\lambda) = -\{2\lambda h'(\lambda) + (p - 2)h(\lambda)\}/\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2},
$$

and assume that it is decomposed as

$$
(3.29) \quad k(\lambda) = k_1(\lambda) + k_2(\lambda),
$$

where $k_1(\lambda)$ is a nondecreasing and differentiable function of $\lambda$ and $k_2(\lambda)$ is a function. Let $k_0$ be a constant such that

$$
(3.30) \quad k_0 \geq \lim_{w \to \infty} \frac{\int C_{p-2}(\lambda, w)k_1(\lambda)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda}{\int C_{p-2}(\lambda, w)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda}.
$$

Combining Theorem 3.2 and Lemmas 3.4 and 3.5, we obtain sufficient conditions given by the following theorem.

**Theorem 3.3.** Assume conditions (A.1'), (A.2) and $p \geq 3$. Also assume that $k_1(\lambda)$ is a nondecreasing and differentiable function such that $\int C_{p-2}(\lambda, w)\cdot |\lambda|d\lambda < \infty$ and $\int C_p(\lambda, w)|\lambda|d\lambda < \infty$ for $k_1(\lambda) = k_1(\lambda)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}$. Then the Bayes equivariant estimator $\hat{\theta}^\pi$ is minimax if $h(\lambda)$ satisfies the inequality

$$
(3.31) \quad k_0 - 2\inf_{\lambda > 0} \left\{ \frac{2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda)\int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}]'}{\int_{\lambda}^{\infty} h'(t)e^{-t/2}dte^{\lambda/2}} \right\} - \inf_{\lambda > 0} k_2(\lambda) \leq \frac{p - 2}{2},
$$

where $[\cdot]'$ denotes the derivative $(d/d\lambda)[\cdot]$.

**Proof.** For notational simplicity, let $I(\lambda, h)(\lambda) = \int_{\lambda}^{\infty} h(t)e^{-t/2}dte^{\lambda/2}d\lambda$ and $I(\lambda, h') = \int_{\lambda}^{\infty} h'(t)e^{-t/2}dte^{\lambda/2}d\lambda$. From (3.12) of Theorem 3.2, it is observed that

$$
(3.32) \quad \frac{\int C_{p-2}(\lambda, w)k_1(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda} - \frac{\int C_{p-2}(\lambda, w)k_2(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda} - 2\left\{ \frac{\int C_p(\lambda, w)\{2\lambda h''(\lambda) + ph'(\lambda)\}d\lambda}{\int C_p(\lambda, w)I(\lambda, h')d\lambda} \right\} \leq \frac{p - 2}{2}.
$$

For the first term in the l.h.s. of (3.31), from Lemma 3.4, it follows that

$$
\sup_w \frac{\int C_{p-2}(\lambda, w)k_1(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda} = \lim_{w \to \infty} \frac{\int C_{p-2}(\lambda, w)k_1(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda} \leq k_0.
$$
The second term is evaluated as
\[
\frac{\int C_{p-2}(\lambda, w)k_2(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda} \geq \inf_{\lambda} k_2(\lambda).
\]

Using (3.27) of Lemma 3.5 for \(d(\lambda) = k_1(\lambda)I(\lambda, h)\), we have the inequality
\[
\frac{\int C_p(\lambda, w)[k_1(\lambda)I(\lambda, h)]d\lambda}{\int C_p(\lambda, w)I(\lambda, h)d\lambda} \leq \frac{\int C_{p-2}(\lambda, w)k_1(\lambda)I(\lambda, h)d\lambda}{\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda}.
\]

Using this inequality, we can see that the third term in the l.h.s. of (3.31) is larger than or equal to
\[
-2\inf_{\lambda > 0} \left\{ \frac{2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda)I(\lambda, h)]'}{I(\lambda, h')} \right\}.
\]

Combining these results, we obtain condition (3.30). \(\square\)

Letting \(k_1(\lambda) = 0\), we get a simple condition from Theorem 3.2.

**Corollary 3.1.** Assume (A.1') and (A.2). Then condition (3.12) holds if \(h(\lambda)\) satisfies the inequality
\[
\sup_{\lambda} \frac{2\lambda h'(\lambda) + (p-2)h(\lambda)}{\int_{\lambda}^{\infty} h(t)e^{-t/2}dt\sqrt{\lambda}} - 2\inf_{\lambda} \frac{2\lambda h''(\lambda) + ph'(\lambda)}{\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt\sqrt{\lambda}} \leq \frac{p-2}{2}.
\]

When \(k(\lambda)\) is nondecreasing, namely, in the case of \(k(\lambda) = k_1(\lambda)\), we get the following proposition.

**Proposition 3.1.** Assume that the function \(h(\lambda)\) satisfies (A.1') and (A.2) for \(p \geq 3\). Also assume that \(k(\lambda) = -\{2\lambda h'(\lambda) + (p-2)h(\lambda)/\{\int_{\lambda}^{\infty} h(t)e^{-t/2}dt\sqrt{\lambda}\}\}\) is nondecreasing in \(\lambda\). Then \(\psi_\pi(w)\) is nondecreasing in \(w\) and the Bayes equivariant estimator \(\hat{\theta}^\pi\) is minimax if \(k_0 \leq (p-2)/2\) for \(k_0\) defined by (3.29).

**Proof.** The monotonicity of \(k(\lambda)\) corresponds to the case of \(k(\lambda) = k_1(\lambda)\) or \(k_2(\lambda) = 0\) in Theorem 3.3. Noting that \([k(\lambda)I(\lambda, h)]' = -ph'(\lambda) - 2\lambda h''(\lambda)\), we observe that
\[
2\lambda h''(\lambda) + ph'(\lambda) + [k(\lambda)I(\lambda, h)]' = 0.
\]

Hence, the condition of Proposition 3.1 is derived from Theorem 3.3. Since \(\psi_\pi(w)\) can be expressed as
\[
\psi_\pi(w) = p - 2 + \int C_{p-2}(\lambda, w)k(\lambda)I(\lambda, h)d\lambda/\int C_{p-2}(\lambda, w)I(\lambda, h)d\lambda,
\]
the monotonicity of \(\psi_\pi(w)\) follows from Lemma 3.4. \(\square\)

Although condition (3.30) given by Theorem 3.3 is simplified, it seems still hard to check, because it includes integrals. When the function \(h(\lambda)\) is three
times differentiable, we can derive a more tractable condition from (3.30). It is assumed that there exists a nondecreasing and differentiable function \( r_1(\lambda) \) and a function \( r_2(\lambda) \) such that

\[
(3.32) \quad r(\lambda) = (p - \lambda)h'(\lambda) + 2\lambda h''(\lambda) \}
\]

\( h(\lambda) = r_1(\lambda) + r_2(\lambda) \).

