A Theory of Semigeostrophic Gravity Waves and its Application to the Intrusion of a Density Current along a Coast

Part 1. Semigeostrophic Gravity Waves*

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Abstract: Semigeostrophic gravity waves associated with a coastal boundary current, which has finite and uniform potential vorticity and is bounded away from the coastline by a density front on the ocean surface, are investigated. It is shown that the semigeostrophic coastal current has two waves which are named here the Semigeostrophic Coastal Wave (SCW) and the Semigeostrophic Frontal Wave (SFW). The SCW becomes an elementary Kelvin wave at some limit while the SFW is caused by the existence of the surface density front. The SCW appears mainly as variations in the upper layer depth at the coast and as alongshore velocity at the density front. On the other hand, the SFW appears mainly as variations in the width of the current. When the weak nonlinearity and ageostrophic effect are included, these semigeostrophic gravity waves satisfy the Kortweg-de Vries equation, which suggests that the local changes in the width and/or velocity of the semigeostrophic coastal current propagate as wave-like disturbances.

1. Introduction

Recently many thermal infrared images of sea surface temperature collected by the environmental satellites have become available (e.g. Legeckis, 1978; Legeckis and Cresswell, 1981) and these show that many ocean currents have thermal fronts exposed on the ocean surface. These thermal fronts can generally be regarded as density fronts and as boundaries for the currents. Since the divergence of horizontal velocity is very large near the surface density front, changes in such a current cannot be described by vorticity dynamics alone.

In this paper, we focus attention on density-driven coastal boundary currents which are bounded away from the coastline by a density front on the ocean surface. There are many examples of such currents in the world ocean, e.g. the Leeuwin current, the East Greenland current and the West Spitzbergen current and the Soya warm current, the Tsushima warm current and the Tsugaru warm current (the latter three flow along the coast of the Japanese Islands).

The first paper treating the dynamics of such a current was that of Stern (1980). He derived the nonlinear evolution equation for semigeostrophic currents which have zero potential vorticity, using a reduced gravity model, where the term "semigeostrophic" means that only one component of horizontal velocity is in geostrophic balance. He showed that the equation has two dependent variables and two kinds of waves. In this paper we reexamine the nonlinear semigeostrophic gravity waves associated with a current in more detail. First, we extend the evolution equation derived by Stern (1980) to the case of finite and uniform potential vorticity, and clarify the relationship between the waves described by this equation and Kelvin waves. Next, we consider the effect of ageostrophic currents induced by the semigeostrophic waves, and show that the evolution of weak nonlinear semigeostrophic gravity waves in the current is governed by the Kortweg-de Vries (K-dV) equation.

The nonlinear evolution of Kelvin waves, which are a type of semigeostrophic gravity wave, was studied by Smith (1972) and Grimshaw (1977). They showed that, in the presence of bottom topography, Kelvin waves have a dispersive nature and are described by the K-dV
equation. In this paper it will be shown that
the dispersive nature of semigeostrophic gravity
waves arises from the existence of a mean cur-
rent.

The assumption of uniform potential vorticity
for the current in this study may place severe
restrictions on the applicability of the present
results to the real world. However, since the
force of gravity is required to bring about
changes in surface density fronts, analysis under
this assumption is a meaningful first step.
Moreover, one of the aims of this paper is to
provide a basis for the interpretation of the ex-
perimental results of Kubokawa and Hanawa
(1984) who investigated the intrusion of a den-
sity current along a coast in a rotating fluid.

2. Formulation

We consider the situation shown in Fig. 1 in
which a fluid of uniform density \( \rho_0 \) flows above
an infinite fluid of density \( \rho_0 + \Delta \rho \). The flow
in the upper layer is bounded by a rigid verti-
cal wall at \( y = 0 \) and by a density front exposed
on the sea surface at \( y = L(x, t) \). Since the
lower layer depth is much larger than that of the
upper layer in this model, fluid motions are con-
centrated in the upper layer. Applying a re-
duced gravity model to this model, the shal-
low water equations and boundary conditions
at \( y = 0 \) and \( y = L(x, t) \) are

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u + f k \times u &= -g' \nabla h, \quad (2-1) \\
h_t + f \times (hu) &= 0, \quad (2-2) \\
v &= 0 \text{ at } y = 0, \\
v = \frac{dx}{dt}, \quad h = 0 \text{ at } y = L(x, t),
\end{align*}
\]

where \( u = (u, v) \) are the \((x, y)\) components of
velocity, \( k \) the vertical unit vector which is
positive upward, \( f \) the horizontal differential
operator \( (\partial/\partial x, \partial/\partial y) \), \( g' \) the reduced gravity
\( g_0 \rho_0^{-1} \) and \( f \) is the Coriolis parameter which
is assumed to be constant. The potential vor-
ticity equation derived from Eqs. (2-1) and (2-2)
is

\[
\frac{d}{dt} \left( \frac{f + \nabla \times u}{h} \right) = 0, \quad (2-3)
\]

When the potential vorticity of the flow is con-
stant, Eq. (2-3) yields

\[
\frac{f + \nabla \times u}{h} = \frac{f}{H}, \quad (2-4)
\]

where \( H \) is the depth of the upper layer when
the fluid is at rest. The assumption of constant
potential vorticity means that the system con-
considered here only permits the existence of
gravity waves.

