Frictionless Generation of a Tidal Eulerian Residual Flow over a Sill in a Narrow Channel

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Using a vertically two-dimensional, two-layer model, we have analytically examined the generation mechanism of a nonzero Eulerian residual flow by strong tide-topography interaction in a narrow channel where the frictional effect is not included. In this case, tidally generated baroclinic disturbances are forced non-uniformly in space and time while being advected by a strong tidal flow over the non-uniform slope of the bottom topography. Consequently, nonzero Eulerian residual flow results when averaged over one tidal period. Although the time average of the velocity field is thus nonzero, the associated Eulerian residual transport in each layer is compensated by a Stokes transport so that no Lagrangian residual transport results in both layers. This warns us that simple time averaging of the velocity data obtained at a fixed mooring station might lead to a spurious material transport.

1. Introduction

Observations of near-shore or channel flow are usually carried out at spatially fixed stations (e.g. Holloway et al., 2001). In most cases, observed flow is dominated by tidal components so that time averaging over a tidal period is used to determine the residual component. An important matter we should bear in mind is that the residual flow thus determined is not necessarily accompanied by material transport. The simple time-averaging of spatially fixed flow data is called Eulerian residual, whereas the residual flow accompanied by material transport is called Lagrangian residual and has been well discussed in terms of the vorticity balance (e.g. Zimmerman, 1981; Robinson, 1981).

In this note, we show analytically that internal wave generation by strong tide-topography interaction in a narrow channel (Hibiya, 1986) leads to the production of nonzero Eulerian residual flow. One remarkable feature of this Eulerian residual flow is that it bears no material transport, although the time average of the velocity field is not zero.

2. Formulation

A flow in a channel along the x-axis is considered. The vertical z-axis is positive upward. The width of the channel is assumed to be sufficiently narrow so that the flow is vertically two dimensional (i.e. no y variation) and the effect of the Earth’s rotation is neglected. We start from the simplest case where a barotropic tidal flow \( U = U(t) \) is superimposed on a two-layer flow over a small topography \( h = h(x) \). When nondimensionalized with typical scales, the tidal flow is of \( O(1) \), while the topography is of \( O(\varepsilon) \) (\( \ll 1 \)). Therefore the baroclinic responses (or internal waves) are expected to be of \( O(\varepsilon) \). Under the rigid-lid approximation, the velocity perturbation in the upper layer \( u_1 \) and in the lower layer \( u_2 \), and the interfacial elevation \( \eta \) are governed by

\[
\left( \partial_t + U \partial_x \right) u_1 = -\partial_z p \tag{1}
\]

\[
-(\partial_t + U \partial_x) \eta + H_1 \partial_z u_1 = 0 \tag{2}
\]

\[
\left( \partial_t + U \partial_x \right) u_2 = -\partial_z p - g' \partial_z \eta \tag{3}
\]

\[
\left( \partial_t + U \partial_x \right) \eta + H_2 \partial_z u_2 = U \partial_z h, \tag{4}
\]

where the unperturbed thicknesses of the upper and lower layers are \( H_1 \) and \( H_2 \), respectively; \( g' \) is the reduced grav-
ity; \( p \) is the pressure divided by density.

The problem can be solved in the same manner as was done by Hibiya (1986), but a different approach is taken here. Following Gill (1982), we introduce quantities \( q \) and \( r \) defined by

\[
q = \frac{H_1 + H_2}{H_1 H_2} c \eta + u
\]

(5)

\[
r = \frac{H_1 + H_2}{H_1 H_2} c \eta - u,
\]

(6)

where \( c \) is the speed of the baroclinic disturbance given by

\[
c = \sqrt{\frac{H_1 H_2}{H_1 + H_2} g'}
\]

(7)

and \( u = u_2 - u_1 \). Then the following expressions for \( q \) and \( r \) are obtained from (1)–(4).

\[
\{ \partial_t + (U + c) \partial_x \} q = \frac{U c}{H_2} \partial_x h(x)
\]

(8)

\[
\{ \partial_t + (U - c) \partial_x \} r = \frac{U c}{H_2} \partial_x h(x).
\]

(9)

These two equations are solved either by applying the Fourier and Laplace transforms (Hibiya, 1986) or changing independent variables from \((x, t)\) to \((x - \int_0^t (U(r') \pm c) dr', t)\). With the initial conditions

\[
q = q_0(x)
\]

(10)

\[
r = r_0(x) \text{ at } t = 0,
\]

(11)

the solutions of (8) and (9) are obtained as

\[
q = q_0 \left( x - \int_0^t U(r') dr' - c t \right)
\]

(12)

\[
+ \int_0^t \frac{c}{H_2} U(\tau) \partial_x h \left( x - \int_\tau^t U(r') dr' - c(t - \tau) \right) d\tau
\]

\[
r = r_0 \left( x - \int_0^t U(r') dr' + c t \right)
\]