**THEOREM 3.4.** Assume conditions (A.1') and (A.2) and that \( h(\lambda) \) is three times differentiable. Then the Bayes equivariant estimator \( \tilde{\theta}^\pi \) is minimax if \( h(\lambda) \) satisfies the inequality

\[
(3.33) \quad k_0 + 2 \sup_{\lambda > 0} \left\{ \frac{(p + 2 - \lambda)h''(\lambda) + 2\lambda h'''(\lambda) - [r_1(\lambda)h(\lambda)]'}{h'(\lambda)} \right\} - \inf_{\lambda > 0} r_2(\lambda) \leq p,
\]

where \( k_0 \) is a constant such that

\[
k_0 \geq \lim_{w \to -\infty} \int C_p(\lambda, w)r_1(\lambda)h(\lambda)d\lambda / \int C_p(\lambda, w)h(\lambda)d\lambda.
\]

If \( r(\lambda) \) is nondecreasing, condition (3.33) holds if \( k_0 \leq p - 2 \).

**PROOF.** Using equation (3.26) gives the expression

\[
k(\lambda) = -\frac{2\lambda h'(\lambda) + (p - 2)h(\lambda)}{\int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}} = \frac{\int_{\lambda}^\infty \{(p - t)h'(t) + 2th''(t)\}e^{-t/2}dt}{\int_{\lambda}^\infty h(t)e^{-t/2}dt} - \frac{p - 2}{2},
\]

which can be decomposed as \( k(\lambda) = k_1(\lambda) + k_2(\lambda) \), where

\[
k_1(\lambda) = \int_{\lambda}^\infty r_1(t)h(t)e^{-t/2}dt / \int_{\lambda}^\infty h(t)e^{-t/2}dt,
\]

and

\[
k_2(\lambda) = \int_{\lambda}^\infty r_2(t)h(t)e^{-t/2}dt / \int_{\lambda}^\infty h(t)e^{-t/2}dt - (p - 2)/2.
\]

Since \( r_1(t) \) is nondecreasing, it can be easily shown that \( k_1(\lambda) \) is nondecreasing. Hence, we can use condition (3.30). From (3.29) and equation (3.13), it is observed that

\[
k_0 \geq \lim_{w \to -\infty} \frac{\int C_{p-2}(\lambda, w)k_1(\lambda) \int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda}{\int C_{p-2}(\lambda, w) \int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} = \frac{\int C_{p-2}(\lambda, w) \int_{\lambda}^\infty r_1(t)h(t)e^{-t/2}dte^{\lambda/2}d\lambda}{\int C_{p-2}(\lambda, w) \int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} = \frac{\int C_p(\lambda, w)r_1(\lambda)h(\lambda)d\lambda}{\int C_p(\lambda, w)h(\lambda)d\lambda}.
\]

Using equation (3.26) again for \( b(\lambda) = 2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda) \int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}]' \), we can show that

\[
2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda) \int_{\lambda}^\infty h(t)e^{-t/2}dte^{\lambda/2}]' = -\int_{\lambda}^\infty \{(p + 2 - t)h''(t) + 2th'''(t) - (p/2)h'(t) - [r_1(t)h(t)]'\}e^{-t/2}dt,
\]
so that
\[
\inf_{\lambda > 0} \left\{ 2\lambda h''(\lambda) + ph'(\lambda) + [k_1(\lambda) \int_{\lambda}^{\infty} h(t)e^{-t^2/2}dt\lambda/2] \right\} e^{-\lambda/2} \\
= \inf_{\lambda > 0} \left\{ -(p + 2 - t)h''(t) - 2th'''(t) + (p/2)h'(t) + [r_1(t)h(t)]' \right\} e^{-t^2/2}dt \\
\geq -\sup_{\lambda > 0} \left( (p + 2 - \lambda)h''(\lambda) + 2\lambda h'''(\lambda) - [r_1(\lambda)h(\lambda)]' \right) \lambda/2.
\]
Also, \( \inf_{\lambda > 0} k_2(\lambda) \geq \inf_{\lambda > 0} r_2(\lambda) - (p - 2)/2 \). Hence from condition (3.30), we obtain condition (3.33). When \( r(\lambda) \) is nondecreasing, condition (3.33) leads to the simple condition \( k_0 \leq p - 2 \), since \( r(\lambda) = r_1(\lambda) \) and \( r_2(\lambda) = 0 \). Therefore, the proof of Theorem 3.4 is complete. \( \square \)

**Example 3.2** (Prior related to a multivariate \( t \)-distribution). Consider a prior distribution with a density proportional to \( (b + \|\theta\|)^{-c}d\theta \) for positive constants \( b \) and \( c \). Let \( \pi(\lambda) = \lambda^{p/2-1}h(\lambda) \) for \( h(\lambda) = (b+\lambda)^{-c} \). To check condition (3.33), we need the first, second and third derivatives of \( h(\lambda) \), given by \( h'(\lambda) = -c(b+\lambda)^{-c-1} \), \( h''(\lambda) = c(c+1)(b+\lambda)^{-c-2} \) and \( h'''(\lambda) = -c(c+1)(c+2)(b+\lambda)^{-c-3} \). Assumption (A.1') is satisfied for all positive \( b \) and \( c \). Then, \( r(\lambda) \) defined by (3.32) has the form
\[
(3.34) \quad r(\lambda) = c\{1 - (b - 2c + p - 2)/(b + \lambda) - 2(c - 1)(b/(b + \lambda))^2\}.
\]
When \( b - 2c + p - 2 \geq 0 \), the function \( r(\lambda) \) is nondecreasing. Taking \( k_0 = c \) for \( k_0 \) defined by (3.29), we see from Theorem 3.4 that the Bayes equivariant estimator is minimax if
\[
(3.35) \quad 0 < c \leq \min\{p - 2, (p - 2 + b)/2\}.
\]
We next consider the case that \( b - 2c + p - 2 < 0 \). In this case, \( r(\lambda) \) is decomposed as \( r(\lambda) = r_1(\lambda) + r_2(\lambda) \) where \( r_1(\lambda) = c\{-A/(b+\lambda)^2 + B/(b+\lambda)\}I(\lambda+b \leq 2A/B) \) and \( r_2(\lambda) = \{-A/(b+\lambda)^2 + B/(b+\lambda)\}I(\lambda+b > 2A/B) \) for \( A = 2c(c-1)b \) and \( B = (2c - b - p + 2)c \). Note that \( r_1(\lambda) \) is increasing. Then it can be seen that
\[
q(\lambda) = \{ (p + 2 - \lambda)h''(\lambda) + 2\lambda h'''(\lambda) - [r_1(\lambda)h(\lambda)]' \} h'(\lambda) \\
= \begin{cases} 
1 - 4b(c + 2)/(b + \lambda)^2 & \text{if } \lambda + b \leq 2A/B, \\
(c + 1)\{1 + B/(b + \lambda) - (A + 6b)/(b + \lambda)^2\} - c & \text{if } \lambda + b > 2A/B,
\end{cases}
\]
so that
\[
\sup_{\lambda > 0} q(\lambda) = (c + 1) \left\{ 1 + \frac{B^2}{4(A + 6b)} \right\} - c = 1 + \frac{(c + 1)(2c - b - p + 2)^2}{8(c + 2)b}.
\]
Since \( k_0 = c \), condition (3.30) can be expressed as
\[
(3.36) \quad c + \frac{(c + 1)(2c - b - p + 2)^2}{4(c + 2)b} \leq p - 2.
\]
Combining the above arguments, we conclude that the Bayes equivariant estimator is minimax if the constants $b$ and $c$ satisfy either (3.35) or (3.36).

