A Theory of Semigeostrophic Gravity Waves and its Application to the Intrusion of a Density Current along a Coast

Part 2. Intrusion of a Density Current along a Coast in a Rotating Fluid*

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Abstract: The intrusion of a density current along a coast in a rotating fluid is investigated both theoretically and experimentally. The theoretical model is a shock wave solution of a semigeostrophic gravity wave which was investigated in Part 1 (Kubokawa and Hanawa, 1984). In the experimental results, the propagation speed of the leading edge of light fluid along a vertical boundary and the current width and depth are nearly equal to those estimated by the shock wave theory. The generation of a frontal wave at the leading edge of the density current is observed. The propagation velocity of these frontal waves agrees well with that of theoretically-predicted semigeostrophic gravity waves.

1. Introduction

Yoon and Suginoara (1977) numerically investigated the behavior of warm water flowing into a cold ocean through a sea strait on a β-plane in the northern hemisphere. They reported that the warm water has a leading edge which advances with the coast on its right at a propagation speed that is nearly equal to the phase speed of a long gravity wave. They also suggested that the dynamics of several coastal boundary currents which have large seasonal variation in mass transport may be similar to that of their numerical model (e.g. the Soya warm current and the Tsugaru warm current along the north-eastern coast of Japan). On the other hand, Chia et al. (1982) using a laboratory experiment showed that, when a surface density front of an anticyclonic current reaches a vertical wall, a front of light fluid progresses along the concave wall cyclonically. The dynamics of progression of a front of light fluid with a coast on the right in the northern hemisphere may be similar for the above two cases even although the two situations are different.

A similar oceanic phenomenon known as “Kyucho”, which may be caused by the latter mechanism described above, occurs in bays along the southern coast of Japan. This phenomenon is characterized by a cyclonic progression of a warm front with a strong alongshore current. The propagation speed of the warm front is 0.5–1.0 m sec⁻¹. This phenomenon was first described in the earlier half of this century (e.g. Miura, 1927; Kimura, 1947; Uda, 1953), because it causes extensive damage to fishing nets. Recently, Matsuyama and Iwata (1977) have reported on the Kyucho in Sagami Bay in detail. This Kyucho is thought to be caused by the intrusion of a warm water mass of the Kuroshio into the bay.

Yamagata (1980) presented the Kelvin shock wave as a model of the Kyucho and of the intrusion of warm water along a coast reported by Yoon and Suginoara (1977). But his model has the following weak points: (1) the current has no density front exposed on the ocean surface, (2) the upper layer depth must be larger in the current than the depth of the fluid at rest, which implies that the intrusion of the current described by his model needs external force, and (3) his model cannot explain the undulation which is often observed after the passing of a warm front. These deficiencies suggest that a new evolution equation of non-linear semigeostrophic gravity waves, which is more realistic than that of the nonlinear Kelvin
wave, is needed.

Stern (1960) presented a nonlinear equation which describes the evolution of the semigeostrophic current which has zero potential vorticity and is bounded away from a coast by a density front on the ocean surface and is concentrated in the upper layer (reduced gravity model). This equation contains two semigeostrophic gravity waves. Kubokawa and Hanawa (1984, cited as Part 1 hereafter) investigated these waves under the assumption of finite and uniform potential vorticity and reported that, when the effect of an ageostrophic current is included, these waves have a dispersive nature. Thus, this equation may be able to explain the undulation behind the warm front.

Stern et al. (1982) proposed a model using the same equation as ours in Part 1 and compared their model with a laboratory experiment. Their model, however, is not the same as ours. The model in Stern et al. (1982) is derived from a supposed restriction on the current and invariance of the Bernoulli function on a steady free (constant pressure) stream line at a vertical coastal boundary. We will show that their model is not acceptable in the context of the long wave theory in Section 3. On the other hand, the model in this paper is based on the law of conservation of semigeostrophic mass transport and momentum, and is called a shock wave model in the context of the long wave theory.

An experiment is carried out in order to show the validity of the theoretical model (in Section 4). In the experiment, waves are generated at the leading edge of a density current, and their propagation velocity is close to that of one of the semigeostrophic gravity waves (SFW in Part 1). The waves propagate upstream relative to the leading edge. These waves can explain the undulation observed after the passing of the warm front in Kyucho.

In Section 5, the theoretical model will be compared with the experimental results including those of Stern et al. (1982), and in Section 6, the theoretical model and experimental results will be compared with the Kyucho phenomenon.

