Topographically Trapped Waves over the Continental Shelf and Slope*

Hiroshi Takeda†

Abstract: General characteristics of topographically trapped subinertia waves are discussed from the viewpoint of an eigenvalue problem and ray theory. Special attention is paid to the slope parameter $S(x) = (dh/dx)/h$, where $x$ denotes the coordinate perpendicular to the shoreline, increasing seaward, and $h(x)$ is the depth which is a measure of the strength of the restoring force of the waves. Three cases for the $S$ distribution are considered, in which $S$ is assumed to be positive at the coast and to tend to zero far from the coast. The first is where $S(x)$ decreases monotonically towards the open ocean. It is found in this case that waves are trapped near the coast. The second is where $S(x)$ does not decrease monotonically, but has a maximum. It is concluded that this case may contain two types of waves, i.e., those trapped near the coast and those trapped near the maximum, and the dispersion curves corresponding to different types may nearly intersect, namely, result in "kissing". The third is where $S(x)$ has a negative region (corresponding to the presence of a trench). It is found in this case that an infinite sequence of waves is trapped in the negative $S$ region which propagate with the coast to their left (right) in the northern (southern) hemisphere besides the waves trapped near the coast.

The topography in the second case corresponds to a typical continental shelf and a typical continental slope. It is shown by model calculation that trapped waves are present over the continental slope as well as over the continental shelf.

Almost the same results are obtained for superinertia waves if $S$ is replaced by $1/h$ which is a measure of the restoring force of superinertia waves.

1. Introduction

For a semi-bounded uniformly rotating ocean over a variable topography, Reid (1958) found topographically trapped subinertia waves on a semi-infinite sloping beach of constant slope. Robinson (1964) discussed the same kind of waves trapped over a shelf of constant slope connected to the infinitely deep ocean, and referred to them as ‘continental shelf waves’. This kind of topographic wave was investigated in more detail by Mysak (1968), Ball (1967), Buchwald and Adams (1968), and Larsen (1969), using models with more realistic topographies. (See Mysak (1980) for detailed review.)

Huthnance (1975) generalized the results of the case studies and derived general properties of the waves. He considered a barotropic ocean of depth $h$ that increases monotonically in a direction (denoted by $x$) towards the open ocean and tends to a constant as $x \to \infty$, and obtained two main conclusions. One is that there is an infinite discrete sequence of topographically trapped subinertia waves of the form $e^{i(kx + \omega t)}$ propagating with the coast to their right (left) in the northern (southern) hemisphere. The other is that the dispersion curves have the property that $\omega/\ell =$ const $> 0$ at the long wavelength limit and $\omega > 0$ at the short wavelength limit if $(dh/dx)/h$ is finite. Hence, there must exist at least one point at which $\partial \omega / \partial \ell = 0$ holds.

After Huthnance, Brooks and Mooers (1977) found two dispersion curves, i.e., one for the wave trapped over the upper slope and the other trapped over the lower slope, using a channel model with two slopes. Kubota (1981) also found different types of topographically trapped subinertia waves using a model computation.

Topographically trapped subinertia waves are formed by stretching and shrinking of the vortex tubes that oscillate over a varying topography. It is natural to suppose, therefore, that the pro-
properties of the waves should be determined by the "slope parameter" \( S=(dh/dx)/h \) which represents the magnitude of the rate of stretching and shrinking of the vortex tubes. The main purpose of the present study is to extend Huthnance's results by paying particular attention to the \( S \) distribution and to try to obtain more detailed knowledge about the properties of the waves.

The governing equations are formulated in Section 2. In Section 3 we treat the equations as an eigenvalue problem and discuss the properties of eigenvalue and eigenfunction qualitatively. In Section 4 "kissing", which occurs between different types of waves, is investigated. In Section 5 we relate the wave trapping with energy by introducing the ray theory and then classify the trapped waves in detail. As an application of Sections 3-5, we consider a model whose topography consists of a typical continental shelf and a typical continental slope (Section 6). A topography with a trench is treated in Section 7. In Section 8 we apply the results of subinertia waves to superinertia waves by replacing \( S(x) \) by \( 1/h(x) \) which is a measure of the strength of the restoring force of superinertia waves.

2. Equations

We use the linearized long-wave equations governing hydrostatic motion of an inviscid, homogeneous, and incompressible water (with surface elevation \( \eta \) above the undisturbed level \( z=0 \)) overlying the sea floor \( z=-h \). We make the \( f \)-plane approximation, taking \( f \) to be constant and neglecting the curvature of the earth's surface.

We shall be primarily concerned with the topography for which the straight coastline lies at \( x=0 \) and the depth \( h \) is independent of \( y \) and tends to a constant as \( x \) increases towards the deep sea. For waves traveling parallel to the coast with the form \( \eta(x) e^{iy-\omega t} \) (where \( l \) is the alongshore wavenumber and \( \omega \) is the frequency), we have the governing equations for \( \eta \) (see Mysak (1980), for example):

\[
\eta'' + S\eta' + K\eta = 0,
\]

where

\[
S = h'/h,
\]

\[
K = \frac{l}{\omega} S - \frac{\delta^2}{h} (1 - \omega^2).
\]

Here a prime indicates \( d/dx \), and the horizontal velocity \((u, v)\), the time \( t \), the horizontal coordinates \((x, y)\), the surface elevation \( \eta \), and the depth \( h \) are non-dimensionalized by the scales \( U \) (representative horizontal velocity), \( f^{-1}, L \) (shelf width), \( fUL/g \), and \( H \) (the depth far from the coast), respectively, and \( \delta^2 = f^2L^2/(gH) \) is the "divergence parameter".