The prior distribution $h(\lambda) = (\lambda + b)^{-c}$ is a special case of generalized Student priors treated in Section 4.3 of Fourdrinier et al. (1998). For parameters $\alpha$, $\beta$ and $\gamma$ used there, $b$ and $c$ correspond to $b = -2\gamma$ and $c = \alpha + p - 2$, and $\beta$ is restricted to $\beta = \lambda + \gamma + (p - 2)/2$. Thus, $\beta$ is expressed as $\beta = -(b - 2c + p - 2)/2$ based on $b$ and $c$. Fourdrinier et al. (1998) derived the same condition to (3.35) for $\beta \leq 0$, but for $\beta > 0$, their condition is $c + (2c - n - p + 2)^2/(4b) \leq p - 2$, which is slightly more restrictive than (3.36).

**4. Expressions based on inverse Laplace transforms**

The general conditions on the function $h(\lambda)$ have been derived in Section 3 for the minimaxity of the Bayes equivariant estimator. When $h(\lambda)$ has an inverse Laplace transform, denoted by $H(t)$, the general conditions can be expressed based on the inverse Laplace transform $H(t)$. This expression is not only useful for checking the minimaxity, but also helpful for constructing prior distributions which result in the generalized Bayes and minimax estimators.

For a nonnegative function $h(\lambda)$, it is assumed that

* (B.1) there exists a function $H(t)$ such that

$$h(\lambda) = \int_0^\infty H(t)e^{-t\lambda}dt,$$

and $H(t)$ satisfies that $H(t)$ is differentiable and absolutely continuous and that $\int (1 + 2t)^{-p/2}|H(t)|dt < \infty$.

The function $H(t)$ can be derived by the inverse Laplace transformation, defined by

$$H(t) = \lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} h(\lambda + i\tau)e^{(\lambda + i\tau)t}d\tau,$$

for $i = \sqrt{-1}$. The inverse Laplace transformation is guaranteed under the integrability $\int_0^\infty |H(t)e^{-t\lambda}|dt < \infty$. Another derivation of $H(t)$ is given by

$$H(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{n+1} h^{(n)} \left(\frac{n}{t}\right),$$

called the Post’s inverse formula, where $h^{(n)}(x) = (d^n/dx^n)h(x)$. Since

$$\int_0^\infty \pi(\lambda)d\lambda = \int_0^\infty H(t)\int_0^\infty \lambda^{p/2-1}e^{-t\lambda}d\lambda dt = \Gamma(p/2) \int_0^\infty t^{-p/2}H(t)dt$$

for $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$, it is seen that the prior $\pi(\lambda)$ is proper if $\int_0^\infty t^{-p/2}H(t)dt < \infty$.

The inverse Laplace transform allows us to rewrite the function $\psi_\pi(w)$ based on an integral expression. The following lemma is useful for the purpose.
Lemma 4.1. For a positive constant $a$ and a function $f(t)$, the following equation holds:

\begin{equation}
\sum_{j=0}^{\infty} c_j^{(a)} \int \int \lambda^{a/2+j-1} e^{-(1+2t)\lambda/2} f(t) d\lambda dt = 2^{a/2} \int \frac{1}{(1+2t)^{a/2}} f(t) e^{w/[2(1+2t)]} dt,
\end{equation}

where $c_j^{(a)} = w^j / [j! \Gamma(a/2+j)^{2j}]$ and it is assumed that $\int_0^\infty (1+2t)^{-a/2} |f(t)| dt < \infty$.

Applying Lemma 4.1 to both the numerator and the denominator of the second term in (2.4), we can rewrite it as

$$
\psi_{\pi}(w) = w - w \int_0^\infty (1+2t)^{-p/2-1} \exp\{w/[2(1+2t)]\} H(t) dt
$$

which is equal to

$$
\psi_{\pi}(w) = w \int_0^\infty 2t(1+2t)^{-p/2-1} \exp\{-wt/(1+2t)\} H(t) dt.
$$

Theorem 4.1. Under assumption (B.1), the Bayes equivariant estimator $\hat{\theta}_{\pi}$ is minimax if the following inequality is satisfied for any $w > 0$:

\begin{equation}
2 \frac{\int K(t) (1+2t)^{-1} q_p(t; w) dt}{\int t (1+2t)^{-1} q_p(t; w) dt} - \frac{\int K(t) q_p(t; w) dt}{\int q_p(t; w) dt} \leq p - 3.
\end{equation}

To prove Theorem 4.1, the following fundamental property of the Laplace transformation is useful: for positive integer $n$,

\begin{equation}
\int t^n H(t) e^{-t\lambda} dt = (-D)^n h(\lambda) = (-1)^n h^{(n)}(\lambda),
\end{equation}

where $D = d/dt$ and $h^{(n)}(\lambda) = d^n h(\lambda)/dt^n$, the $n$-th derivative of $h(\lambda)$.

Proof. Using Lemma 4.1 and (4.4), we can see that each term in condition (3.1) can be expressed as

$$
\sum_{j=0}^{\infty} c_j^{(p)} \int \lambda^{p/2+j-1} h(\lambda) e^{-\lambda/2} d\lambda = 2^{p/2} \int H(t) q(t; w) dt,
$$
\[
\sum_{j=0}^{\infty} c_j^{(p+2)} \int \lambda^{p/2+j} h'(\lambda) e^{-\lambda/2} d\lambda = -2^{p/2+1} \int \frac{t}{1 + 2t} H(t) q(t; w) dt,
\]
\[
\sum_{j=0}^{\infty} 4j c_j^{(p)} \int \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} d\lambda = -2^{p/2+1} \int \frac{t}{1 + 2t} H(t) q(t; w) dt,
\]
\[
\sum_{j=0}^{\infty} 4j c_j^{(p+2)} \int \lambda^{p/2+j} h''(\lambda) e^{-\lambda/2} d\lambda = 2^{p/2+2} \int \frac{t^2}{(1 + 2t)^2} H(t) q(t; w) dt,
\]

which imply that assumption (A.1) is satisfied by assumption (B.1). By integration by parts, it is observed that
\[
\int wt H(t) e^{-\lambda t} dt = \int \frac{tH(t)}{(1 + 2t)^{p/2+1}} \frac{d}{dt} \left(-e^{w/[2(1+2t)]}\right) dt
\]
\[
= \int \left\{1 - (p - 4)t + (1 + 2t)t \frac{H'(t)}{H(t)}\right\} q(t; w) dt.
\]

Similarly,
\[
\int wt^2 H(t) e^{-\lambda t} dt
\]
\[
= \int \left\{2 - (p - 4)t + (1 + 2t)t \frac{H'(t)}{H(t)}\right\} \frac{t}{1 + 2t} q(t; w) dt.
\]

Hence, condition (4.3) can be obtained by rewriting condition (3.1) based on these expressions, and the proof of Theorem 4.1 is complete.