(13)

\[
+ \int_0^t \frac{c}{H_2} U(\tau) \partial_x h \left( x - \int_\tau^t U(r') dr' + c(t - \tau) \right) d\tau.
\]

The first terms on the right-hand sides of (12) and (13) represent the effect of the initial condition. For the case of a localized initial condition (i.e. \( q_0(x) \rightarrow 0, r_0(x) \rightarrow 0 \) as \( x \rightarrow |\infty| \)), these terms become negligible as \( t \rightarrow \infty \) and so will not be considered here. Hereafter, horizontal and vertical lengths and horizontal velocity are nondimensionalized by the horizontal scale of the bottom topography \( b, H_2, \) and \( c, \) respectively. Then (12) and (13) become, in the nondimensional form,

\[
q = \int_0^t U(\tau) \partial_x h \left( x - \int_\tau^t U(r') dr' - (t - \tau) \right) d\tau
\]

(14)

\[
r = \int_0^t U(\tau) \partial_x h \left( x - \int_\tau^t U(r') dr' + (t - \tau) \right) d\tau,
\]

(15)

with the tidal flow defined by

\[
U(t) = F_r \sin(2\pi t / T),
\]

(16)

where \( F_r \) is the internal Froude number and \( T \) is the dimensionless tidal period.

3. Eulerian Residual Flow

For the tidal flow in (16), we can get the following relations:

\[
q(x, t + T)
\]

(17)

\[
= \int_0^T U(\tau) \partial_x h \left( x - \int_\tau^{T+\tau} U(r') dr' - (t + T - \tau) \right) d\tau
\]

\[
= \int_0^T U(\tau) \partial_x h \left( x - \int_\tau^{T} U(r') dr' - (t - \tau) \right) d\tau
\]

\[
= q(x, t) + \int_0^T U(\tau) \partial_x h \left( x - \int_\tau^{T} U(r') dr' - (t - \tau) \right) d\tau.
\]

When the bottom topography \( h(x) \) is localized around \( x = 0 \), the second integral is an initial transient and is negligible around \( x \sim 0 \) for \( t >> T \), because the parameter \( x - \int_0^T U(r') dr' \sim 0 \) for \( t >> T \). This implies that the solution becomes periodic within the time scale over which baroclinic disturbances pass the bottom topography. This time scale is \( O(1) \) and is much shorter than the time scale over which our leading order solutions (14) and (15) break down \( (O(\epsilon^{-1})) \) or the time scale over which the hydrostatic approximation breaks down \( (O(c^2/\sqrt{bg'})) \). Of course, the same periodicity also holds for \( r \).

It is therefore sufficient to take the time average over a tidal period \( T \) to determine the Eulerian residual components such that,
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\[
\begin{align*}
\bar{q} &= \frac{1}{T} \int_{t''}^{t''+T} \int_{t''}^{t''+T} U(\tau) \partial_x h \left( x - \int_{t''}^{t''+T} U(t') dt' - (t - \tau) \right) dt d\tau, \\
\bar{r} &= \frac{1}{T} \int_{t''}^{t''+T} \int_{t''}^{t''+T} U(\tau) \partial_x h \left( x - \int_{t''}^{t''+T} U(t') dt' + (t - \tau) \right) dt d\tau, 
\end{align*}
\]

where \( t'' \) is arbitrary so long as \( t'' > 1 \). Figure 1 shows an example of the Eulerian residual components obtained by numerically evaluating (18) and (19) where the bottom topography is \( h_0 \exp(-x^2/b^2) \) in its dimensional form, and \( T = 8.5 \).

Residual components result from a nonzero oscillatory tidal flow. The shape of the residual components is not similar to that of the bottom topography. Equations (18) and (19) show that if the slope of the bottom topography is uniform, namely, \( \partial_x h \equiv \text{const.} \), the residual components vanish so long as the tidal average of \( U(t) \) is zero. Thus, the existence of the non-uniform slope of the bottom topography is essential for the generation of nonzero Eulerian residual flow in a narrow channel.

4. Generation of the Eulerian Residual Flow

4.1 Wave evolution along characteristics

The parameters \( x - \int U dt' \equiv (t - \tau) \) in (14) and (15) show the positions of baroclinic disturbance at time \( \tau \) along the trajectories (characteristics) \( dx/dt = U \pm 1 \) which pass through \((x, t)\). Therefore, \( q \) and \( r \) result from the accumulation of \( U \partial_x h \) propagating along each trajectory. This idea, first proposed by Hibiya (1986), is most clearly seen in (8) and (9). The left-hand side of (8) or (9) describes the evolution of \( q \) or \( r \) during the propagation along each trajectory, whereas the right-hand side represents the “elementary wave”, in other words, forcing at each instant of time. The time dependency of the elementary wave can be formally removed by rewriting (8) and (9) such that

\[
\begin{align*}
\left\{ \partial_x + (U + 1) \partial_{x'} \right\} (q - h) &= -\partial_x h, \\
\left\{ \partial_x + (U - 1) \partial_{x'} \right\} (r - h) &= \partial_x h.
\end{align*}
\]