The boundary conditions \( hu = 0 \) at \( x = 0 \) and \( \eta \rightarrow 0 \) as \( x \rightarrow \infty \) for trapped waves are

\[
h\left( \eta' + \frac{l}{\omega} \eta \right) = 0 \text{ at } x = 0,
\]

\[
\eta \rightarrow 0 \text{ as } x \rightarrow \infty.
\]

In subinertia waves the horizontal divergence effect is unessential because their restoring force is not stretching and shrinking of the vortex tubes due to the fluctuation of sea surface, but that of the vortex tubes oscillating over the inclined bottom. Hence, we assume non-divergence \((\partial u/\partial x = 0)\) in the discussion of subinertia waves and use the stream function \( \psi \) \((hu = \partial \psi/\partial y, \)

\( hv = -\partial \psi/\partial x \) as the dependent variable instead of \( \eta \). Then Eqs. (2.1) and (2.2) reduce to

\[
\psi'' - S\psi' + K\psi = 0,
\]

\[
k = \frac{l}{\omega} S - \frac{\delta^2}{h} (1 - \omega^2),
\]

\[
\psi = 0 \quad (x = 0),
\]

\[
\psi \rightarrow 0 \quad (x \rightarrow \infty).
\]

If we rewrite (2.3) in the standard form using the transformation of the independent variable

\[
\xi = \int_0^x hdx,\]

we have

\[
\frac{d^2\psi}{d\xi^2} + G\psi = 0,
\]

(2.5a)

There are an infinite number of ways to rewrite (2.3) in the standard form if we transform both the independent and dependent variables. Only the transformation (2.4), however, brings (2.3) into the same standard form as that which would be obtained from the equation for \( \eta \) (2.1). Hence, the transformation of the dependent variable (2.4) was used.
Continental Shelf Waves and Continental Slope Waves

Fig. 1. Distribution of the slope parameter $S$, and the sign of $G$ when $S(x)$ is a monotonically decreasing function.

\[ G = h^{-2}K = h^{-2} \left( \frac{I}{\omega} S - l^2 \right) \]
\[ = h^{-2} \frac{I}{\omega} (S - \omega l), \quad (2.5b) \]
\[ \psi = 0 \quad (\xi = 0), \quad (2.5c) \]
\[ \psi \to 0 \quad (\xi \to \infty). \]

We assume $f > 0$ (corresponding to the northern hemisphere) and $l > 0$ for convenience.

3. Consideration from the viewpoint of an eigenvalue problem

In this section we discuss the properties of waves by treating Eq. (2.5) as an eigenvalue problem. Although we treat the problem on the $\xi$-$S$ plane, we can apply the results obtained on the $\xi$-$S$ plane to the $x$-$S$ plane immediately by using (2.4) (note that $\xi$ is a monotonically increasing function of $x$).

Since Eq. (2.5) represent the eigenvalue problem of Sturm-Liouville type for fixed $l$, they have an infinite number of eigenfunctions whose eigenvalues $1/\omega$ are real. In addition, when $S(x)$ is positive in the entire region, the frequency $\omega$ must be positive because the region where the coefficient $G$ is positive may exist only for positive $\omega$ (if $G$ is negative in the entire region, no trapped waves exist; see Footnote 2 for details). These results have been obtained by Huthnance (1975) in the divergent case. In Sections 3–6 we consider the case in which $S(x)$ is positive (i.e., $h(x)$ increases monotonically towards the open ocean), and hence assume $\omega > 0$ in these sections.

We consider the following two classes of $S(x)$ distribution.

(i) When $S(x)$ is a monotonically decreasing function

We denote $S$ at $\xi = 0$ by $S_b$ and $\xi$ for the point at which $S$ is equal to $\omega l$ by $\xi_1$ (Fig. 1) ($\omega l$ must be smaller than $S_b$). Since $G$ is negative in the region $\xi > \xi_1$, the absolute value of the solution of (2.5) decreases monotonically as $\xi$ increases in this region. Hence, roughly speaking, waves may be concluded to be trapped within the region $(0, \xi_1)$. More precise definition of the wave-trapped region will be given in Section 5.

(ii) When $S(x)$ has a maximum

We denote the minimum and maximum points by $(\xi_a, S_a)$ and $(\xi_b, S_b)$, respectively, and the $\xi$-component of the points at which $S = \omega l$, if any, by $\xi_1$, $\xi_2$, and $\xi_3$ ($\xi_1 < \xi_2 < \xi_3$). We assume, for convenience, that $S$ takes the greatest value $S_b$ at $\xi = 0$ (Fig. 2). There are three cases for decreasing $\omega l$ in relation to $S$ value as follows:

(a) $\omega l < S_b$

Waves trapped in the region $(0, \xi_1)$ may exist because $G$ is positive in the region $(0, \xi_1)$ and negative in the region $\xi > \xi_1$.

(b) $S_b < \omega l < S_a$

Since $G$ is negative in the region $(\xi_1, \xi_2)$, Eq. (2.5a) has two linear independent solutions in this region; one whose absolute value increases.

Footnote: If $\omega l > S_b$, that is, $G$ is negative in the entire region, the trapped wave solution of (2.5a) must be a monotonically decreasing function of $\xi$. Such a solution, however, cannot satisfy the boundary condition at $\xi = 0$ (2.5c). Therefore, $\omega l$ must be smaller than $S_b$. 

Fig. 2. Distribution of the slope parameter $S$, and the sign of $G$ when $S(x)$ has a maximum.
monotonically in this region ($\psi_1$) and the other whose absolute value decreases monotonically ($\psi_2$). The eigenfunction in this region is, therefore, expressed by a linear combination of the two solutions, and hence, we have the following two cases. One is when the eigenfunction consists mainly of $\psi_1$ in this region. In this case the absolute value of the eigenfunction is large in the region $(\xi_1, \xi_3)$, and as a result, waves trapped near the maximum may exist. The other is when the eigenfunction consists mainly of $\psi_2$. In this case the absolute value is large in the region $(0, \xi_1)$ and waves trapped near the coast exist. Hereafter, we represent the wave trapped near the coast by CW (Coastal Wave) and that trapped near the maximum by MW (Maximum Wave) for convenience.

(c) $S_0 > \omega l > 0$

Two cases are possible according to whether the solutions are oscillatory or non-oscillatory\(^3\) in the vicinity of $\xi_0$. First for $G$ whose absolute value is small the solutions may be regarded as non-oscillatory in this vicinity. Hence, the two monotonical solutions still exist, and two types of waves may exist as in Case (b). The waves are, however, trapped in the entire region $(0, \xi_1)$ if $S(x)$ varies slowly (see Section 5 (ii) (c)). In contrast, the solutions will be oscillatory even in this vicinity as $\omega l$ decreases (i.e., as $G$ in this vicinity increases). In this case, therefore, the waves are again trapped in the entire region $(0, \xi_1)$.