Assume that \( K(t) \) is decomposed as
\[ K(t) = K_1(t) + K_2(t), \]
where \( K_1(t) \) is a nonincreasing function of \( t \) and \( K_2(t) \) is a function. Let \( K_0 \) be a constant such that
\[ K_0 \geq \lim_{w \to \infty} \int K_1(t) q(t; w) dt / \int q(t; w) dt. \]

**Theorem 4.2.** Assume (B.1) and that
(B.2) \( \int tH(t)e^{-\lambda t} dt \) is a nonnegative function.
Then the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax if the following inequality is satisfied:
\[ K_0 + 2 \sup_w \frac{\int K_2(t)(1 + 2t)^{-p/2-1} H(t) \exp\{w/[2(1 + 2t)]\} dt}{\int (1 + 2t)^{-p/2-1} H(t) \exp\{w/[2(1 + 2t)]\} dt} \]
\[ - \inf_w \frac{\int K_2(t)(1 + 2t)^{-p/2} H(t) \exp\{w/[2(1 + 2t)]\} dt}{\int (1 + 2t)^{-p/2} H(t) \exp\{w/[2(1 + 2t)]\} dt} \leq p - 3. \]

Further, if
\( (B.2') \) \( H(t) \) is a nonnegative function, then inequality (4.6) is satisfied under the condition

\[
K_0 + 2 \sup_t K_2(t) - \inf_t K_2(t) \leq p - 3.
\]

In the case that \( K(t) = K_1(t) \), \( \hat{\theta}^\pi \) is minimax when \( K_0 \leq p - 3 \).

**Proof.** From Lemma 3.1 and the monotonicity of \( K_1(t) \), it follows that

\[
\frac{\int K_1(t)t(1+2t)^{-1}q(t,w)dt}{\int t(1+2t)^{-1}q(t,w)dt} \leq \frac{\int K_1(t)q(t,w)dt}{\int q(t,w)dt},
\]

so that condition (4.3) holds if for any \( w > 0 \),

\[
\frac{\int K_1(t)q(t,w)dt}{\int q(t,w)dt} + 2\frac{\int K_2(t)t(1+2t)^{-1}q(t,w)dt}{\int t(1+2t)^{-1}q(t,w)dt} - \frac{\int K_2(t)q(t,w)dt}{\int q(t,w)dt} \leq p - 3.
\]

Since the ratio of integrals \( \int K_1(t)q(t,w)dt / \int q(t,w)dt \) is nondecreasing in \( w \), it is noted that

\[
\sup_{w>0} \int K_1(t)q(t,w)dt / \int q(t,w)dt = \lim_{w \to \infty} \int K_1(t)q(t,w)dt / \int q(t,w)dt.
\]

Therefore, condition (4.6) in Theorem 4.2 is obtained from (4.8) and (4.9). It can be easily verified that inequality (4.7) implies inequality (4.6). □

It is noted that assumption (B.2') is equivalent to the function \( h(\lambda) \) being completely monotone (see Feller (1971)). Then, condition (4.7) is similar to that of Fourdrinier et al. (1998). When \( K(t) \) is nonincreasing, namely, in the case of \( K(t) = K_1(t) \), we get the following proposition from Theorem 4.2.

**Proposition 4.1.** Assume that the function \( H(t) \) satisfies conditions (B.1) and (B.2') for \( p \geq 3 \). Also assume that \( K(t) = -(p-4)t + t(1+2t)H'(t)/H(t) \) is nonincreasing in \( t \). Then \( \psi_\pi(w) \) is nondecreasing in \( w \) and the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax if \( K_0 \leq p - 3 \) for \( K_0 \) defined by (4.5).

Theorem 4.2 provides a class of prior distributions such that the resulting Bayes equivariant estimators can be minimax. Let \( K_1(t) \) be a nonincreasing function and assume that there exists a constant \( K_0 \) such that

\[
K_0 \geq \lim_{w \to \infty} \int K_1(t)q(t,w)dt / \int q(t,w)dt.
\]

Let \( K_2(t) \) be an integrable function satisfying condition (4.6) or (4.7). Then, denote \( K(t) = K_1(t) + K_2(t) \) and solve the differential equation

\[
-(p-4)t + t(1+2t) \frac{d}{dt} \log H(t) = K(t)
\]
for a positive function $H(t)$. A solution of this equation is given by

$$H(t) = (1 + 2t)^{(p-4)/2} \exp \left\{ \int_{t_0}^{t} \frac{K(x)}{x(1+2x)} \, dx \right\},$$

where $t_0$ is a positive constant. Then the Bayes equivariant estimator against the prior $h(||\theta||^2) = \int H(t) \exp\{-||\theta||^2 t\} \, dt$ is minimax if $H(t)$ satisfies (B.1).

**Example 4.1** (Scale mixture of a normal distribution). Consider the scale mixture of the normal distribution

$$\theta | t \sim \mathcal{N}_p(0, (2t)^{-1} I_p),$$

$$t \sim t^{b-2}(1 + 2t)^{-a} \nu(t) \, dt, \quad t > 0,$$

for constants $a$, $b$ and a function $\nu(t)$ satisfying the conditions

- **(NM-1)** $a \geq b$ and $1 - p/2 < b \leq (p - 2)/2$,
- **(NM-2)** the function $\nu(t)$ is nonnegative, differentiable and bounded.

The function $h(\lambda)$ is given by $h(\lambda) = \int_0^\infty H(t) e^{-\lambda t} \, dt$ for $H(t) = t^{p/2+b-2}(1 + 2t)^{-a} \nu(t)$ where the normalization constant is omitted. It can be verified that assumptions (B.1) and (B.2') are satisfied under conditions (NM-1) and (NM-2). From the arguments between (2.5) and (2.6), the generalized Bayes estimator against prior (4.10) is the Bayes equivariant estimator $\hat{\theta}_\pi$ against the prior $\pi(\lambda) = \lambda^{p/2-1} h(\lambda)$. The function $K(t)$ defined by (4.2) may be written as

$$K(t) = 2(b-a)t + (p/2 + b - 2) + t(1 + 2t)\nu'(t)/\nu(t).$$

When $(1+2t)\nu'(t)/\nu(t)$ is nonincreasing in $t$, the function $K(t)$ is nonincreasing under conditions (NM-1) and (NM-2). Noting that the constant $K_0$ defined by (4.5) is given by $K_0 = p/2 + b - 2$, we see from Proposition 4.1 that the Bayes equivariant estimator $\hat{\theta}_\pi$ against prior (4.10) is minimax if $(1 + 2t)\nu'(t)/\nu(t)$ is nonincreasing in $t$.