2. Formulation

The model configuration considered in this paper is similar to that in Fig. 1 of Part 1, in which a fluid of uniform density $\rho_0$ flows above an infinite fluid of density $\rho_0 + \Delta \rho$. The flow in the upper layer is bounded by a rigid vertical boundary at $y = 0$ and by a density front on the ocean surface at $y = L(x, t)$. The depth of the lower layer is taken to be much greater than that of the upper layer and so the motion of fluid is concentrated in the upper layer. Since the force of gravity is essential to the progression of a density front on the ocean surface, we remove the influence of the potential vorticity distribution. That is, we suppose that the potential vorticity in the upper layer is uniform.

We define nondimensional variables after Part 1 as follows,

$$
\begin{align*}
t &= t_* f_0, & x &= x_* (g' h_0)^{-1/2} f_0, & y &= y_* (g' h_0)^{-1/2} f_0, \\
u &= u_* (g' h_0)^{-1/2}, & v &= v_* (g' h_0)^{-1/2} \delta^{-1},
\end{align*}
$$

$$
\begin{align*}
h &= h_* h_0^{-1}, & \kappa^2 &= \kappa_0 H_*^{-1},
\end{align*}
$$

where $g'$ is a reduced gravity $g\Delta \rho \rho_0^{-1}$, $f$ the Coriolis parameter, $h_0$ the typical depth of the fluid, $H_*$ the fluid depth at rest, and $\delta$ is a small parameter corresponding to the ratio of the offshore and alongshore scales. The parameters with asterisks are dimensional.

When we consider the situation at the limit of $\delta \to 0$, the following nonlinear equations which govern the semigeostrophic motion in the upper layer are obtained.

$$
(U - L)_t + UU_x = 0, \quad (2-1)
$$

$$
\cosh \kappa L + \sinh \kappa L = \cosh \kappa L
$$

$$
\begin{align*}
+ \frac{1}{2\kappa}(\kappa U \sinh 2\kappa L - \sinh 2\kappa L + 2 \sinh \kappa L)
+ \kappa U \sinh \kappa L \frac{1}{2\kappa}(1 + \kappa^2 U^2) \sinh 2\kappa L
- 2\kappa U \cosh 2\kappa L - 2 \sinh \kappa L
+ 2 \kappa U \cosh \kappa L)_L x = 0,
\end{align*}
$$

$$
\begin{align*}
(2-2)
\end{align*}
$$

where $U(x, t)$ is the alongshore velocity at $y = L(x, t)$. Using $U(x, t)$ and $L(x, t)$, the alongshore velocity component $u(x, y, t)$ and the depth of the upper layer $h(x, y, t)$ can be written as,

$$
\begin{align*}
u(x, y, t) &= -\frac{1}{\kappa} \sinh \kappa (L(x, t) - y) \\
+ U(x, t) \cosh \kappa (L(x, t) - y),
\end{align*}
$$

$$
\begin{align*}
h(x, y, t) &= \frac{1}{\kappa^2} [1 - \cosh \kappa (L(x, t) - y)]
+ \kappa U(x, t) \sinh \kappa (L(x, t) - y)].
\end{align*}
$$

$$
\begin{align*}
(2-3)
\end{align*}
$$

$$
\begin{align*}
(2-4)
\end{align*}
$$
The details of the derivation of these equations are given in Part 1. Then the offshore velocity component \( v(x, y, t) \) can be written as

\[
\tau(x, y, t) = \frac{1}{\kappa^2 h} \left[ u_t + uu_x + h_x \right].
\]

These equations represent two kinds of waves. As in Part 1, we call these waves SCW (Semigeostrophic Coastal Wave) whose propagation velocity is \( \lambda_+(U, L) \) and SFW (Semigeostrophic Frontal Wave) whose propagation velocity is \( \lambda_-(U, L) \). In Part 1, we showed that

\[
\lambda_+(U, 0) = \lambda_+(U, 0) = U,
\]

\[
\lambda_+(U, L) > \lambda_-(U, L) \geq 0, \text{ for } L \neq 0,
\]

and both waves have a dispersive nature when ageostrophic effects \( O(\delta^3) \) are considered. The coefficient of dispersion term is positive for SCW and negative for SFW.

Even if \( \delta = O(1) \), when \( \nu = O(1) \), Eqs. (2-1) and (2-2) are satisfied. Therefore the motion of fluid described by Eqs. (2-1) and (2-2) is independent of the time scale.