We define the quantity

$$F = \begin{cases} 0 \sqrt{G} \frac{d\xi}{d\xi} & \text{for Case (i) and (ii) (a), (c)}, \\ \int_0^{\xi_1} \sqrt{G} \frac{d\xi}{d\xi} + \int_{\xi_1}^{\xi_2} \sqrt{G} \frac{d\xi}{d\xi} & \text{for Case (ii) (b)}, \end{cases}$$

(3.1)

which is a measure of the intensity of oscillation (i.e., number of node) of the eigenfunctions. We denote the $n$-th mode of $F$ by $F_n$. Since the $n$-th mode eigenfunction has the same node $n$, $F_n$ must be kept almost constant with change of wavenumber $l$. In addition we have the relation

$$F_1 < F_2 < \ldots < F_n < F_{n+1} < \ldots$$

(3.2)

because higher modes have more nodes. When $\omega$ decreases for fixed $l$, the integrated region of (3.1) (i.e., the oscillatory region of the eigenfunction) becomes wider and the value of $G$ there increases according to (2.5b), and as a result, $F$ increases. It is concluded from (3.2), therefore, that the higher modes have the wider wave-trapped region for fixed $l$.

Next we look at the $n$-th mode eigenfunction. Using (2.5b), we can rewrite (3.1) as

$$F_n/l = \int_0^{\xi_n} \sqrt{G/\omega l} - 1/\eta d\xi$$

(3.3)

for Case (i) and (ii) (a), (c) ($\xi_n$ and $\omega_n$ denote $\xi_1$ and $\omega$ of the $n$-th mode). A similar equation can be obtained also for Case (ii) (b) by changing the integrated region $(0, \xi_n)$ to regions $(0, \xi_n)$ and $(\xi_{2n}, \xi_{3n})$. When $l$ increases, the right-hand side of Eq. (3.3) must be smaller ($F_n$ is kept almost constant against the change of $l$ as stated before). Since the decrease of the integrated region reflects the increase of $\omega_n l$, the wave is found to be trapped in a narrower region as the wavenumber increases. At the high wavenumber limit, moreover, $\omega_n l$ approaches to $S_0$ (for CW's) or $S_0$ (for MW's), and the waves are trapped in the infinitesimal region that includes the coast or the maximum because CW's and MW's may exist for $\omega_l$ smaller than $S_0$ and $S_0$, respectively. In other words, the dispersion curves for the waves trapped near the coast (maximum) have asymptote $\omega l = S_0$ ($S_0$) for high wavenumbers. If $S_0$ ($S_0$) is finite, therefore, the frequency $\omega_n$ for CW's (MW's) approaches zero at the high wavenumber limit. This may be explained physically as follows; if $S_0$ or $S_0$ is finite, the waves can no longer have the restoring force due to stretching and shrinking of the vortex tubes that oscillate in the infinitesimal region, and the frequency drops to zero. If $S_0$ or $S_0$ is infinite (i.e., $\eta \to 0$ at $x = 0$ or $dh/dx \to \infty$ at $x = x_0$), in contrast, the waves may have the restoring force so far as they are trapped in the region containing $x = 0$ or $x_0$, and the frequency may remain finite.

Here, we will give an example for Case (ii), using the bottom profile

$$h(x) = \begin{cases} H_1 e^{\alpha x} & (0 \leq x \leq L_1), \\ H_2 e^{\beta(x-L_1)} & (L_1 \leq x \leq L_2), \\ H_3 e^{\gamma(x-L_2)} & (L_2 \leq x \leq 1), \\ 1 & (1 \leq x), \end{cases}$$

(3.4)
where $H_1e^{p_1x}x=H_2$, $H_2e^{p_2x}x=H_3$ and $H_3$ $e^{p_3x}x=1$ so that the depth is continuous at $x=L_4$, $L_5$ and $1$. The corresponding $S$ distribution is shown in Fig. 3. Figure 4 shows the analytically derived dispersion relations of the first eight modes for $b_1=5$, $b_2=1$, $b_3=3$, $L_4=0.2$, and $L_5=0.6$. Curves $al=b_1$, $al=b_2$, and $al=b_3$ are also shown by dotted lines. The domain below $al=b_1$ and above $al=b_2$ corresponds to Case (a) and that below $al=b_3$ and above $al=b_2$ to Case (b) and that below $al=b_2$ corresponds to Case (c). Two types of dispersion curves can be seen below the curve $al=b_2$. Between them the dispersion curves which have the asymptote $al=b_2$ for high wavenumbers correspond to the waves trapped within the region $(0, L_4)$ (CW’s) and those which have the asymptote $al=b_2$ correspond to the waves trapped within the region $(L_4, 1)$ (MW’s) according to the conclusions obtained above.

4. “Kissing”

Let us consider the $S$ distribution in Section 3 (ii) and the case in which $al$ coincides with $S_b$, and let us define the quantity

$$F_b = \int_0^{t_b} \sqrt{S} \, d\xi$$

$$= \int_0^{t_b} \sqrt{S/S_b-1} \, d\xi,$$  \hspace{1cm} (4.1)

by using (2.5b) and the relation $al=S_b$. Since MW’s may exist for $al$ smaller than $S_b$, $F_b$ for MW’s must be larger than $F_b$ ($F$ is a monotonically decreasing function of $\omega$ for fixed $l$; see Section 3 (ii)). According to (4.1), $F_b$ is an increasing function of $l$. It is concluded, therefore, that MW’s may appear in higher modes for larger wavenumbers. We can see in Fig. 4, for example, that the lowest mode for MW’s, which was the 1st mode at $l=10$, changes into the 2nd mode at $l=12$, and the dispersion curves are intersected in between. Since the eigenfunctions of the eigenvalue problem of Sturm-Liouville type do not degenerate, the two dispersion curves do not intersect each other, but “kissing” (i.e., nearly intersect).

Next we consider the eigenfunctions when “kissing” occurs, taking the eigenfunctions of the 1st and the 2nd modes at $l=11.6$, $12.6$, and $13.6$ (these values are indicated by three vertical dotted lines in Fig. 4) shown in Fig. 5 as an example. It can be seen in Fig. 5 that the properties of the eigenfunctions of the 1st (solid lines) and the 2nd (dotted lines) modes are reversed when $l$ is changed slightly. Moreover at $l=12.6$ (when “kissing” occurs), the eigenfunctions are of such a form that they are trapped both near the coast and near the maximum. We can see this tendency more clearly in the energy density distribution $E(x) = (1/2) \int_0^\infty h(u^2 + v^2) dx$ (normalized by $E(x=1)$) in Fig. 5. We separate the wave-trapped region into three,
Region I or III although their absolute values are almost the same. Namely, the two eigenfunctions may be concluded to have quite different characteristics throughout the entire wave-trapped region.