When $(1 + 2t)\nu'(t)/\nu(t)$ does not have a monotonicity property, let $K_1(t) = 2(b-a)t + (p/2 + b - 2)$ and $K_2(t) = t(1 + 2t)\nu'(t)/\nu(t)$. From condition (4.7), it follows that the Bayes equivariant estimator is minimax if

$$b + 2 \sup_t ((1 + 2t)\nu'(t)/\nu(t)) - \inf_t ((1 + 2t)\nu'(t)/\nu(t)) \leq (p - 2)/2. \tag{4.11}$$

For a suitable function $m(t)$, solve the differential equation $(1 + 2t)\nu'(t)/\nu(t) = m(t)$. A solution of this equation is given by

$$\nu(t) = \exp \left\{ \int_{t_0}^{t} \frac{m(s)}{(1+2s)s} \, ds \right\},$$

for a positive constant $t_0$. Take the function $m(t)$ such that $\nu(t)$ is bounded for any $t$. Then, $\nu(t)$ satisfies the condition (NM-2), and condition (4.11) is expressed by

$$b + 2 \sup_t m(t) - \inf_t m(t) \leq (p - 2)/2. \tag{4.12}$$
For example, consider the function \( m(t) = -2ct/(1+t^2) \) for \( c > 0 \). Then, \( \nu(t) \leq 1 \), namely, \( \nu(t) \) is bounded, and condition (4.12) holds if \( b+c \leq (p-2)/2 \).

In the case that \( m(t) = (1+2t)\nu'(t)/\nu(t) \) is nondecreasing in \( t \), condition (4.11) can be used. However, we can derive a better condition from (4.6) by using the monotonicity property of \( m(t) \). From the monotonicity of \( K_2(t) = m(t) \), it is noted that

\[
\int \frac{m(t)(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}}dt
\]

is nonincreasing in \( w \), so that

\[
\sup_w \int \frac{m(t)(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}}dt = \int \frac{m(t)(1+2t)^{-p/2-1}H(t)dt}{t(1+2t)^{-p/2-1}H(t)dt}.
\]

Similarly,

\[
\inf_w \int \frac{m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}dt = \lim_{w \to -\infty} \int \frac{m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}dt.
\]

Since \( K_0 = p/2 + b - 2 \), condition (4.6) can be rewritten as

\[
b + 2\int \frac{m(t)(1+2t)^{-p/2-1}H(t)dt}{t(1+2t)^{-p/2-1}H(t)dt} - \lim_{w \to -\infty} \int \frac{m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}dt \leq (p-2)/2.
\]

For example, consider the increasing function \( m(t) = 2ct/(1+2t) \) for \( c > 0 \). Then, the function \( \nu(t) \) is written by

\[
\nu(t) = \exp \left\{ 2c \int_{t_0}^t (1+2s)^{-2}ds \right\} = C_0 \exp\{-c/(1+2t)\},
\]

which is bounded. It can be verified that

\[
\lim_{w \to -\infty} \int \frac{m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}{t(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}}dt = 0.
\]

Hence, condition (4.13) can be expressed as

\[
b + 2c \int_0^1 \frac{z^{a-b}(1-z)^{p/2+b}}{z^{a-b}(1-z)^{p/2+b-1}} \exp\{-cz\}dz \leq (p-2)/2,
\]
after making the transformation $z = 1/(1 + 2t)$. On the other hand, (4.12) yields the condition that $b + 2c \leq (p - 2)/2$, which is not better than (4.14), although we need to resort to numerical computation to check condition (4.14).

Strawderman-type and Shifted inverted gamma priors treated in Sections 4.1 and 4.2 of Fourdrinier et al. (1998) are included in the framework of (4.10), (NM-1) and (NM-2). It is noted that (4.13) and (4.14) are conditions which have not been developed in the literature.

5. Derivation of Stein’s super-harmonic condition

In this section, we provide another expression of condition (3.12) and clarify the relationship between condition (3.12) and the Stein super-harmonic condition of prior densities.

**Theorem 5.1.** Assume condition (A.1'). Then condition (3.12) is equivalent to

\begin{equation}
\frac{2}{\int C_p(\lambda, w) h(\lambda) d\lambda} \left( \frac{\int C_p(\lambda, w) s(\lambda) d\lambda}{\int C_p(\lambda, w) \int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} + \frac{\int C_p(\lambda, w) \{-\lambda h'(\lambda) + s(\lambda)\} d\lambda}{\int C_p(\lambda, w) h(\lambda) d\lambda} \right) \geq 0,
\end{equation}

where

\begin{equation}
s(\lambda) = ph'(\lambda) + 2\lambda h''(\lambda).
\end{equation}

**Proof.** From equation (3.26), it is seen that

$$\{2\lambda h'(\lambda) + (p - 2)h(\lambda)\}e^{-\lambda/2} = -\int_\lambda^\infty \{s(t) - th'(t) - (p - 2)h(t)/2\}e^{-t/2}dt,$$

so that, from Lemma 3.2,

$$\frac{\int C_{p-2}(\lambda, w) \{2\lambda h'(\lambda) + (p - 2)h(\lambda)\} d\lambda}{\int C_{p-2}(\lambda, w) \int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} = -\frac{\int C_{p-2}(\lambda, w) \int_\lambda^\infty \{s(t) - th'(t)\}e^{-t/2}dte^{\lambda/2}d\lambda}{\int C_{p-2}(\lambda, w) \int_\lambda^\infty h(t)e^{-t/2}dte^{\lambda/2}d\lambda} + \frac{p - 2}{2},$$

which is equal to

$$-\frac{\int C_p(\lambda, w) \{s(\lambda) - \lambda h'(\lambda)\}e^{-\lambda/2}d\lambda}{\int C_p(\lambda, w) h(\lambda) e^{-\lambda/2}d\lambda} + \frac{p - 2}{2}.$$

Hence, condition (5.1) is obtained from condition (3.12). □

Assume that $s(\lambda)$ is decomposed as

$$s(\lambda) = ph'(\lambda) + 2\lambda h''(\lambda) = s_1(\lambda) + s_2(\lambda),$$

where $s_1(\lambda) \leq 0$ and $s_2(\lambda) > 0$ for any $\lambda > 0$. Then from Theorem 5.1, we get the following condition.
Proposition 5.1. Assume (A.1') and (A.2). Then, the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax if

\[
(5.3) \quad 2 \inf_{\lambda} \left\{ \frac{s_2(\lambda)e^{-\lambda/2}}{\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt} \right\} + \inf_{\lambda} \left\{ -\lambda h'(\lambda) - s_1(\lambda) + s_2(\lambda) \right\} \geq 0.
\]