3. Theoretical model

When a light fluid is intruding over a heavy fluid along a coast, \( U \) and \( L \) are evidently zero in areas the current has not yet reached. Since \( \lambda_+ = 0 \) where \( U = L = 0 \), and \( \lambda_+ > 0 \) and \( \lambda_- > 0 \) where \( U \neq 0 \) and \( L \neq 0 \), SCW must steepen and offshore velocity \( v \) is induced at the leading edge of the current. Because waves show no dispersion in the region of \( U = L = 0 \) (see Part 1), \( v \) becomes large and the semigeostrophic assumption is completely broken at the leading edge of the current. At this stage, we must return to the primitive equations (2-1) and (2-2) in Part 1. However, if the law of conservation of semigeostrophic mass transport and momentum is satisfied, the intrusion of a density current along a coast is represented by the shock wave solution of the semigeostrophic equations (2-1) and (2-2).

The shock wave solution is given by the law of conservation of semigeostrophic mass transport and momentum on both sides of the shock plane under an evolution condition (see Appendix A). From Eqs. (2-1) and (2-2), the conservation law can be written as

\[
\frac{\partial}{\partial t} \left( U - L \right) + \frac{\partial}{\partial x} \left( \frac{U^2}{2} \right) = 0,
\]

\[
\frac{\partial}{\partial t} \left[ -\frac{1}{\kappa} \sinh \kappa L + U \cosh \kappa L \right] + \frac{\partial}{\partial x} \left[ \frac{1}{4\kappa^2} (1 + \kappa^2 U^2) \cosh 2\kappa L \right.
\]

\[
\left. - \frac{U}{2\kappa} \sinh 2\kappa L - \frac{1}{\kappa^2} \cosh \kappa L \right) + \frac{U^2}{4\kappa^2} \sinh \kappa L + \frac{U^2}{4} = 0.
\]

If Eqs. (3-1) and (3-2) are satisfied on both sides of the shock plane whose propagation speed is a constant \( \lambda \), using \( U = L = 0 \) at the downstream side of the shock plane, we can get

\[
\lambda \left[ U_0 - L_0 \right] + \frac{U_0^2}{2} = 0,
\]

\[
\lambda \left[ -\frac{1}{\kappa} \sinh \kappa L_0 + U_0 \cosh \kappa L_0 \right] + \left[ \frac{1}{4\kappa^2} (1 + \kappa^2 U_0^2) \cosh 2\kappa L_0 \right.
\]

\[
\left. - \frac{U_0}{2\kappa} \sinh 2\kappa L_0 - \frac{1}{\kappa^2} \cosh \kappa L_0 \right) + \frac{U_0^2}{4\kappa^2} \sinh \kappa L_0 + \frac{U_0^2}{4} + \frac{3}{4\kappa^2} = 0,
\]

where the subscript "\(_0\)" denotes quantities on the upstream side of the shock plane. From Appendix (A), the evolution condition, which is required by a continuous dependence of the solution on the boundary conditions, is

\[
\lambda_- < \lambda < \lambda_+,
\]

and this condition uniquely determines the solution with physical meaning from the solutions of Eqs. (3-3) and (3-4). Since Eqs. (3-3) and (3-4) have three unknown variables, when one of them is given, the others can be determined.

The dependence of \( \lambda \) on \( \kappa^2 h(0) \) (a fluid depth multiplied by \( \kappa^2 \) at the vertical boundary) is shown in Fig. 1 in which the broken line denotes the phase speed of a long gravity wave \( \kappa \sqrt{h(0)} \) for a fluid of constant depth \( \kappa^2 h(0) \). When \( \kappa^2 h(0) < 0.5 \), \( \lambda \) becomes approximately

\[
\lambda \approx (1.45 \pm 0.02) \sqrt{h(0)}.
\]

The validity of this shock wave solution should be examined by comparison with experiments.
Fig. 1. Dependence of the propagation speed of the shock plane multiplied by \( \kappa \) on \( \kappa \) and \( h(0) \) on the current depth at the wall. The phase speed of a long gravity wave \( \kappa \sqrt{h(0)} \) for a fluid of a constant depth \( \kappa h(0) \) (broken line).

However, as the theory proposed by Benjamin (1968), which discussed a similar phenomenon in the inertial system using the invariance of the Bernoulli function on a steady free (constant pressure) stream line, has essentially been supported up to now (e.g. Simpson, 1982) and as the discussion using the invariance of the Bernoulli function is consistent with the conservation law of the shock wave (see Appendix (B)), the model proposed here is thought to be valid.