5. Application of "ray theory"

In this section the meaning of wave trapping is clarified by introducing the ray theory, i.e., wave trapping is considered from the viewpoint of energy because the energy of 'elementary waves' is carried along a ray path. We adopt topographic Rossby waves (Veronis, 1966) of the form \( \phi = h^{1/2} e^{i(x+yt+\alpha t)} \) which exist for constant \( S \) (exponential topography) as elementary waves. The dispersion relation is

\[
\frac{l}{\omega} S - \frac{1}{4} S^2 - k^2 - \beta^2 = 0.
\]

The condition that the waves exist (i.e., a real wavenumber in the \( x \)-direction, \( k \), exists) for a given frequency \( \omega \) and wavenumber in the \( y \)-direction \( l \), can be obtained by replacing \( k \) and the equal sign in (5.1) by 0 and inequality, respectively. Then we have

\[
\Omega_1 < S < \Omega_3,
\]

where

\[
\Omega_1 = 2 \frac{l}{\omega} \left( 1 - \sqrt{1 - \omega^2} \right),
\]

\[
\Omega_3 = 2 \frac{l}{\omega} \left( 1 + \sqrt{1 - \omega^2} \right).
\]

For small \( \omega \), \( \Omega_1 \) and \( \Omega_3 \) can be approximated by

\[
\Omega_1 \approx \omega l,
\]

\[
\Omega_3 \approx 4 \frac{l}{\omega} - \omega l,
\]

using the approximation

\[
\sqrt{1 - \omega^2} \approx 1 - \frac{1}{2} \omega^2.
\]

Here, we assume that \( S(x) \) varies slowly within a wavelength of elementary waves and that trapped waves may be constructed by the elementary waves. Then, according to the ray theory (see LeBlond and Mysak (1978) for details), we obtain the equation

\[
\frac{l}{\omega} S - \frac{1}{4} S^2 - k^2 - \beta^2 = 0.
\]
\[
\frac{dx}{dt} = -\frac{\partial \omega}{\partial k}, \tag{5.3}
\]

where \(x\) denotes the \(x\)-component of the ray path (i.e., the path of the elementary waves) and \(d/dt\) means the time derivative along the path. In addition the equations

\[
\frac{dk}{dt} = \frac{\partial \omega}{\partial S} \frac{\partial S}{\partial x}, \tag{5.4a}
\]

\[
\frac{dl}{dt} = \frac{\partial \omega}{\partial S} \frac{\partial S}{\partial y}, \tag{5.4b}
\]

\[
\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} \tag{5.4c}
\]

hold. Equations (5.4b, c) show that the alongshore wavenumber \(l\) and the frequency \(\omega\) are conserved along the path because \(S\) is a function of \(x\) only. Hence, we can regard them as equal to those for the trapped wave.

Substituting (5.1) into (5.3) and (5.4a), we obtain, after some rearrangement, the equations

\[
\frac{dx}{dt} = 2\frac{\omega^3}{lS}, \tag{5.5}
\]

and

\[
\frac{d^2x}{dt^2} = R, \tag{5.6}
\]

where

\[
R(x) = \frac{2\omega^3}{lS}\frac{S'(\Omega_2 - S)},
\Omega_2 = 2\omega l.
\]

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Fig. 6. Distribution of the slope parameter \(S\), and the sign of \(R\) when \(S(x)\) is a monotonically decreasing function.

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These equations describe the velocity and the acceleration of the elementary waves in the direction normal to the coast.

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Fig. 7. Distribution of the slope parameter \(S\), and the sign of \(R\) when \(S(x)\) has a maximum.
We consider three different cases for the $S(x)$ distribution.

(i) When $S(x)$ is a monotonically decreasing function (Fig. 6)

We denote $x$ for the points at which $S=\Omega_i$ by $x_i$ ($i=1, 2, 3$) (Fig. 6). The elementary waves exist in the region $(x_3, x_1)$ according to (5.2), and they are accelerated in the positive direction in the region $(x_3, x_2)$ and in the negative direction in the region $(x_2, x_1)$ according to (5.6).

In addition, the wave velocity in the $x$-direction is zero at $x=x_3$ and $x_1$ according to (5.5) because $k=0$ at these points. We can conclude, therefore, that the elementary waves propagate within the region $(x_3, x_1)$ changing their direction at $x_1$ and $x_3$, and as a result, trapped waves may be formed in the region $(x_3, x_1)$. From (5.2) the inequalities

$$\frac{\partial \Omega_1}{\partial \omega} = \frac{2I}{\omega^2 \sqrt{1-\omega^2}} (1 - \sqrt{1-\omega^2}) > 0, \quad (5.7a)$$

$$\frac{\partial \Omega_3}{\partial \omega} = \frac{2I}{\omega^2 \sqrt{1-\omega^2}} (-1 - \sqrt{1-\omega^2}) < 0 \quad (5.7b)$$

are derived. Hence, if we define a wave-trapped region as the region where elementary waves exist, we can conclude that for fixed $I$ it is wider for smaller $\omega$, namely for higher modes because the decrease of $\Omega_1$ reflects the increase of $x_1$ and the increase of $\Omega_3$ reflects the decrease of $x_3$. This result coincides with that obtained in Section 3. When $\omega$ becomes smaller, $\Omega_3$ becomes greater than $S_6$. Then the elementary waves are reflected at the coast and the waves are trapped in the region $(0, x_1)$. In this case the wave-trapped region is found to agree approximately with the region defined in Section 3 because $\Omega_1$ is almost equal to $\omega l$.

(ii) When $S(x)$ has a maximum

We denote the minimum and maximum points by $(x_a, S_a)$ and $(x_b, S_b)$, respectively. We consider the following three cases for decreasing $\omega$ in relation to $S$ value.

(a) $\Omega_1 > S_6$ (Fig. 7a)

Only CW's may exist because the elementary waves propagate in the region $(x_3, x_1)$ (or the region $(0, x_1)$).

(b) $S_b > \Omega_1 > S_6$ (Fig. 7b)

Since the elementary waves exist in two separate regions; near the coast and near the maximum, both CW's and MW's are possible.