As we are interested in the long waves whose
alongshore scale is much larger than the offshore
scale, we introduce the small parameter \( \delta = \frac{H}{L} \) \((\ll 1)\) and define the nondimen-
sional variables as follows,

\[
\begin{align*}
t &= t \delta, \quad x_\ast = x(g'h_0)^{-1/2} \delta^2 t, \quad y_\ast = y(g'h_0)^{-1/2} f, \\
h_\ast &= h_0, \quad u_\ast = u(g'h_0)^{-1/2}, \quad v_\ast = v(g'h_0)^{-1/2} \delta^{-1}, \\
\kappa^2 &= \frac{h_0}{H h},
\end{align*}
\]

where variables with asterisks are nondimen-
sional, \( h_0 \) is a typical depth of the upper layer
and \( (g'h_0)^{-1/2} \delta^{-1} \) is the radius of deformation.
When \( \delta \) is infinitely small, only the alongshore
component of velocity is in geostrophic balance
and such a flow is called a semigeostrophic flow.

Upon dropping the asterisks for convenience
Eqs. (2-1), (2-2) and (2-4) become

\[
\begin{align*}
\frac{\partial u}{\partial t} + uu_x + v(u_y - 1) &= -h_x, \quad (2-5) \\
\delta^2 (v_t + uu_y + vv_y) + u &= -h_y, \quad (2-6) \\
h_t + (hu)_x + (hv)_y &= 0, \quad (2-7) \\
\delta^2 u_x - u_y + 1 &= \kappa^2 h. \quad (2-8)
\end{align*}
\]

We will separate the solutions of the above
equations into two parts, that is,
\[
\begin{pmatrix}
  u \\
  v \\
  h
\end{pmatrix} = \begin{pmatrix}
  u_a \\
  v_a \\
  h_a
\end{pmatrix} + \partial \begin{pmatrix}
  u_a \\
  v_a \\
  h_a
\end{pmatrix},
\]
where the quantities with the subscript "s" are the semigeostrophic parts of solutions which are exact solutions when \( \partial^2 \) equals zero, and the quantities with the subscript "a" are the ageostrophic parts of solutions which represent the ageostrophic or finite wave-length effect. Setting \( \partial^2 \) equal to zero in Eqs. (2-6) and (2-8), \( u_a \) and \( h_a \) take the forms

\begin{align}
\tag{2-9}
& u_a(x, y, t) = -\frac{1}{\kappa} \sinh \kappa(L(x, t) - y) \\
& + U(x, t) \cosh \kappa(L(x, t) - y), \\
& h_a(x, y, t) = \frac{1}{\kappa^2} [1 - \cosh \kappa(L(x, t) - y)] \\
& + \kappa U(x, t) \sinh \kappa(L(x, t) - y), \\
\tag{2-10}
& h_a(x, t) = \frac{1}{\kappa} [1 - \cosh \kappa L + \kappa U \sinh \kappa L] \geq 0.
\end{align}

If \( h_a(x, t) < 0 \), the current must separate from the coastal boundary and thus it is not a coastal boundary current. When we substitute Eqs. (2-9) and (2-10) into Eq. (2-5) and evaluate the results at \( y = 0 \) and \( y = L \), we get the following nonlinear equations for \( U(x, t) \) and \( L(x, t) \),