For example, Fig. 2 shows the trajectories of \( q' = q - h \) (nondimensional) for the case \( F_r = 0 \). We can see that \( q' \)
propagates along \( \frac{dx}{dt} = 1 \) without any stagnation in the forcing region so that the amplification of \( q' \) becomes identical for all the trajectories. The resulting amplitude of \( q' \) compensates the steady response in (20) and (21) so that no velocity perturbation nor interfacial displacement can be observed. When \( F_r = 0.75 \), in contrast, some trajectories are seen to be stagnant in the forcing region so that the amplification of \( q' \) becomes non-uniform (Fig. 3).

The non-uniform amplification of \( q' \) can be most clearly traced in Fig. 4 where the time evolution of \( q' \) propagating along the trajectory \( \frac{dx}{dt} = U + 1 \) is shown for the case \( F_r = 1.5 \). As an example, let us take the solid trajectory indicated by the arrow. For \( t < -0.5 \), it is advected by the positive tidal flow. When the trajectory passes around \( x \approx -1 \), negative elementary waves are superimposed so that the value of \( q' \) becomes negative, as seen in the upper panel. For \( -0.5 < t < 0 \), the trajectory is carried back by the negative tidal flow and hence stagnates at around \( x \approx 1 \) where positive elementary waves are efficiently superimposed. The value of \( q' \) therefore rapidly grows positive during this time period. Similar interpretations hold for other trajectories as well. As is seen in the upper panel, more trajectories end up with positive values of \( q' \) so that a positive residual component is expected to occur when averaged over one tidal period. The shape of the residual component of \( \eta \) or \( u \) is thus sensitive to the shape of the bottom topography as well as the functional form of the tidal flow \( \bar{U}(t) \). It is therefore difficult to propose “a rule of thumb” for the shape of the residual components (see Fig. 1).

4.2 Physical interpretation and significance of the residual flow

We take the Eulerian time average of the continuity equations (2) and (4) such that

\[
-U \partial_x \bar{\eta} + H \partial_x \bar{u}_1 = 0
\]

\[
\bar{U} \partial_x \bar{\eta} + H \partial_x \bar{u}_2 = 0.
\]

These equations imply that the Eulerian residual velocity \( \bar{u}_i \) \((i = 1, 2)\) is generated if the tidal flow \( U \) and the interface displacement \( \eta \) are correlated \((\bar{U} \eta \neq 0)\) over the localized bottom topography. Actually, the solutions (14) and (15) assure that nonzero correlation exists between \( U \) and \( \eta \), namely, \( \bar{U} \eta > 0 \) over the left-hand side slope and \( \bar{U} \eta < 0 \) over the right-hand side slope such that negative (positive) velocity anomaly results in the upper (lower) layer over the right-hand side slope and positive (negative) velocity anomaly results in the upper (lower) layer over the left-hand side slope (see Hibiya, 1986).

The fact that this Eulerian residual flow is not accompanied by a net transport has interesting implications. Academically, it serves as an interesting example of the relationship

\[
\text{Lagrangian residual velocity} = \text{Eulerian residual velocity} + \text{Stokes drift},
\]

which has been discussed in detail by Zimmerman (1979).
In textbooks (e.g. Lighthill, 1978), an oscillatory flow with zero Eulerian residual velocity is shown to possess a material transport. In the present case, however, two terms on the right-hand side cancel each other so that no Lagrangian residual velocity results. In practical terms, it warns us against any “apparent tidal residual”; a spurious material transport might result from simple time averaging of the velocity data obtained at a fixed mooring station.

5. Summary

Using a vertically two-dimensional, two-layer model, we have analytically examined the generation mechanism of nonzero Eulerian residual flow by strong tide-topography interaction in a narrow channel where the frictional effect is not included so that the traditional vorticity approach is not applicable. The generation of the tidal residual component over the bottom topography has been discussed using characteristics which describe the propagation of tidally generated baroclinic disturbances under the effect of strong tidal advection. We have found that tidally generated baroclinic disturbances are forced non-uniformly in space and time while being advected by strong tidal flow over the non-uniform slope of the bottom topography so that nonzero tidal residual flow results when averaged over one tidal period.

Although the time average of the velocity field is thus nonzero, the associated Eulerian residual transport in each layer is compensated by a Stokes transport so that no Lagrangian residual transport results in both layers. This warns us that simple time averaging of the velocity data obtained at a fixed mooring station might lead to a spurious material transport.

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