(c) $S_2 > \Omega_1$ (Fig. 7c)

The elementary waves propagate in the region $(x_3, x_1)$, and as a result, trapped waves may

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Fig. 8. (a) Distribution of the slope parameter $S$ when it is expressed by a polynomial of the third degree in the shelf slope.

(b) Dispersion relations of the first eight modes for the $S$ distribution in (a) with $S_a=10$, $S_b=1.54$, $S_6=3.86$, $x_a=0.35$, and $x_b=0.75$. Curves $\Omega_1=S_6$, $\Omega_1=S_a$, and $\Omega_3=S_6$ are shown by dotted lines.
again exist in the entire region \((x_3, x_1)\). This is, however, only valid on the assumption that \(S(x)\) varies slowly. If \(S(x)\) varies quickly, two types of waves may exist even in this case (see Section 3 (ii) (c)).

As an example we use the \(S\) distribution expressed by a polynomial of the third degree to represent the continental shelf slope (Fig. 8a). Figure 8b shows the analytically derived dispersion relations (series solution exist in the shelf slope for this \(S\) distribution; see Brauer and Nohel (1973) for details) of the first eight modes for \(S_0=10\), \(S_0=1.54\), \(S_0=3.86\), \(x_0=0.35\), and \(x_0=0.75\). Curves \(\Omega_1=S_0\), \(\Omega_1=S_0\), and \(\Omega_1=S_0\) are shown by dotted lines. The domain above \(\Omega_1=S_0\) corresponds to Case (a) and that below \(\Omega_1=S_0\) and above \(\Omega_1=S_0\) to Case (b) and that below \(\Omega_1=S_0\) corresponds to Case (c). We can clearly see two types of dispersion curves below \(\Omega_1=S_0\) and above \(\Omega_1=S_0\). Among them the curves that have the asymptote \(\omega/L_1=S_0\) for high wavenumbers correspond to MW's and those that have the asymptote \(\omega/L_1=S_0\) to CW's according to the conclusion obtained in Section 3. Figure 9 shows the eigenfunctions and the energy density distribution for the first eight modes at \(l=10\). This value is indicated by a dot-dashed line in Fig. 8b. There are no eigenfunctions corresponding to Case (a) (see Fig. 8b). According to Fig. 8b, the eigenfunctions for the first three modes correspond to Case (b) and the others to Case (c). We can see in Fig. 9 that among the first three modes (the modes for Case (b)), the 2nd mode corresponds to CW's and the others to MW's.

(d) \(S_0>\Omega_0>\Omega_1>S_0\) (Fig. 7d)

The case where \(S_0>\Omega_0>\Omega_1>S_0\) is possible for sufficiently large \(S_0\) and sufficiently small \(S_0\). In this case there are three separate regions where elementary waves exist; the right-hand side of \(x_0\), the left-hand side of \(x_0\), and near the coast. Hence, three types of trapped waves may exist in each region.

As an example the \(S\) distribution shown in Fig. 10a was chosen. There are no waves trapped near the coast in this model. Figure 10b shows the analytically derived dispersion relation for \(S_0=80\) and \(x_0=0.5\). The domain above \(\Omega_3=S_0\) (a dotted line) corresponds to Case (d) according to (5.7b) \((S_0=0\) in the present \(S\) distribution). Two types of dispersion curves can

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Fig. 9. Distributions of stream function \(\psi\) and energy density \(E\) for the first eight modes at \(l=10\) in Fig. 8b. Stream function and energy density are normalized by \(\int_0^\infty Edx=1\).
Fig. 10. (a) Distribution of the slope parameter $S$ taken up as an example for Case (d). (b) Dispersion relations of the first eight modes for the $S$ distribution in (a) with $S_b=80$ and $x_0=0.5$. Curve $Q_5=S_5$ is shown by a dotted line.

Fig. 11. Distribution of energy density for the first six modes at $l=10$ in Fig. 10b. Energy density is normalized by $\int_0^\infty Edx=1$.

be seen there. Figure 11 shows the energy density distribution for the first six modes at $l=10$. According to Fig. 10b, the first four modes correspond to Case (d). Figure 11 shows that among them the 2nd and 4th modes correspond to the waves trapped on the right-hand side of the maximum and the others correspond to the waves trapped on the left-hand side. Note that in Fig. 8 there are no waves corresponding to Case (d).

(iii) When $S(x)$ has a stationary inflection point

We denote the stationary inflection point by $(x_0, S_0)$. First, we give an example, taking the $S$ distribution shown in Fig. 12a which is expressed by a polynomial of the third degree. The corresponding dispersion relations for $x_0=0.6, S_0=10$, and $S_b=2.29$ are shown in Fig. 12b and the energy density distributions for the first six modes at $l=10$ in Fig. 13. Note that in Fig. 13 the 2nd mode corresponds to the wave trapped in the vicinity of $x_0$. The dispersion curve corresponding to this mode can be seen just below
the dashed line $Q_1=S_a$ in Fig. 12b. Near the stationary inflection point, the elementary waves are hardly accelerated in the $x$-direction according to (5.6) because $S'$ is almost zero in this vicinity ($S'$ and $S''$ are zero at $x_a$). It can be supposed, therefore, that elementary waves may exist that propagate almost straight and parallel to the coast near $x_a$, and trapped waves consisting of such elementary waves may be formed. The elementary wave velocity in the $x$-direction is equal to zero when $\omega$ and $l$ of the trapped wave satisfy the relation $Q_1=S_a$ exactly. Therefore, the dispersion curves for these trapped waves appear just below the curve $Q_1=S_a$ because the elementary waves that form these waves should have little velocity in the $x$-direction.

6. Continental shelf waves and continental slope waves

We consider the following bottom profile that consists of a typical continental shelf and a typical continental slope:

$$h(x) = \begin{cases} \quad p_1 x & (0 \leq x \leq L_1), \\ \quad H_1 + p_2 (x-L_1) & (L_1 \leq x \leq 1), \\ \quad 1 & (1 \leq x). \end{cases} \quad (6.1)$$

The topography is illustrated in Fig. 14a and the corresponding $S$ distribution in Fig. 14b. Since $S(x)$ in Fig. 14b has a maximum at $x=L_1$, two types of waves may exist, i.e., those trapped over the continental shelf (near the coast) and those trapped over the continental slope (near the maximum).