If \( \{(d/d\lambda)[s_2(\lambda)e^{-\lambda/2}]\}/\{h'(\lambda)e^{-\lambda/2}\} \) is nonincreasing, then

\[
(5.4) \quad \inf_{\lambda} \left\{ \frac{s_2(\lambda)e^{-\lambda/2}}{\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt} \right\} = \lim_{\lambda \to 0} \left\{ \frac{s_2(\lambda)e^{-\lambda/2}}{\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt} \right\}.
\]

Proof. Noting that

\[
\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt = -h(\lambda)e^{-\lambda/2} + \frac{1}{2} \int_{\lambda}^{\infty} h(t)e^{-t/2}dt \geq -h(\lambda)e^{-\lambda/2},
\]

we observe that

\[
\frac{\int C_p(\lambda, w)s_1(\lambda)d\lambda}{\int C_p(\lambda, w) \int_{\lambda}^{\infty} h'(t)e^{-t/2}dt e^{\lambda/2}d\lambda} \geq \frac{-\int C_p(\lambda, w)s_1(\lambda)d\lambda}{\int C_p(\lambda, w)h(\lambda)d\lambda}.
\]

Hence from Theorem 5.1, it suffices to show that

\[
2 \frac{\int C_p(\lambda, w)s_2(\lambda)d\lambda}{\int C_p(\lambda, w) \int_{\lambda}^{\infty} h'(t)e^{-t/2}dt e^{\lambda/2}d\lambda} + \frac{\int C_p(\lambda, w)\{-\lambda h'(\lambda) + s(\lambda) - 2s_1(\lambda)\}d\lambda}{\int C_p(\lambda, w)h(\lambda)d\lambda} \geq 0,
\]

which shows that the sufficient condition (5.3) is sufficient.

To verify equality (5.4), we show that the ratio \( g(\lambda)/\int_{\lambda}^{\infty} h'(t)e^{-t/2}dt \) is nondecreasing in \( \lambda \) for \( g(\lambda) = s_2(\lambda)e^{-\lambda/2} \). Since \( g(\lambda) = -\int_{\lambda}^{\infty} g'(t)dt \), the derivative of the ratio is proportional to

\[
g'(\lambda) \int_{\lambda}^{\infty} h'(t)e^{-t/2}dt + g(\lambda)h'(\lambda)e^{-\lambda/2}
\]

which is nonnegative if \( g'(t)/\{h'(t)e^{-t/2}\} \) is nonincreasing in \( t \). Therefore, Proposition 5.1 is established. \( \Box \)

When \( s(\lambda) \leq 0 \) for any \( \lambda > 0 \), this condition implies that \( h'(\lambda) \leq 0 \) for any \( \lambda > 0 \). In fact, whenever \( h'(\lambda) > 0 \), the derivative of \( h'(\lambda) \) is negative since \( h''(\lambda) \leq -ph'(\lambda)/(2\lambda) \). This fact means that \( h'(\lambda) \leq 0 \), and assumption (A.2) holds. Then from Proposition 5.1, we get

Corollary 5.1. If \( s(\lambda) \leq 0 \) for any \( \lambda > 0 \), then the Bayes equivariant estimator \( \hat{\theta}^\pi \) is minimax under assumption (A.1').
Stein (1981) showed that the Bayes equivariant estimator is minimax if the prior density \( h(||\theta||^2) \) is super-harmonic, namely \( \sum_{i=1}^{p}(\partial^2/\partial \theta_i^2)h(||\theta||^2) \leq 0 \) for any \( \theta \). Since \( \sum_{i=1}^{p}(\partial^2/\partial \theta_i^2)h(||\theta||^2) \) is identical to \( s(\lambda) \), it is seen that the condition \( s(\lambda) \leq 0 \) corresponds to the super-harmonic condition.

To express the Stein super-harmonic condition based on the inverse Laplace transform \( H(t) \) of \( h(\lambda) \), note that \( s(\lambda) = \int S(t)e^{-\lambda t}dt \) where \( S(t) = -(p-4)tH(t) + 2t^2H'(t) \). The super-harmonic condition \( s(\lambda) \leq 0 \) holds if \( S(t) \leq 0 \) for any \( t > 0 \).

Applying the super-harmonic condition to the prior distribution treated in Example 3.2, we see that the super-harmonic condition for the minimaxity is not positive if \( \alpha t > 0 \). This is quite restrictive in comparison with conditions (3.35) and (3.36). Although the Stein super-harmonic condition is more restrictive in this example, it can provide nice and simple conditions for the minimaxity as demonstrated in the following example.

**Example 5.1 (Prior based on the arctan function).** Let us treat a prior distribution of the form

\[
\theta \sim \frac{1}{\alpha} \left\{ \tan^{-1} \frac{\alpha}{\|\theta\|^2} \right\} d\theta,
\]

where \( \alpha \) is a positive constant. In this case, the function \( h(\lambda) \) is written by

\[
h(\lambda) = \frac{1}{\alpha} \tan^{-1} \frac{\alpha}{\lambda} = \frac{1}{\alpha} \left( \frac{\pi}{2} - \tan^{-1} \frac{\lambda}{\alpha} \right) = \frac{1}{\alpha} \int_{\lambda/\alpha}^{\infty} \frac{1}{1+x^2} dx,
\]

and the inverse Laplace transform is given by \( H(t) = (\sin \alpha t)/(\alpha t) \), namely, \( h(\lambda) = \int \{ \sin \alpha t/(\alpha t) \} e^{-\lambda t}dt \). It is noted that \( H(t) \) goes to zero while taking positive and negative values periodically as \( t \) tends to infinity. Since \( h(\lambda) \) is rewritten as \( h(\lambda) = \int_{\lambda/\alpha}^{\infty} (\alpha^2 + s^2)^{-1} ds \) by making the transformation \( s = \alpha x \), it is observed that \( h'(\lambda) = -((\alpha^2 + \lambda^2)^{-1} \) and \( h''(\lambda) = 2\lambda(\alpha^2 + \lambda^2)^{-2} \). Since \( H(t) \) takes negative values periodically, the conditions given in Section 4 do not work well. The conditions derived in Section 3 can give a feasible but somewhat restrictive condition on \( \alpha^2 \) and \( p \). However, the Stein super-harmonic condition provides a nice condition for the minimaxity. That is, the function \( s(\lambda) \) defined by (5.2) can be written as

\[
s(\lambda) = -((p-4)\lambda^2 + p\alpha^2)/(\alpha^2 + \lambda^2)^2,
\]

which is not positive if \( p \geq 4 \). Hence from Corrolary 5.1, the Bayes equivariant estimator \( \hat{\theta}_\pi \) against prior (5.5) is minimax for \( p \geq 4 \). Although it is interesting to clarify whether \( \hat{\theta}_\pi \) is minimax for \( p = 3 \), it is not easy to show.