The model proposed by Yamagata (1980) is consistent with the model which is given by Eqs. (3–1) and (3–2) when \( \kappa U = 1 + J \), \( \kappa L \to \infty \) and \( J \to 0 \), where \( J \) satisfies \( J \exp(\kappa L) = O(1) \) as \( \kappa L \to \infty \) at the upstream side of the shock plane which has a constant progression velocity and \( \kappa U = 1 \) and \( \kappa L \to \infty \) at the downstream side of the shock plane.

Recently Stern et al. (1982) presented a model which uses Eqs. (2–1) and (2–2). They gave a special meaning to the \( \kappa U - \kappa L \) relation in which the Riemann invariant is constant. They supposed that the \( \kappa U - \kappa L \) relation in which \( \kappa U \) vanishes at \( \kappa L = 0 \), can represent the bore phenomenon and is satisfied at the initial state. They also supposed that the intrusion of the density current is stationary in a coordinate system moving with the leading edge, so the boundary condition on the far upstream side continues to satisfy the above \( \kappa U - \kappa L \) relation. They modeled this phenomenon using the \( \kappa U - \kappa L \) relation and the invariance of the Bernoulli function on a steady free stream line at \( y = 0 \).

The initial condition given by Stern et al., however, has no physical basis. Even if the initial condition is realized, their idea that the initial condition keeps on governing the evolution of the intrusion as the upstream boundary condition, is unacceptable in the context of the long wave theory as follows. Following Appendix (A), if the upstream condition does not satisfy the conservation law of the shock wave, SFW is generated on the shock plane and propagates upstream relative to the shock plane and ultimately modifies the upstream condition. This indicates that the free stream line which connects the leading edge of the density current to infinitely upstream is not steady, because of the progression of SFW. Therefore the initial condition cannot keep on governing the evolution of the intrusion as the upstream boundary condition. The upstream condition which directly governs the propagation of the leading edge must be one which satisfies the conservation law of the shock wave.

4. The laboratory experiment

4.1. Experimental apparatus and procedure

A rectangular tank made of transparent acrylic resin was mounted on a rotating turntable as

Fig. 2. Experimental apparatus: tank and turntable (a), top view of the tank (b), where A is the reservoir for the light fluid, B the reservoir for the heavy fluid, C a wall of tank A, D is a watergate and E is a dividing wall, and side view of wall E (c).
shown in Fig. 2(a). The tank was 45 cm deep, 100 cm long and 60 cm wide and was divided diagonally into two parts by a vertical wall E as shown in Fig. 2(b). One tank, tank A, was used as a reservoir of fresh water and the other, tank B, was filled up with saline water. The depth $H_*$ of tank A was variable so that the potential vorticity $f/H_*$ of fresh water could be controlled. A vertical slideable watergate D placed at a corner of the dividing wall connected tanks A and B (Fig. 2c). In order to restrict the influence of the lower layer on the intrusion of fresh water, tank B was filled with saline water to a depth of 40 cm.

The experimental procedure was as follows. After the tanks were filled up with tap water to the same level, several hundreds grams of salt (NaCl) was added to tank B and several grams of rhodamine B for the visualization of the intruding water was added to tank A. After the water was sufficiently mixed, the density of the fluid in each tank was measured. Next the water was spun up in a counter clockwise direction for at least 45 min (7-8 times the minimum required spin-up time). After the rigid rotation was accomplished, the watergate D was opened. The fresh water, which intruded into tank B from tank A, began to flow along the vertical wall C. The behavior of the intruding water was photographed from the top and side of the tank continuously by two synchronized 35 mm cameras. The time interval of photographing was varied for each Run (1.1-1.33 sec) since the time scale of the phenomenon varied with variation in the parameters. We carried out 15 Runs changing parameters $f$, $g'$ and $H_*$ (Table 1). The errors in measurement of parameters were $\pm 0.1$ cm sec$^{-2}$ for $g'$, $\pm 0.01$ sec$^{-1}$ for $f$ and $\pm 0.1$ cm for $H_*$.  

<table>
<thead>
<tr>
<th>Run No.</th>
<th>$f_0$ (sec$^{-1}$)</th>
<th>$g'$ (cm sec$^{-2}$)</th>
<th>$H_*$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.84</td>
<td>1.3</td>
<td>15.8</td>
</tr>
<tr>
<td>2</td>
<td>0.82</td>
<td>2.3</td>
<td>16.8</td>
</tr>
<tr>
<td>3</td>
<td>0.83</td>
<td>2.7</td>
<td>9.6</td>
</tr>
<tr>
<td>4</td>
<td>0.86</td>
<td>1.1</td>
<td>26.4</td>
</tr>
<tr>
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<td>0.86</td>
<td>1.7</td>
<td>14.2</td>
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<tr>
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<td>3.1</td>
<td>15.2</td>
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<td>15</td>
<td>0.99</td>
<td>1.6</td>
<td>26.7</td>
</tr>
</tbody>
</table>

$^a$ Coriolis parameter.

