Journal of Oceanography, Vol. 54, pp. 343 to 346. 1998

Capillary Waves Understood by an Elementary Method

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(Received 26 March 1998; in revised form 1 June 1998; accepted 30 June 1998)

The central physics of capillary waves (or ripples) can be understood by an elementary method which makes use of the balance of static and dynamic pressure differences along the surface streamline between crest and trough, in the steady reference frame, and conservation of mass through vertical cross-sections beneath crest and trough. Basically Einstein's (1916) model of surface gravity waves is adapted for the purpose of explaining the existence of capillary waves of infinitesimal amplitude. One product of the physical understanding, the phase speed of capillary waves, is derived as a function of the wave length and surface tension, and the result agrees exactly with that obtained by the classical mathematical procedure. In the elementary method it is not necessary to assume irrotational flow, upon which the classical theory is founded, nor are perturbation expansions of the nonlinear fluid equations employed. The extension to capillary-gravity waves, by including the acceleration of gravity in the physical model, is straightforward, and the calculated phase speed of these waves is identical to what is found in the text books as well.

Keywords:
· Capillary waves,
· ripples.

1. Introduction

How is it that capillary waves, or ripples, can exist? The most elementary answer to this question apparently is not to be found in the scientific literature, although a rather complicated and lengthy answer is available. In order to understand the standard answer considerable mathematical preparation is needed. It is to be hoped that the physical approach given below will facilitate a quicker, and at the same time, more fundamental appreciation of one of the more fascinating and important types of waves on the water's surface. One qualitative measure of the geophysical importance of capillary waves, for example, is that it appears to be necessary that they must be generated first by the wind before the more energetic gravity waves can form.

Gravity acts over all length scales, but surface tension is most effective at short scales, about one centimeter or less for air over water. Surface tension behaves like a stretched membrane (Batchelor, 1967) by tending to flatten out bulges in a curving surface. Thus surface tension acts as a restoring force: crests try to become lower and troughs higher. Gravity is a restoring force as well.

In the next section gravity is temporarily neglected, so that surface tension is the only restoring force in the model. One demonstration of the physical understanding that the model gives us is that it provides a simple way to derive the formula for the phase speed of pure capillary waves, i.e. to find its functional dependence on the wave length and surface tension. Then in the section following that gravity is reintroduced and combined with surface tension to obtain easily the propagation speed of capillary-gravity waves.

The central method used in the present discussion involves incorporating surface tension into Einstein is (1916) model of surface gravity waves, which is a straightforward and relatively short procedure. By this means the physics of capillary waves can be understood clearly in an elementary way without the necessity of assuming from the start that the motion is irrotational or using perturbation expansions on nonlinear fluid dynamics equations, in accordance with the usual mathematical methods employed in the text books.

Einstein's gravity wave model makes use of the steady reference frame in which the observer moves at the speed of the wave. In this reference frame the wave shape is stationary and the fluid beneath the surface flows by the observer from left to right, say. The steady reference frame is the one that is most convenient for the purposes here also. Einstein was not the first nor the only one to make use of the technique of the steady reference frame, but his succinct statement of the method makes it the easiest to imitate. Now Einstein's stimulating short paper, written originally in German, is available in English for the first time (Kenyon and Sheres, 1997).

Basically, Einstein showed that for surface gravity waves in the steady frame there is a balance between the oppositely directed static and dynamic pressure differences between the crest and trough along the surface streamline. The dynamic pressure is related to the flow speed, through Bernoulli's equation, and the static pressure is related to the acceleration of gravity and to the height of a fluid column (the wave height). For example, at the wave trough the static pressure is greatest, because of the hydrostatic head of the column of fluid of height equal to the wave height, and the dynamic pressure is least by Bernoulli's law, because the perturbed flow speed is greatest due to conservation of mass. By adjusting the mean flow speed the static and dynamic pressures at the trough can be made to balance. This is the fundamental physics of the gravity wave.

For a capillary wave the same type of understanding comes about when the dynamic pressure is still determined by the flow speed but the static pressure is now determined by surface tension. Calculating the wave phase speed is just used below as an example of one way to show that we do understand the physics of capillary waves. Einstein did not compute the phase speed of surface gravity waves, but he nevertheless captured the full physics of these waves in a nutshell.

2. Capillary Waves

Consider waves of infinitesimal amplitude $H/2$. First, Laplace's equation for the pressure difference across the air-water interface due to the effect of surface tension, which is applicable to plane or long-crested (i.e. two-dimensional) waves, is

$$\Delta p = \frac{T}{R} \tag{1}$$

where Δp is the pressure differential across the interface, T is the surface tension (taken constant for air and water at a given temperature and salinity) and R is the principal radius of curvature of the surface. (For short-crested waves there would be two principle radii of curvature.) By convention the pressure is higher on the concave side of the curving surface (e.g. Landau and Lifshitz, 1959). Equation (1) is independent of any motion of the fluid below the surface.