The Stein super-harmonic condition can be applied to another type of prior distributions. When the prior of \( \theta \) is \( (\alpha^2 + ||\theta||^4)^{-1} d\theta \), it corresponds to the case that \( h(\lambda) = (\alpha^2 + \lambda^2)^{-1} \) and \( H(t) = \sin(\alpha t) \), namely, \( h(\lambda) = \int \sin(\alpha t) \exp\{-\lambda t\} dt \). Then,

\[
s(\lambda) = -\frac{2\lambda}{(\alpha^2 + \lambda^2)^2} \left\{ p - 6 + \frac{8\alpha^2}{\alpha^2 + \lambda^2} \right\},
\]
which is not positive if \( p \geq 6 \). Hence, the Bayes equivariant estimator is minimax for \( p \geq 6 \).

6. Admissibility of the Bayes equivariant estimators

The conditions for the minimaxity have been investigated for the Bayes equivariant estimators. Another interesting topic is to provide a characterization of prior distributions for the admissibility. Using Brown’s admissibility condition, in this final section, we derive conditions on priors for the admissibility of the Bayes equivariant estimator. The results given here may be helpful for checking the admissibility for general priors, though most of them are known in the literature.

We begin with stating Brown’s admissibility condition, which is known as a very useful tool for checking the admissibility in the Stein problem. As noted in Section 2, the generalized Bayes estimator against a prior distribution with the spherically symmetric density \( h(\|\theta\|^2)d\theta \) is the Bayes equivariant estimator \( \hat{\theta}^\pi \) against the prior \( \pi(\lambda) = \lambda^{p/2-1}h(\lambda) \) for \( \lambda = \|\theta\|^2 \). Define \( A(h) \) by \( A(h) = \int_1^\infty \{r^{p-1}f_h(r)\}^{-1}dr \), where \( f_h(\|x\|) = \int (2\pi)^{-p/2}\exp\{|x-\theta|^2/2\}h(\|\theta\|^2)d\theta \).

**Theorem 6.1** (Brown (1971)). The Bayes equivariant estimator \( \hat{\theta}^\pi \) is inadmissible if \( A(h) < \infty \). When \( f_h(\|x\|) \) and \( \|\hat{\theta}^\pi - x\| \) are uniformly bounded with respect to \( x \), \( \hat{\theta}^\pi \) is admissible if \( A(h) = \infty \).

It is noted that \( f_h(\|x\|) \) is the marginal density with respect to \( dx \) while

\[
g_\pi(w) = \frac{1}{2^{p/2}} \sum_{j=0}^\infty \frac{w^{p/2+j-1} \exp\{-w/2\}}{j!\Gamma(p/2 + j)2^{2j}} \int \lambda^{p/2+j-1}e^{-\lambda/2}h(\lambda)d\lambda
\]

is the marginal density with respect to \( dw \) for \( w = \|x\|^2 \). It is thus seen that \( g_\pi(w) = w^{p/2-1}f_h(\sqrt{w}) \), or \( f_h(r) = r^{2-p}g_\pi(r^2) \) for \( r = \sqrt{w} \), and \( A(h) \) is written as \( A(h) = \int_1^\infty (rg_\pi(r^2))^{-1}dr \). Since \( \|\hat{\theta}^\pi - x\| = \psi_\pi(w)/\sqrt{w} \), Theorem 6.1 is rewritten in the following.

**Lemma 6.1.** Assume that there exists a constant \( \delta \) such that \( g_\pi(r^2) \sim C_0r^{\delta} \) for some generic constant \( C_0 \) as \( r \to \infty \). If \( \delta > 0 \), the Bayes equivariant estimator \( \hat{\theta}^\pi \) is inadmissible. When \( g_\pi(w)/w^{p/2-1} \) and \( \psi_\pi(w)/\sqrt{w} \) given by (2.4) are uniformly bounded, \( \hat{\theta}^\pi \) is admissible if \( \delta \leq 0 \).

Through this section, we use the notations \( C, C', C_0, C_1 \) and \( C_2 \) as generic positive constants, namely, for example we use the same notation \( C \) for different constants without anything confusing.

It may be hard to check the conditions in Lemma 6.1. However, the use of the inverse Laplace transform of \( h(\lambda) \) can make them tractable. Since \( h(\lambda) = \int H(t)e^{-\lambda t}dt \), \( g_\pi(w) \) is expressed as

\[
g_\pi(w) = w^{p/2-1} \int \frac{H(t)}{(1+2t)^{p/2}} \exp\left\{-\frac{2t}{2(1+2t)}w\right\}dt.
\]
Making the transformation \( z = 2t/(1 + 2t) \) gives the expression \( g_\pi(w) = w^{p/2 - 1}G_\pi(\sqrt{w})/2 \) where

\[
G_\pi(r) = \int_0^1 (1 - z)^{p/2 - 2}H\left(\frac{z}{2(1 - z)}\right) e^{-rz^2/2}dz.
\]

(6.1)

Also, the function \( \psi_\pi(w) \) is written by

\[
\psi_\pi(w) = w\int_0^1 z(1 - z)^{p/2 - 2}H(z/[2(1 - z)]) \exp\{-wz/2\}dz
\]

\[
= w\int_0^1 z(1 - z)^{p/2 - 2}H(z/[2(1 - z)]) \exp\{-wz/2\}dz.
\]

(6.2)

Then Lemma 6.1 is expressed in the following.

**Lemma 6.2.** Assume that there exists a constant \( \delta \) such that \( G_\pi(r) \sim C_0r^{2-p+\delta} \) for some generic constant \( C_0 \) as \( r \to \infty \). If \( \delta > 0 \), the Bayes equivariant estimator \( \hat{\theta}_r^\pi \) is inadmissible. When \( G_\pi(r) \) and \( \psi_\pi(r^2)/r \) are uniformly bounded, \( \hat{\theta}_r^\pi \) is admissible if \( \delta \leq 0 \).

When \( H(t) \) is a positive function, it follows from (6.2) that \( \psi_\pi(w) \leq w \). If \( \psi_\pi(w) \) is bounded, namely, \( \psi_\pi(w) \leq C \) for a constant \( C \), then it is observed that \( \psi_\pi(r^2)/r \leq \min\{r, C/r\} \leq \sqrt{C} \), so that \( \psi_\pi(r^2)/r \) is uniformly bounded. Hence, the boundedness of \( \psi_\pi(r^2) \) is sufficient for the boundedness of \( \psi_\pi(r^2)/r \). The following lemma is useful for checking the conditions in Lemma 6.2.

**Lemma 6.3.** For positive constants \( a, b \) and \( d \), there are constants \( C_1 \) and \( C_2 \) such that

\[
C_1\int_0^1 z^{a-1}e^{-dz}dz \leq \int_0^1 z^{a-1}(1 - z)^{b-1}e^{-dz}dz \leq C_2\int_0^1 z^{a-1}e^{-dz}dz.
\]

This implies that \( \int_0^1 z^{a-1}(1 - z)^{b-1}e^{-dz}dz \sim C_0/d^a \) for a constant \( C_0 \) as \( d \to \infty \).