$^b$ reduced gravity.

$^c$ depth of the fluid at rest.

for $f$ and $\pm 0.1$ cm for $H_*$.

4.2. Qualitative description of experimental results

After opening the watergate D, the light fluid of tank A intruded into tank B and the heavy fluid in tank B moved slowly into tank A under the light fluid. The light fluid, which intruded into tank B, stretched vertically and formed an anticyclonic eddy which enlarged slowly near gate D. When the front of the light fluid reached wall C, the light fluid began to flow along wall C forming a narrow jet. The width and depth of the current rapidly increased, waves were observed in the offshore front, and a head formed at the leading edge of the current (Fig. 3). Simultaneously with the formation of the

![Image](https://via.placeholder.com/150)

$\Delta T = 1.18$ s

$10$ cm

$f = 0.99$ s$^{-1}$  \hspace{1cm}  $g' = 1.6$ cm s$^{-2}$  \hspace{1cm}  $H_* = 26.7$ cm

Fig. 3. Behavior of the intruding water as viewed from above: formation of the head and generation of the wave (Run 15). The photographs taken at constant intervals of 1.18sec are arranged in order from left to right.
head, enlargement of the width and depth of current ceased. The waves propagated upstream relative to the head, and the head began to extend and a new wave was generated as the head propagated (Fig. 4). This wave also propagated upstream relative to the head and a similar phenomenon to that described above was repeated. At the upstream position (near gate D), wave generation was also observed as shown in Fig. 4.

The waves tended to break backwards, but when the Reynolds number was relatively small the wave propagated conserving its wave form. The position of the wave crests in depth coincided with those in width. The amplitude of the wave generated at the head tended to decrease as the head propagated.

![Image](image)

Fig. 4. Behavior of the intruding water as viewed from above: generation and propagation of the waves, where the position of the leading edge in each frame is arranged along the one vertical line (Run 15).

![Graph](graph)

Fig. 5. Example of the temporal variation of the propagation speed of the leading edge, where the mark "↓" denotes the time when the head was formed and the mark "↑" denotes the time when wave generation at the head was observed. Measurements were carried out after the density front of the light fluid began to propagate along wall C. The origin of the time scale is arbitrary.

The leading edge of the light fluid was initially accelerated, but after formation of the head, the propagation speed decreased gradually and seemed to approach a constant value (Fig. 5). The amplitude of waves generated at the head also decreased.

Each Run was continued until the head reached the end of wall C.

### 5. Quantitative experimental results and comparison with the theory

#### 5.1. Relationships between the propagation speed of the head and the width and depth of the current

In order to compare the theoretical model with the experimental results, the current depth at the wall C and current width behind the head were measured from photographs. Since the current depth and width tend to increase towards the water gate D, we defined the current width \( L \) and depth \( h(0) \) as shown in Fig. 6. These values are not necessarily consistent with those used in the theoretical model, but the dependence of propagation speed of the head and current width on the current depth can be clarified.

The propagation speeds of the head, current width and depth were measured after the head was formed and the acceleration of the leading edge of the current ceased. Measurements were continued until the last frame of film (we used 36-frame film), or until the head reached about 10–20 cm from the end of wall C. All available data were averaged for each Run, and were

![Diagram](diagram)

Fig. 6. Definition of \( L \) and \( h(0) \) used in the analysis.
Table 2. Experimental results for the propagating head and the current.

<table>
<thead>
<tr>
<th>Run No.</th>
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<th>κ²h(0)</th>
<th>κL₀</th>
<th>κδ²</th>
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<td>—</td>
<td>—</td>
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<td>168</td>
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a Reynolds number.

b nondimensional current depth at the wall.

c nondimensional current width.

d nondimensional propagation speed of the head.

nondimensionalized using \( \sqrt{g/H} \) for speed, \( \sqrt{gH_f}f^{-1} \) for width and \( H_b \) for depth. These nondimensional values correspond to \( \kappa L_\theta \), \( \kappa L \), \( \kappa^2 h(0) \) of the theoretical model, respectively. The experimental results are shown in Table 2.