\begin{align}
\tag{2-11}
& (U - L) + U U_x = 0, \\
& \cosh \kappa L \cdot U_t + (\kappa U \sinh \kappa L - \cosh \kappa L) U_x \\
& + \frac{1}{2\kappa} (\kappa U \cosh 2\kappa L - \sinh 2\kappa L) \\
& + 2 \sinh \kappa L U_x \\
& + \frac{1}{2\kappa} (1 + \kappa^2 U_x^2) \sinh 2\kappa L \\
& - 2\kappa U \cosh 2\kappa L - 2 \sinh \kappa L \\
& + 2\kappa U \cosh \kappa L L_t = 0. \\
\tag{2-12}
& \text{These equations can be rewritten as,} \\
& U_t + A(U) U_x = 0, \\
& U = (U \in L), \quad A(U) = \begin{pmatrix}
  A_{11}(U) \\
  A_{12}(U) \\
  A_{21}(U) \\
  A_{22}(U)
\end{pmatrix}, \\
& A_{11}(U) = \frac{1}{\kappa U \sinh \kappa L} \left[ -\frac{1}{2\kappa} \sinh 2\kappa L \\
& + \frac{U}{2} (\cosh 2\kappa L + 1) \\
& + \frac{1}{\kappa} (1 + \kappa^2 U^2) \sinh \kappa L - U \cosh \kappa L \right], \\
& A_{12}(U) = \frac{1}{\kappa U \sinh \kappa L} \left[ \frac{1}{2\kappa} (1 + \kappa^2 U^2) \sinh 2\kappa L \\
& - U \cosh 2\kappa L - \frac{1}{\kappa} \sinh \kappa L + U \cosh \kappa L \right], \\
& A_{21}(U) = A_{11}(U) - U, \\
& A_{22}(U) = A_{11}(U).
\end{align}

We will examine the waves governed by this equation in the next section and investigate the ageostrophic effects \( O(\partial^2) \) in Section 4.

3. Semigeostrophic gravity waves

3.1. Propagation velocity

When Eq. (2-13) is multiplied by the left eigenvector \( L_\lambda \) of the matrix \( A \), we have

\begin{align}
\tag{3-1}
& L_\lambda (U_t + \lambda_\lambda U_x) = 0, \\
& \lambda_\lambda = \frac{1}{2} (A_{11} + A_{22}) \{1 \pm \sqrt{1 - 4 \lambda_1 U/(A_{11} + A_{22})}\},
\end{align}

where \( \lambda_\lambda \) is the eigenvalues of the matrix \( A \). \( \lambda_\lambda \) are called the characteristic velocities of Eq. (2-13) and represent the propagation speeds of \( \int L_\lambda dU \),

which are called the Riemann invariants. If Eq. (2-13) is linearized, \( \lambda_\lambda \) correspond to the phase velocities of the semigeostrophic gravity waves. The distributions of the values \( \lambda_\lambda \) on the \( \lambda \in \kappa U \) plane are shown in Figs. 2 (a) and (b). Since Eq. (2-13) has two dependent variables, two waves exist in the system. \( \lambda_+ = U \) for \( u_a(x, 0, t) = 0 \) as shown in Appendix (A) and it is larger than \( U \) for \( u_a(x, 0, t) > 0 \) and \( L \neq 0 \). \( \lambda_- \) is always smaller than \( U \) for \( L \neq 0 \). These results suggest that the wave with the propagation velocity of \( \lambda_+ \) propagates downstream and the other wave with the propagation velocity of \( \lambda_- \) moves upstream relative to the mean current.

For \( \kappa U \geq 1 \) and \( \kappa L \to \infty \), \( u_a(x, y, t) \) and \( h_a(x, y, t) \).
y, t) become for \( y = O(1) \),

\[
\begin{align*}
  u_\delta(x, y, t) &= \frac{1}{2\kappa}(\kappa U - 1) e^{\xi z} e^{-\kappa y}, \\
  h_\delta(x, y, t) &= \frac{1}{\kappa^2} [1 + \kappa u_\delta(x, y, t)].
\end{align*}
\]

The dimensional depth profile is

\[
h(x_0, y_0, t_0) = H \left[ 1 + \frac{1}{\sqrt{g}} u_\delta(x_0, y_0, t_0) \right],
\]

where the variable with asterisks are dimensional. Because this profile is coincident with that of the boundary current which has a uniform potential vorticity and no surface front, one of the waves must have the form of a Kelvin wave at this limit. When \( \kappa U \geq 1 \) and \( \kappa L \to \infty \), \( \lambda_+ \) becomes

\[
\lambda_+ = \frac{1}{\kappa} [1 + \kappa u_\delta(x, 0, t)]
\]

as shown in Appendix (A). This is the same as the phase speed of Kelvin waves advected by a boundary current. It will be shown in the next subsection that this wave is trapped near the coast under this condition. That is, the wave with propagation velocity of \( \lambda_+ \) can be regarded as a generalized Kelvin wave. On the other hand, at this limit, \( \lambda_- \) becomes

\[
\lambda_- = \frac{1}{\kappa} [\kappa u_\delta(x, L, t) - 1],
\]

as shown in Appendix (A). This is the same as the phase speed of a Kelvin wave when the front is regarded as a coast. That is, the wave with the phase speed of \( \lambda_- \) is the one which requires the existence of the surface density front. We will show in the next subsection that this wave is trapped near the surface density front under this condition. For convenience, we call the wave with the propagation velocity of \( \lambda_+ \) the Semigeostrophic Coastal Wave (hereafter abbreviated to SCW) and the wave with the propagation velocity of \( \lambda_- \) the Semigeostrophic Frontal Wave (SFW).