**Proof.** In the case of \( 0 < b < 1 \), \((1 - z)^{b-1}\) is increasing and \( 1 \leq (1 - z)^{b-1} \), so that the left inequality holds. Since \((1 - z)^{b-1} \) and \( e^{-dz} \) are monotone in opposite directions, Lemma 3.1 implies that

\[
E[(1 - Z)^{b-1}e^{-dZ}] \leq E[(1 - Z)^{b-1}]E[e^{-dZ}],
\]

where \( Z \) is a random variable having the density \( az^{a-1} \). Inequality (6.3) is expressed as

\[
\int_0^1 z^{a-1}(1 - z)^{b-1}e^{-dz}dz \leq a \int_0^1 z^{a-1}(1 - z)^{b-1}dz \int_0^1 z^{a-1}e^{-dz}dz,
\]

which yields the right inequality. In the case of \( b \geq 1 \), \((1 - z)^{b-1}\) is decreasing and \((1 - z)^{b-1} \leq 1 \). The same arguments can be used to get the inequalities in Lemma 6.3.

Since \( \int_0^1 z^{a-1}e^{-dz}dz = d^{-a} \int_0^d x^{a-1}e^{-x}dx \), the inequalities in Lemma 6.3 imply that \( C_1 \leq \lim_{d \to \infty} d^{-a} \int_0^1 z^{a-1}(1 - z)^{b-1}e^{-dz}dz \leq C_2 \), which means that \( \int_0^1 z^{a-1}(1 - z)^{b-1}e^{-dz}dz \sim C_0/d^a \) for \( C_1 \leq C_0 \leq C_2 \) as \( d \to \infty \). 

\( \square \)
We conclude this section with checking the conditions in Lemma 6.2 for the admissibility of the Bayes equivariant estimators treated in examples in the previous sections.

For the prior distribution discussed in Examples 3.1 and 3.2, \( H(t) \) is given by \( H(t) = t^{c-1}e^{-bt} \). For some generic constant \( C_0 \), \( G_\pi(r) \) may be written as

\[
G_\pi(r) = C_0 \int_0^1 z^{c-1}(1 - z)^{p/2-c-1}e^{-b z/(1-z) - (r^2/2)z} dz,
\]

which can be seen to be uniformly bounded with respect to \( r \) for \( 0 < c < p/2 \) and \( b \geq 0 \). It can be shown that

\[
C_1 \int_0^1 z^{c-1}(1 - z)^{p/2-c-1}e^{-(r^2/2)z} dz \\
\leq G_\pi(r) \leq C_0 \int_0^1 z^{c-1}(1 - z)^{p/2-c-1}e^{-(r^2/2)z} dz.
\]

Combining inequalities (6.4) and Lemma 6.3 gives that

\[
G_\pi(r) \sim C \int_0^1 z^{c-1}(1 - z)^{p/2-c-1}e^{-(r^2/2)z} dz \sim C' r^{2-p+(p-2-2c)}
\]

as \( r \to \infty \). Inequality (6.4) and Lemma 6.3 are again used to evaluate \( \psi_\pi(w) \) as

\[
\psi_\pi(w) = w \int_0^1 z^{c}(1 - z)^{p/2-c-1}e^{-bz/(1-z) - (w/2)z} dz \\
\leq C \int_0^\infty x^{c}e^{-x/2} dx \\
\leq C \int_0^\infty x^{c-1}e^{-x/2} dx,
\]

which is bounded. Since \( G_\pi(r) \) is bounded, we can use Lemma 6.2 for \( \delta = p-2-2c \) and \( 0 < c < p/2 \). Hence, the Bayes equivariant estimator \( \tilde{\theta}_\pi \) is inadmissible for \( 0 < c < (p-2)/2 \) and \( b \geq 0 \), and admissible for \( (p-2)/2 \leq c < p/2 \) and \( b > 0 \).

When \( (p-2)/2 \leq c \leq \min\{p-2,(p-2+b)/2\} \), \( c < p/2 \) and \( b \geq 0 \), \( \tilde{\theta}_\pi \) is admissible and minimax. This result suggests to take \( c = (p-2)/2 \), because the resulting Bayes equivariant estimator is admissible and minimax for any \( b \geq 0 \).

For the prior distribution in Example 4.1, \( H(t) \) is given by \( H(t) = t^{p/2+b-2}(1+2t)^{-a}\nu(t) \). Then,

\[
G_\pi(r) = C_0 \int_0^1 z^{p/2+b-2}(1 - z)^{a-b}\nu\left(\frac{z}{2(1-z)}\right)e^{-(r^2/2)z} dz,
\]

which is bounded when \( 1 - p/2 < b < a + 1 \) and \( \nu(t) \) is bounded as \( 0 < \nu_1 \leq \nu(t) \leq \nu_2 \) for some positive constants \( \nu_1 \) and \( \nu_2 \). Since

\[
C_0\nu_1 \int_0^1 z^{p/2+b-2}(1 - z)^{a-b}e^{-(r^2/2)z} dz \\
\leq G_\pi(r) \leq C_0\nu_2 \int_0^1 z^{p/2+b-2}(1 - z)^{a-b}e^{-(r^2/2)z} dz,
\]
Lemma 6.3 implies that $G_\pi(r) \sim C r^{2-p-2b}$ as $r \to \infty$. The boundedness of $\psi_\pi(w)$ can be verified by the same arguments as in (6.5). Hence from Lemma 6.2, it is concluded that $\hat{\theta}_\pi$ is inadmissible for $1 - p/2 < b < \min(0, a + 1)$, and admissible for $0 \leq b < a + 1$. When $0 \leq b < a$ and $1 - p/2 + a < b \leq (p - 2)/2$, the Bayes equivariant estimator $\hat{\theta}_\pi$ is admissible and minimax, where $\nu(t)$ is a nonnegative and differentiable function such that $0 < \nu_1 \leq \nu(t) \leq \nu_2$ and $(1 + 2t)\nu'(t)/\nu(t)$ is nonincreasing in $t$.

For the prior distribution treated in Example 5.1, $H(t)$ is given by $H(t) = (\sin \alpha t) / (\alpha t)$. Since $(\sin \alpha t)/t \leq 1$, it is observed that

\[
\int_0^1 (1 - z)^{p/2-2} e^{-(r^2/2)z} dz \geq G_\pi(r) \geq C \int_0^1 e^{-(r^2/2)z} dz \sim C' r^{2-p+(p-4)}
\]

where

\[
G_\pi(r) = \int_0^1 (1 - z)^{p/2-2} \frac{\sin(\alpha z/[2(1-z)])}{\alpha z/[2(1-z)]} e^{-(r^2/2)z} dz.
\]

Hence, $G_\pi(r) \sim C r^{2-p+(p-4)}$ as $r \to \infty$. Since $\psi_\pi(w)$ can be verified to be bounded, it is concluded from Lemma 6.2 that the Bayes equivariant estimator $\hat{\theta}_\pi$ is inadmissible for $p > 4$, and admissible for $p = 3$ and $p = 4$. The estimator $\theta_\pi$ is admissible and minimax for $p = 4$, though it is minimax, but inadmissible for $p \geq 5$.

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