We can obtain the dispersion relation of the waves that exist in the bottom profile (6.1) analytically (see Appendix). Figure 15 shows the dispersion relations of the first eight modes for $H_1=0.05$ and $L_1=0.4$ (solid lines). According to the conclusions obtained in Section 3, the dispersion curves for which $\omega$ remains finite at $l \rightarrow \infty$ correspond to the waves trapped over the continental shelf (continental shelf waves) and those for which $\omega \rightarrow 0$ as $l \rightarrow \infty$ correspond to the waves trapped over the continental slope (continental slope waves).

We may take Mysak’s model (1968) where the continental slope is approximated by a cliff as a special case of this model. Letting $L_1$ approach 1 in our model, we find that only the frequency for the lowest mode of continental slope waves remains finite and that for higher modes goes down to zero. This can be confirmed easily by computations for various values of $L_1$. It can be found in the dispersion relation in Fig. 1 of Mysak’s paper (1968) that only the 1st mode corresponds to the continental slope wave and the modes equal to or higher than the 2nd correspond to continental shelf waves. (Compare the limiting frequency values for infinitely large wavenumbers with those in
Fig. 13. Distribution of energy density $E$ for the first six modes at $l=10$ in Fig. 12b. Energy density is normalized by $\int_0^\infty Edx = 1$.

Reid's (1958) and Larsen's (1969) models.)

Furthermore, we can obtain Robinson's model (1964) if we replace the open ocean in Mysak's model (1968) by the infinitely deep ocean. However, an infinitely strong restoring force occurs in this model due to stretching and shrinking of the vortex tubes that oscillate near the infinitely high cliff. Therefore, no solutions for continental slope waves (waves trapped near the cliff) exist in Robinson's formulation in which the eigenfunctions are assumed to be finite in the entire region.

Fig. 14. (a) The bottom profile expressed by (6.1) and (b) the corresponding $S$ distribution.

Fig. 15. Dispersion relations (A6) of the first eight modes for $h$ and $S$ profiles in Fig. 14 with $L_z=0.4$, $H_z=0.05$, $p_z=0.05$, $p_x=1.58$, and $\delta^2=0$ (solid lines) and $\delta^2=0.1$ (dotted lines).
divergence effect of the sea surface. Therefore, the rate of stretching and shrinking of the vortex tubes in superinertia waves is proportional to $1/h$ while that of subinertia waves is proportional to the slope parameter $\xi$. Hence, we may anticipate that we will obtain similar results for superinertia waves if we replace $\xi$ by $1/h$.

First we consider the properties of waves from the viewpoint of an eigenvalue problem. As the governing equations, we use the Eqs. (2.1) and (2.2) written in dimensional form:

$$\frac{d^2}{dt^2} + \frac{K}{h} \frac{d}{dt} + K\eta = 0,$$

$$h^a K \approx h^b\left(\frac{\omega^2-f^2}{gh} - \ell^2\right), \quad (8.1)$$

$$h(f\eta_p + \omega^2) = 0 \quad at \quad x=0,$$

$$|\eta| < \infty \quad at \quad x=\infty.$$

By transforming the independent variable, we bring (8.1) into the standard form

$$\frac{d^2\eta}{dt^2} + G\eta = 0,$$

where

$$G = h^a K \approx h^b\left(\frac{\omega^2-f^2}{gh} - \ell^2\right) = \frac{h}{\ell}\left(\omega^2-f^2-gh\ell^2\right). \quad (8.2)$$

In deriving the above equation, we neglected the second term in $K$. Since this term is due to the rotation of the earth, it may be regarded as small in edge waves for which the rotational effect is unessential.

We consider, first, the case where $h(x)$ increases monotonically (i.e., $1/h$ decreases monotonically) (Fig. 18). There are three cases for increasing $\omega^2-f^2$ in relation to the value of $gh\ell^2$. First for negative $\omega^2-f^2$, $G$ is negative in the entire region, and hence, only the zero mode (the mode for Kelvin type waves) may exist. When $0<\omega^2-f^2<gh\ell^2$ (where $H$ is the depth at $x=\infty$), waves trapped in the region $(0, \xi_1)$ may exist because $G$ is positive in the region $(0, \xi_1)$ and negative in the region $\xi_1<\xi$, where $\xi_1$ denotes the $\xi$-component of the point at which $gh\ell^2=\omega^2-f^2$. For $\omega^2-f^2$ greater than $gh\ell^2$, $G$ is positive in the entire region. Hence, the dispersion curves for this case are continuous (i.e., not discrete) spectrum because there are two independent finite solutions at $x=\infty$.

When $h(x)$ has a minimum (i.e., $1/h(x)$ has a maximum), two types of waves are possible, i.e., those trapped near the coast and those trapped near the minimum as in subinertia waves.

Next we apply the ray theory to the superinertia waves. We proceed in a similar way as for subinertia waves. Adopting gravity waves at a constant depth as elementary waves, we have the following equations corresponding to (5.1), (5.5), and (5.6):

$$\frac{dx}{dt} = -\frac{k}{\omega} h,$$

$$\frac{d^2x}{dt^2} = R,$$ \hspace{1cm} (8.3)

where

$$R = \frac{\omega^2-f^2}{gh} - \ell^2 > 0.$$ \hspace{1cm} (8.4)

The condition that the elementary waves exist for a given frequency, $\omega$, and wavenumber in the $y$-direction, $l$, is

$$\frac{\omega^2-f^2}{gh} - \ell^2 > 0.$$ \hspace{1cm} (8.4)

We denote the $x$-component of the points at which $gh\ell^2=\omega^2-f^2$ and $gh\ell^2=(\omega^2-f^2)/2$ by $x_1$ and $x_2$, respectively (Fig. 19). According to (8.3) and (8.4), the elementary waves exist within the region $(0, x_1)$ when $0<\omega^2-f^2<gh\ell^2$. In addition they are accelerated in the positive direction in the region $(0, x_2)$ and in the negative
divergence effect of the sea surface. Therefore, the rate of stretching and shrinking of the vortex tubes in superinertia waves is proportional to 1/h while that of subinertia waves is proportional to the slope parameter S. Hence, we may anticipate that we will obtain similar results for superinertia waves if we replace S by 1/h.