The dependence of propagation speed of the head on the depth \( \kappa^2 h(0) \) is shown in Fig. 7(a). In this figure, the results of Stern et al. (1982) are also drawn along with those in this experimental study. The symbols used depend on the range of Reynolds number \( Re=\nu^{-1/2}g^{1/2}H_b(0)^{3/2} \), where \( \nu \) is the molecular viscous coefficient. The solid line represents the theoretical prediction and the broken line represents \( \kappa \sqrt{h(0)} \) which is the phase speed of a long gravity wave without rotation when the depth of the fluid is \( \kappa^2 h(0) \).

The dependence of the current width on the current depth is shown in Fig. 7(b). Again symbols depend on the range of Reynolds number. The solid line represents the theoretical prediction and the broken line represents \( \kappa \sqrt{h(0)} \) which is the Rossby radius of deformation when the depth of fluid is \( \kappa^2 h(0) \).

The experiment of Stern et al. (1982) was carried out by a method in which the light fluid, which had been retained in one side of a rectangular tank, was released. Because of the difference in the method, \( \kappa^2 h(0) \) of their experiment was larger than that of this study, and as their tank was larger than ours, high Reynolds number experiments could be carried out. The \( h_b(0) \) of Stern et al. was defined by the depth at the neck behind the head. As a result, their experimental results for the propagation speed of the head as a function of \( \kappa^2 h(0) \) might be slightly larger than those as a function of \( \kappa^2 h(0) \) defined in Fig. 6. It should also be noted here that our definition of the Reynolds number, differed from theirs \( (\nu^{-1/2}g^{1/2}H_b(0)^{3/2}) \) which was based on the depth of the fluid at rest.

![Fig. 7. Comparison of the experimental results with the theoretical model: relationship between the propagation speed of the head and the current depth, where SWH denotes the experimental results of Stern et al. (1982) and Re is the Reynolds number (a), and relationship between the current width and depth (b). The solid lines denote prediction of the theoretical model and the broken lines denote \( \kappa \sqrt{h(0)} \) which is the phase speed of a long gravity wave with constant depth \( \kappa^2 h(0) \) (a) and the radius of deformation (b), respectively.](image-url)
The estimated propagation speed of the head is slightly smaller than that expected by the theory, but the dependence on $\kappa^2 h(0)$ is consistent with the theory. The reason for this difference seems to lie in the viscosity because the four data of Stern et al. with $Re > 1000$ coincide with the predictions of the model. The dependence of $\kappa^2 h(0)$ in Stern et al. and this study shows the same trend inspite of the differences in experimental design, which adds support to the view that this phenomenon can be regarded as a shock wave which has no direct dependence on the initial conditions. The nondimensional current width is slightly wider than that predicted by the theoretical model, but its dependence on the current depth is consistent with that of the model, though the parameter range is not so wide. The reason for this difference also seems to stem from the effects of viscosity, though there is a problem in the definition of the current width. The Ekman layer produced by friction with the lower layer or air has a tendency to increase the current width. Therefore the effective width which contributes to the progression of the head cannot be determined precisely from the experiment, but the general agreement between the model and experimental results supports the theoretical model.

Recently, Griffiths and Hopfinger (1983) who carried out experiments similar to those of Stern et al. reported that the observed propagation speed of the head was about $1.3 \sqrt{gH_s}$ for a Reynolds number ($= \nu^{-1} \rho^{0.5} F_{Hs}^{3/2}$) > 1000, where $h_s$ was the depth of the light fluid far upstream on the wall. Their experimental results also support the theoretical model.

5.2. The propagation of waves

We reported in Section 4 that waves are generated at the head and propagate upstream from the head. The propagation speed of these waves is discussed here.

The propagation speed was estimated using waves which could be traced in at least four frames of the photographs, and was defined as the propagation speed of the center between one trough and the next as viewed from above. Waves showing rapid changes in propagation speed were not included, because such rapid change was thought to be caused by some other mechanism (e.g. interaction with other waves).

<table>
<thead>
<tr>
<th>Run No.</th>
<th>$L_s$ (cm)</th>
<th>$h_s(0)$ (cm)</th>
<th>wave length (cm)</th>
<th>$\kappa^2$</th>
<th>$\kappa$</th>
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<td>6.7</td>
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<tr>
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<td>6.0</td>
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<td>0.19</td>
<td>0.16</td>
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</table>

Run 4 and Runs 11-13 were not used, because the photographs of Run 4 were not clear, and no wave could be traced for four frames in the photographs of Runs 11-13. Other parameters, e.g. the propagation speed of the head, and the width and depth of the current, were measured from the same frames as those used to estimate the propagation speed of waves. The results are shown in Table 3.