For simplicity take the fluid flow in the steady reference frame to be zero. Applying Laplace' s equation (1) to the pressure difference between crest and trough gives

$$\Delta p_1 = 2\frac{T}{R} \tag{2}$$

where Δp_1 is the pressure at the crest minus that at the trough, the assumption being made that the radius of curvature is the same at crest and trough (infinitesimal amplitudes). Also the atmospheric pressure is assumed to be constant over the entire surface. Thus the left side of Eq. (2) just involves a difference in fluid (water) pressures between crest and trough. Equation (2) gives the static pressure difference between crest and trough.

Now assume for the moment that surface tension is not acting and let the fluid flow past the observer from left to right (Fig. 1). Below the trough in the steady frame the flow speed is a little faster than the mean speed, and below the crest the speed is a little less than the average speed. This is due to the disturbance in the flow caused by the wave shape, and the disturbed flow is confined in depth to the vicinity of the wave. Because of the Bernoulli effect the perturbed flow tries to make the trough lower and the crest higher, or in other words to make the waves grow bigger in amplitude.

Next, Bernoulli's equation for steady frictionless and incompressible flow is

$$p = \text{const} - \frac{1}{2}\rho u^2 \tag{3}$$

from which is derived the dynamic pressure difference between crest and trough. Where the speed is greatest the pressure is least, according to Eq. (3). Therefore below the trough the pressure is least and below the crest it is greatest. The constant in Eq. (3) means constant along a streamline, and the crest and trough are on the same streamline.

Forming the dynamic pressure difference, Δp_2, between crest and trough gives

$$\Delta p_2 = \frac{1}{2}\rho\left(u + \frac{\Delta u}{2}\right)^2 - \frac{1}{2}\rho\left(u - \frac{\Delta u}{2}\right)^2 = \rho u \Delta u \tag{4}$$

where $\Delta u/2$ is the perturbation in the flow speed caused by the shape of the wave. The perturbed flow speed is assumed to be the same at crest and trough (infinitesimal amplitudes).

Evidently the static pressure difference between crest and trough caused by surface tension has the opposite sign from that due to the dynamic effect of the fluid flow. For a balance between the two oppositely directed pressure differences we have

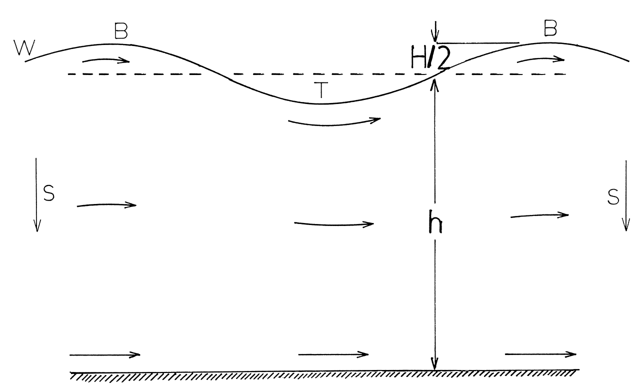

Fig. 1. Flow in the steady frame beneath the wave shape W with crests B and trough T, the wave has infinitesimal amplitude $H/2$. The perturbation in the flow speed due to the presence of the wave diminishes with increasing depth and vanishes at the depth of wave influence h. The acceleration of gravity points in the direction S. Adapted from Einstein (1916; Fig. 4).

$$\Delta p_1 = \Delta p_2 = 2\frac{T}{R} = \rho u \Delta u. \tag{5}$$

In order to evaluate Eq. (5) further, the perturbed flow speed must be expressed in terms of more easily measured quantities. For this purpose conservation of mass through vertical cross-sections is used between crest and trough

$$\left(u + \frac{\Delta u}{2}\right)\left(h - \frac{H}{2}\right) = \left(u - \frac{\Delta u}{2}\right)\left(h + \frac{H}{2}\right) \tag{6}$$

where H is the wave height and h is the depth of wave influence (Fig. 1). It is assumed that the perturbed flow speed, Δu, is constant over the depth of wave influence. This assumption is not expected to change the final results qualitatively (Kenyon, 1983). Equation (6) condenses to

$$uH = \Delta u h. \tag{7}$$

The next ingredient needed is an expression for the depth of wave influence, which is adapted from the gravity wave result of Kenyon (1983)

$$h = \frac{\lambda}{2\pi} \tag{8}$$

where λ is the wave length. By definition the wave motion and wave pressure are confined to depths above the depth of wave influence. According to Eq. (8) the depth of wave influence is proportional to the wave length and independent of the wave height. It is as if a rigid plate were to be put at the depth h (Fig. 1).

Notice that Eq. (8) is also independent of the acceleration of gravity. What this means is that if there is some other cause of acceleration besides gravity, and surface tension provides such a cause, then the result Eq. (8) will still be valid. Therefore Eq. (8) is true for capillary waves as well. Alternatively, one can follow the arguments in Kenyon (1983) step by step, except that the static pressure differences between crest and trough are now due to surface tension instead of gravity, and the vertical acceleration of the fluid due to gravity is replaced by the vertical acceleration due to surface tension. In summary, the depth of wave influence for capillary waves is proportional to the wave length and independent of both the wave height and surface tension.