First we consider the properties of waves from the viewpoint of an eigenvalue problem. As the governing equations, we use the Eqs. (2.1) and (2.2) written in dimensional form:

\[ \eta'' + \frac{h'}{h} \eta' + K \eta = 0, \]
\[ K = \frac{\omega^2 - f^2}{gh} + \frac{fl}{h} - l^2, \]
\[ h(\omega \eta + \omega \eta') = 0 \text{ at } x=0, \]
\[ |\eta| < \infty \text{ at } x \to \infty. \]

By transforming the independent variable, we bring (8.1) into the standard form

\[ \frac{d^2 \eta}{dz^2} + G \eta = 0, \]

where

\[ G = \frac{h}{\omega^2} = \frac{h}{(\omega^2 - f^2) - ghP}. \]

In deriving the above equation, we neglected the second term in K. Since this term is due to the rotation of the earth, it may be regarded as small in edge waves for which the rotational effect is unessential.

We consider, first, the case where \( h(x) \) increases monotonically (i.e., 1/h decreases monotonically) (Fig. 18). There are three cases for increasing \( \omega^2 - f^2 \) in relation to the value of \( ghP \). First for negative \( \omega^2 - f^2 \), \( G \) is negative in the entire region, and hence, only the zero mode (the mode for Kelvin type waves) may exist. When \( 0 < \omega^2 - f^2 < ghP \) (where \( H \) is the depth at \( x \to \infty \)), waves trapped in the region \( (0, \xi) \) may exist because \( G \) is positive in the region \( (0, \xi) \) and negative in the region \( \xi > \xi_1 \), where \( \xi_1 \) denotes the \( \xi \)-component of the point at which \( ghP = \omega^2 - f^2 \). For \( \omega^2 - f^2 \) greater than \( ghP, \) \( G \) is positive in the entire region. Hence, the dispersion curves for this case are continuous (i.e., not discrete) spectrum because there are two independent finite solutions at \( x \to \infty \).

When \( h(x) \) has a minimum (i.e., 1/h(x) has a maximum), two types of waves are possible, i.e., those trapped near the coast and those trapped near the minimum as in subinertia waves.

Next we apply the ray theory to the superinertia waves. We proceed in a similar way as for subinertia waves. Adopting gravity waves at a constant depth as elementary waves, we have the following equations corresponding to (5.1), (5.5), and (5.6):

\[ \omega^2 = f^2 + g(k^2 + P)h, \]
\[ \frac{dx}{dt} = -g \frac{\omega^2}{\omega f} h, \]
\[ \frac{d^2 x}{dt^2} = R, \]

where

\[ R = \frac{\omega^2}{gh} \left( \frac{\omega^2 - f^2}{2} - ghP \right). \]

The condition that the elementary waves exist for a given frequency, \( \omega \), and wavenumber in the \( y \)-direction, \( k \), is

\[ \omega^2 = f^2 + g(k^2 + P)h > 0. \]

We denote the \( x \)-component of the points at which \( ghP = \omega^2 - f^2 \) and \( ghP = (\omega^2 - f^2)/2 \) by \( x_1 \) and \( x_2 \), respectively (Fig. 19). According to (8.3) and (8.4), the elementary waves exist within the region \( (0, x_1) \) when \( 0 < \omega^2 - f^2 < ghP \). In addition they are accelerated in the positive direction in the region \( (0, x_2) \) and in the negative
direction in the region \((x_0, x_1)\). Therefore, the ray theory provides the same result as the eigenvalue problem. When \(\omega^2 - f^2 > gH^2\), the elementary waves exist in the entire region, and the wave-trapped region becomes infinite. This corresponds to the case in which the dispersion curves are continuous.

We give an example of the case where \(h(x)\) has a minimum using the following bottom profile:

\[
h(x) = \begin{cases} 
H_1 & (0 \leq x \leq L_1), \\
H_2 & (L_1 \leq x \leq L_2), \\
H_3 & (L_2 \leq x \leq L), \\
H & (L \leq x).
\end{cases}
\]  

(8.5)

Figure 20 shows the analytically derived dispersion relations for \(H_1/H = 0.1\), \(H_2/H = 0.8\), \(H_3/H = 0.4\), \(L_1/L = 0.2\), \(L_2/L = 0.6\), and \(\delta^2 = f^2L^2/(ghH) = 0.5\) (solid lines). Curves \(\omega^2 - f^2 = gHL^2\) and \(\omega^2 - f^2 = gH^2\) are also shown by dot-dashed lines and the dispersion relations for the non-rotational case are shown by dotted lines for comparison. Two types of dispersion curves can be seen between the curves \(\omega^2 - f^2 = gHL^2\) and \(\omega^2 - f^2 = gH^2\). Among them, the dispersion curves with slower phase velocity, which can also be seen between the two curves \(\omega^2 - f^2 = gHL^2\), correspond to the waves trapped in the region \((0, L_1)\). It can be seen that the dispersion curves for superinertia waves are hardly influenced by the rotational effect even for the large value \(\delta^2 = 0.5\) as anticipated before.

9. Summary and discussion

We have investigated various properties of topographically trapped waves by paying particular attention to the distribution of the slope parameter \(S\) or the depth \(h\). The conclusions obtained are as follows.

First for subinertia waves, waves are trapped near the coast when \(S(x)\) decreases monotonically. When \(S(x)\) has a maximum, two types of waves may exist; those trapped near the coast and those trapped near the maximum (Section 3, Section 5).

The dispersion curves for each type of wave have an asymptote \(\omega = S_0\) or \(\omega = S_h\) for high wavenumbers. All the waves with a finite mode number are trapped near the coast (maximum) at the high wavenumber limit if \(S_0(S_h)\) is greater than \(S_h(S_0)\). In addition, each type of wave is trapped within the infinitesimal region including the coast or the maximum at this limit (Section 3).

When \(S(x)\) has a maximum, the dispersion
curves for different types of waves may result in "kissing". When "kissing" occurs, both waves are trapped both near the coast and near the maximum. The eigenfunctions for the two waves are almost the same in magnitude in both regions, but have different signs in one of the regions (Section 4).

For a topography that consists of a typical continental shelf and a typical continental slope, waves trapped over the continental slope exist as well as those trapped over the continental shelf (Section 6).

Almost the same conclusions are obtained for superinertia waves if $S$ is replaced by $1/h$ (Section 8).

We may here remark on the effect of horizontal divergence in subinertia waves which has been ignored in this paper. In Fig. 15 the dispersion relations for the topography in Section 6 with $\alpha = 0.1$ are shown by dotted lines. The other parameters are the same as for the nondivergent case. Although the value of $\alpha = 0.1$ is larger than the typical value in middle latitudes, such value was chosen to show qualitative tendencies more clearly. Since continental shelf waves are trapped in shallower regions compared with continental slope waves, they are influenced more strongly by the horizontal divergence according to (2.1). This can be seen, for example, in the case of the 1st and the 2nd modes near $l = 33$ (near the point at which "kissing" occurs) in Fig. 15.