In Subsection 5.1, we mentioned that quantitative estimation of the effective width and depth of the current, which are thought to influence the propagation speed of the head, is difficult, but the propagation speed of the head and the waves can be easily estimated. Therefore, we examined the dependence of the propagation speed of the waves on that of the head. The dependence of the propagation speed of the waves on that of the head is shown in Fig. 8, where a broken line and a solid line represent the propagation speed of the semigeostrophic gravity waves SCW and SFW as a function of $\kappa^2$. 
Fig. 8. Propagation speed of the waves observed in the experiment as a function of the propagation speed of the head. The broken line denotes $\xi\lambda_0(\xi\lambda)$ and the solid line denotes $\xi\lambda(\xi\lambda)$.

6. Conclusion and oceanic applications

We investigated the intrusion of a density current along a coast in a rotating fluid using a long wave theory and a laboratory experiment.

The main results of this study are as follows: (1) the density current intrudes along a vertical coast as a shock wave of the Semigeostrophic Coastal Wave (SCW), (2) the head which is formed at the leading edge of the density current becomes a wave source, because the semigeostrophic assumption breaks down at this point and (3) the SFWs generated at the head, which may have a dispersive nature, propagate upstream from the head.

One of the oceanic phenomena to which this study can be applied is Kyucho. Figure 9(a) shows the variations of water temperature along the coast of Sagami Bay, Japan, between 22 to 24 April 1975 (source: Matsuyama and Iwata, 1977). The location of the stations is shown in Fig. 9(b). It can be seen that a warm front progressed from Misaki to Hayakawa with a propagation speed of about 1 m sec$^{-1}$. Yamagata (1980) supposed that the current depth at the coast was 50 m and $\phi'$ was $10^{-2}$ m sec$^{-2}$, in order to compare his model with the Kyucho. If we use the same values in our model as Yamagata (1980), we get from Eq. (3-5),

$$\lambda_s \approx (1.45 \pm 0.02) \sqrt{g' h_s(0)} \approx 1 \text{ m sec}^{-1}.$$

Therefore, the model in this paper is able to explain the observed propagation speed of the warm front.

Undulation was observed after the passing of the warm front in the time series of temperature
Semigeostrophic Gravity Waves and the Intrusion of a Density Current 2.

at Hayakawa where the sensor depth was 25 m. Takeuchi and Suzuki (1976), who observed a Kyucho at Kumanonada on the southern coast of Japan between 11 to 12 February 1974, also reported undulation after the passing of the warm front. These observational results are consistent with the experimental results.

At this stage, we cannot be certain whether the Kyucho phenomenon is the same as the experiment in this paper or not. However, if the propagation speed of the undulation can be estimated from the observational data, a comparison between the Kyucho and the experiment is possible and the applicability of the shock wave model could thus be tested.

In this section, we compared the model and experimental results with a Kyucho phenomenon that had a time scale of several days. However, since Eqs. (3-1) and (3-2) are independent of the time scale, the results in this paper are also applicable to phenomena which have much longer time scales.

Acknowledgements

We would like to thank Prof. Yoshiaki Toba and Dr. Kuniaki Okuda and other members in our laboratory for their valuable comments and many interesting discussions during the course of this study.

References


Appendix

(A) Evolution condition

The solution with physical meaning continuously depends on the boundary condition. In order to seek this condition, we will consider the shock wave solution of Eqs. (2-1) and (2-2) with a time-dependent boundary condition at the far upstream position. Following Part 1, Eqs. (2-1) and (2-2) can be written as

\[ U_t + A(U) U_x = 0, \quad (A-1) \]

where

\[ U = \begin{pmatrix} U \\ L \end{pmatrix}, \quad A(U) = \begin{pmatrix} A_{11}(U) & A_{12}(U) \\ A_{21}(U) & A_{22}(U) \end{pmatrix}. \]