Using Eq. (8) to eliminate h and Eq. (7) to eliminate Δu, Eq. (5) becomes

$$u^2 = \frac{2T}{\rho R k H}. \tag{9}$$

Finally, the convenient formula for the radius of curvature is (Burington, 1948)

$$R = \left[\frac{\partial^2 \zeta}{\partial k^2}\right]^{-1} \tag{10}$$

for infinitesimal amplitudes, i.e. $(\partial\zeta/\partial k)^2$ is much smaller than one. The surface elevation for a sinusoidal progressive wave in the steady frame is

$$\zeta = a \sin kx \tag{11}$$

and $a = H/2$ is the wave amplitude. With Eq. (11) the magnitude of radius of curvature at crest and trough becomes (we ignore the sign convention)

$$R = \frac{1}{k^2 a} \tag{12}$$

where (ka) is proportional to the mean wave slope.

Putting Eq. (12) into Eq. (9) gives the end result

$$u^2 = \frac{Tk}{\rho}. \tag{13}$$

Equation (13) gives the square of the phase velocity for pure capillary waves can be seen by changing reference frames from the steady frame to the fixed frame. By this interpretation $c = u$, i.e. the phase velocity in the fixed frame equals the mean flow speed in the steady frame. In the fixed frame the waves move by the observer from right to left, and the mean flow speed is zero everywhere.

Equation (13) agrees exactly with the classical result (e.g. Lamb, 1932), when the air density is neglected compared to the water density. The larger the surface tension the greater the phase speed, as expected, since surface tension acts like a restoring force. Also Eq. (13) shows that shorter wave length capillary waves move faster than longer ones (the phase speed increases with increasing wave number). This is expected too because surface tension is more effective for smaller scales. However, the result that the square of the phase speed depends linearly on both the surface tension and the wave number is not obvious a priori.

If the fluid were to flow past the observer, who is attached to the steady frame, from right to left, instead of from left to right, as assumed above, then in switching back to the fixed frame the waves would be found to move past the observer from right to left, and the formula for the wave speed would be exactly the same. In other words, capillary waves can propagate equally well in either of the two horizontal directions. (There is a type of surface gravity wave, called the roll wave, that can only move in one direction, i.e. down the slope, and the Einstein method can be applied to these waves also.)

3. Capillary-Gravity Waves

Let gravity act now in concert with surface tension. In the steady reference frame the static pressure difference between crest and trough, due to gravity alone, is

$$\Delta p_3 = \rho g H \tag{14}$$

which is independent of the motion of the fluid. The static pressure is higher at the trough than at the crest, just the opposite of the dynamic pressure, which is lower at the trough and higher at the crest.

The static pressure between crest and trough due to surface tension alone is still given by Eq. (2), and it too produces a higher pressure at the trough than at the crest. Thus the two separate static pressure differences, Eqs. (2) and (14), which are independent of each other, can be added together to give the total static pressure difference between crest and trough, as is done on the left side of Eq. (15).

For a balance among all three pressure effects between crest and trough in the steady frame Eq. (5) now becomes

$$\Delta p_1 + \Delta p_3 = \Delta p_2 \tag{15}$$

which can be expressed in terms of the mean flow speed as

$$u^2 = \frac{Tk}{\rho} + \frac{g}{k} \tag{16}$$

when the square of the wave slope is small compared to one.

Equation (16) agrees with the classical result (Lamb, 1932) after converting back to the fixed frame such that $c = u$ by the usual interpretation and neglecting the air density compared to the water density. The phase speed increases with both increasing and decreasing wave number, as Eq. (16) shows, so the phase speed must have a minimum value at one particular wave length, which turns out to be about 23 cm/sec at the wave length of about 1.7 cm for water and air at room temperature. ($T \approx 75$ in c.g.s. units for room temperatures and typical ocean salinities.) For very short scales, or when gravity can be neglected, Eq. (16) reduces to the expression (13) found above for pure capillary waves. For very long waves, or neglecting surface tension, Eq. (16) reduces to the wellknown formula for the phase speed of surface gravity waves in the deep water limit, which has been derived earlier by a similar method (Kenyon, 1983).

4. Conclusion

The phase speed for capillary-gravity waves has now been obtained in Eq. (16) by an elementary physical method that is more direct than the usual mathematical one. In the process of computing the phase speed the very existence of capillary waves has been made understandable and by a way that is quite short and should be easy to follow. Pure surface gravity waves were explained before by Einstein (1916), and then Kenyon (1983) later added one small piece to the puzzle solved by Einstein. The irrotational assumption is sometimes useful, but it is not necessary to account for basic properties of capillary or gravity waves. The perturbation technique provides a convenient program for dealing with the nonlinear equations of fluid motion, but in very few cases has it been possible to check independently the results produced by perturbation analyses. However, a qualitative example of verifying the predictions of perturbation theory, founded on irrotationality, is given by the above approach for capillary waves, at least in the infinitesimal amplitude limit.

Acknowledgements

The journal review produced several helpful comments.

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