We have so far considered a semi-infinite ocean. However, we could, in a similar manner, deal with the case where the ocean spreads from $x = -\infty$ to $\infty$ such as an escarpment (Longuet-Higgins, 1968a, b).

Bane (1980) found two types of topographically trapped subinertia waves in a model for the ocean off North Carolina. Although his model includes the mean current and front, he regarded the waves as topographic waves. He then classified them as continental shelf waves and quasi-geostrophic edge waves, considering that the former are generated by the presence of the continental slope with finite width and the latter by the presence of the constant-slope shelf. However, the present results which show that the character of topographically trapped subinertia waves is determined by the distribution of the slope parameter suggests that Bane's classification is not appropriate.

Kubota (1982) computed waves generated off the Fukushima coast using a similar model to that in Section 6. He then arrived at the conclusion that near the coast the 2nd mode wave was excited most strongly in the long wavelength range (non-dispersive range). The fact that his model may contain both continental shelf waves and continental slope waves would imply that in his result the 1st mode corresponds to continental slope waves and the 2nd mode corresponds to continental shelf waves in the long wavelength range.

Continental shelf waves have been detected several times over the past two decades since Shoji (1961) and Hamon (1962) observed them along the Japanese and Australian coasts. The detection of continental slope waves, however, has hardly been reported because most reports were based on records near the coast such as tidal level records. (Continental slope type waves seem to appear in Kubota, et al. (1981) observational results.) Since continental slope waves accompany larger motion over the continental slope and smaller motion near the coast, more observations over the continental slope will be necessary for the detection of continental slope waves.

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References


Appendix

Derivation of the dispersion relation of the waves existing in the bottom profile (6.1)

The governing equations can be written separately in three regions.

(i) \(0 \leq x \leq L_1\)

Transforming the governing equation (2.1) with the bottom profile (6.1) by \(\eta = \zeta e^{-i\nu} \) an \(x = 2Lx\), we obtain the confluent hypergeometric (or Kummer's) differential equation (see Mysak (1968)):

\[
X \frac{d^2\zeta}{dx^2} + (1 - X) \frac{d\zeta}{dx} + \nu_1 \zeta = 0,
\]

where

\[
\nu_1 = \frac{1}{2} \left(1 - \frac{\partial^2}{\partial \nu^2} (1 - \omega^2) - 1 \right).
\]

(ii) \(L_1 \leq x \leq 1\)

We make the coordinate transformation \(x' = x - (L_1 - H_1/p_2)\) and \(x' = x\). Then the region \(L_1 \leq x \leq 1\) is transformed into \(L_1' \leq x' \leq L_1'\), where \(L_1' = H_1/p_2\) and \(L_1' = 1 - L_1 - L_1'\), and the governing equation reduces to the same form equation as that in (i). Therefore, transforming by \(\eta = \zeta e^{-i\nu} \) and \(X = 2Lx'\), we obtain

\[
X \frac{d^2\zeta}{dx'^2} + (1 - X) \frac{d\zeta}{dx'} + \nu_2 \zeta = 0,
\]

where

\[
\nu_2 = \frac{1}{2} \left(1 - \frac{\partial^2}{\partial \nu^2} (1 - \omega^2) - 1 \right).
\]

(iii) \(1 \leq x\)

we have

\[
\frac{d^2\eta}{dx^2} - \alpha^2 \eta = 0,
\]

where

\[
\alpha = \sqrt{\partial^2 + \partial^2 (1 - \omega^2)}.
\]
The boundary conditions for \( \eta \) are

\[
\eta: \text{regular at } x=0, \\
\eta \to 0 \text{ as } x \to \infty,
\]

and the connecting conditions at \( x=L_1 \) and 1 are

\[
\eta: \text{continuous}, \\
\frac{d\eta}{dx}: \text{continuous}.
\]

We denote a regular solution at \( X=0 \) by \( M_i(X) \) (\( i=1, 2 \); where \( i=1 \) represents the quantity in \( 0 \leq x \leq L_1 \) and \( i=2 \) represents that in \( L_1 \leq x \leq 1 \)) and a non-regular one by \( U_i(X) \) between the two linear independent solutions of (A1) and (A2) (see Abramowitz and Stegun (1965) for precise expressions of \( M_i(X) \) and \( U_i(X) \)). From (A1)-(A5), we obtain the following dispersion relation:

\[
\left\{ M_1(2L_1') \frac{dM_2(2L_1')}{dX} - \frac{dM_1(2L_1)}{dX} M_2(2L_1') \right\} \\
\left\{ \frac{1}{2} \left( 1 - \frac{\alpha}{1} \right) U_2(2L_1') - \frac{dU_2(2L_1')}{dX} \right\} \\
- \left\{ M_1(2L_1) \frac{dU_2(2L_1')}{dX} - \frac{dM_1(2L_1)}{dX} U_2(2L_1') \right\} \\
\left\{ \frac{1}{2} \left( 1 - \frac{\alpha}{1} \right) M_2(2L_1') - \frac{dM_2(2L_1')}{dX} \right\} = 0,
\]

where \( \nu_2 \equiv 1, 2, 3, \ldots \).

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大陸棚および大陸斜面に捕捉された地形性波動

竹田 宏*

要旨：地形に捕捉された超慣性波動の一般的性質を、固有価問題およびレイ理論の概念を用いることにより考察する。ここでは、波動の復元力の指数と考えられるスロープ・パラメーター \( S(x) \) の性質を水深 \( h(x) \) の影響を考慮することによって、3つの \( S \) の分布について考察する。ただし、\( S \) は岸において正で、岸から遠く離れた所では 0 になるものとする。最初に、\( S(x) \) が極値に向かって単調に減少する場合を扱う。この場合において、波動は岸近くに捕捉されることが示される。2 番目は、\( S(x) \) が単調に増加せず、極大を持つ場合を扱う。この場合において、岸近くに捕捉された波動は極大付近に捕捉された波動の 2 つのタイプの波動の存在が得られる。また、異

* 東京大学工学部物理工学科

〒 113 東京都文京区本郷 7-3-1