As the temporal variation of the boundary condition must be propagated to the shock plane by SCW and/or SFW, we must treat \( U \) and \( L \) at the upstream side as a function of \( x \) and \( t \), and \( \lambda \) as a function of \( t \), that is,

\[ U = U_0 + \varepsilon U_1(x,t), \quad L = L_0 + \varepsilon L_1(x,t) \quad \text{and} \quad \lambda = \lambda_0 + \varepsilon \lambda_1(t), \]

where subscript "0" denotes a constant value and \( \varepsilon \ll O(1) \). When we introduce the new coordinate system moving with the shock plane, namely \( \xi = x - \lambda_0 t \) and \( \tau = t \), Eq. (A-1) becomes

\[ U_t + (A(U) - \lambda_0 I) U_x = 0, \quad (A-2) \]

where \( I \) is a unit matrix (\( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)). When Eq. (A-2)
is linearized for $U$, we get
\[ U_{1r} + (A(U_0) - \lambda_0 I) U_1 = 0. \] (A-3)

If we use the right eigenvectors $R_s$ of matrix $A(U_0) - \lambda_0 I$, $U_1$ can be written as
\[ U_1 = R_s \phi_s(x, \tau). \] (A-4)

On the other hand, Eq. (A-2) can be rewritten as
\[ U_r + (F - \lambda_0 U_0) \tau = 0, \] (A-5)
where
\[ F_s = A(U) U_s. \] (A-6)

When the shock plane is at $\xi = 0$, since $U = L = 0$ for $\xi > 0$, the following condition at $\xi = 0$ is obtained,
\[ \lambda_0 U = (A(U_0) - \lambda_0 I) U, \quad \text{at} \quad \xi = 0. \] (A-7)

When Eq. (A-7) is linearized, we obtain
\[ \lambda_0 U_0 = (A(U_0) - \lambda_0 I) U_1, \quad \text{at} \quad \xi = 0. \] (A-8)

Substitution of Eq. (A-4) into Eq. (A-8) gives
\[ \lambda_0 U_0 = (A(U_0) - \lambda_0 I) R_s \phi_s(0, \tau). \] (A-9)

Because Eq. (A-9) has three unknown variables, when one is given, the others can be determined.

This implies that one wave (SCW) is determined by Eq. (A-3) with the boundary condition at infinitely upstream and the other (SFW) is determined by Eq. (A-3) with the boundary condition at $\xi = 0$. In other words, the time variation of the far upstream condition must be propagated to the shock plane by SCW, and when the upstream condition does not satisfy the conservation law of the shock wave, SFW is generated at the shock plane. Therefore the evolution condition is
\[ \lambda_0 < \lambda < \lambda_1, \]
where $\lambda_0$ corresponds to $\lambda$ in the text.

B) Invariancy of the Bernoulli function and conservation law of the shock wave

Following Stern et al. (1982), the Bernoulli function $B$ in the coordinate system moving with constant velocity $\bar{\lambda}$ in a rotating one-layer fluid (reduced gravity model) is written as
\[ B = \frac{1}{2} u^2 + \bar{\lambda} y + \bar{g} z, \] (B-1)

where $u$ is a fluid velocity in this coordinate system, and $B$ is a constant on the steady free (constant pressure) stream line.

If we consider the invariancy of the Bernoulli function on the two steady free stream lines at $y = 0$ and $y = L_0$, when $u_1 = 0$ for $x > x_0$ and $u_1 = u - \bar{\lambda}$ for $x < x_0$, we get
\[ \frac{1}{2} (u(0) - \bar{\lambda})^2 + \bar{g} h(0) = -\frac{1}{2} \bar{\xi}^2, \] (B-2)
\[ \frac{1}{2} (u(L_0) - \bar{\lambda})^2 + \bar{g} L_0 = -\frac{1}{2} \bar{\xi}^2. \] (B-3)

Substitution of Eqs. (2-3) into Eqs. (B-2) and (B-3), after Eqs. (B-2) and (B-3) are nondimensionalized, gives Eqs. (3-4) and (3-3). Therefore, the invariancy of the Bernoulli function on the steady free stream line is consistent with the conservation law of the shock wave. The same result, of course, can be obtained for an inertial system.

半地衡性重力波とその岸に沿う密度流の侵入への適用

2. 回転系での岸に沿う密度流の侵入

久保川厚*, 花輪公雄*

要旨: 回転系での岸に沿う密度流の侵入現象を理論並びに実験的に調べた。本論文で提案した理論モデルはPart 1 (Kubokawa and Hanawa, 1984) で述べた半地衡性重力波の衝撃波である。侵入する軽い流体の先端部の進

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行速度並びに流れの幅の実験結果は衝撃波理論からの予

説と大きく違っていた。実験では、先端部から波動

が発生し、その波動が先端部に相対的に上流に伝播して

いくのが観察されたが、その波動の伝播速度を半地衡性

前続波 (Part 1 参照) のそれとはほぼ一致していた。