### 3.2. Wave structures

The structures of these waves are described by Eqs. (2-9) and (2-10). Since these equations have two unknown variables \( U(x, t) \) and \( L(x, t) \), the relationships between \( U(x, t) \) and \( L(x, t) \) must be determined in order to clarify the structures of these waves. In this subsection, we will examine these relationships.

We will separate \( U \) and \( L \) into two parts i.e. \( U = U_0 + \epsilon U_1, \ L = L_0 + \epsilon L_1 \), where \( \epsilon \ll 0(1) \). The variables with the subscript "0" represent the basic field while those with the subscript "1" represent the disturbed field. The linearized equation for \( U_1 \) and \( L_1 \) is

\[
U_{1t} + A(U_0)U_{1x} = O(\epsilon), \quad (3-3)
\]

where

\[
U_1 = \begin{pmatrix} U_1 \\ L_1 \end{pmatrix},
\]

\( u_\delta \) and \( h_\delta \) become
\[ u_s = u_{s0}(y) + \varepsilon u_{s1}(x, y, t), \]
\[ h_s = h_{s0}(y) + \varepsilon h_{s1}(x, y, t), \]
where
\[ u_{s0}(y) = u_s(U_0, L_0, y), \]
\[ h_{s0}(y) = h_s(U_0, L_0, y), \]
\[ u_{s1}(x, y, t) = U_1 \cdot \nabla_x u_{s0} + O(\varepsilon), \]
\[ h_{s1}(x, y, t) = U_1 \cdot \nabla_x h_{s0} + O(\varepsilon), \]
where \( \nabla_x = (\partial/\partial x, \partial/\partial L_0) \). Since the propagation velocities of the waves are \( \lambda_s(U_0) + O(\varepsilon) \), we obtain
\[ (-\lambda_s(U_0) + A(U_0)) U_{1s} = O(\varepsilon). \] (3-4)
The right eigenvectors \( R_s = \begin{pmatrix} r_s^1 \\ r_s^2 \end{pmatrix} \) of matrix \( A(U_0) \) corresponding to \( \lambda_s(U_0) \) satisfy
\[ (-\lambda_s(U_0) + A(U_0)) R_s = 0. \] (3-5)
Equations (3-4) and (3-5) require
\[ U_{1s} = R_s \phi_{s0} + O(\varepsilon). \] (3-6)
If \( U_1 \) approaches \( \phi_{s0} R_s (\phi_{s0} \text{ is a constant}) \) as \( x \to \infty \), we obtain
\[ U_1 = R_s \phi_s + O(\varepsilon), \] (3-7)
where \( \phi_s \) are functions of \( x \) and \( t \). The change of depth at \( y = 0 \) is
\[ h_{s1}(x, 0, t) = -\frac{1}{\kappa} \left( r_{s1}^2 - r_{s2}^2 \right) \sinh \kappa L_0 \]
\[ + r_{s2} \kappa U_0 \cosh \kappa L_0 \phi_s + O(\varepsilon), \]
\[ = H_s(\kappa U_0, \kappa L_0) \phi_s + O(\varepsilon). \] (3-8)
Equations (3-6) and (3-7) show the relationships between \( U_1 \), \( L_s \) and \( h_{s1}(x, 0, t) \) are described in terms of \( R_s \) and \( H_s \).

The distributions of the values of \( r_{s1}, r_{s2}, r_{s1} \) and \( r_{s2} \) on the \( \kappa L_0 - \pi U_0 \) plane are shown in Figs. 3(a), (b), (c) and (d), and their explicit forms are shown in Appendix (B), where \( R_s \) are normalized as \( \| R_s \| = 1 \). In the figure, the contour lines of \( r_{s2} = 0 \) and \( r_{s1} = 0 \) are coincident with the line of \( u_{s0}(0) = 0 \) as shown in Appendix (C). Since \( r_{s1} \times r_{s2} \) is positive for \( u_{s0}(0) > 0 \) and is negative for \( u_{s0}(0) < 0 \), the variations of \( U_1 \) and \( L_s \) of SCW are in phase for \( u_{s0}(0) > 0 \) and out of phase for \( u_{s0}(0) < 0 \). That is, for \( u_{s0}(0) > 0 \), the current velocity \( U \) is largest at the wave crest where the current width is largest, and smallest at the wave trough. The fact that \( r_{s2} \) is larger than \( r_{s1} \) where \( h_{s0}(0) \) is not nearly zero, implies that the amplitude of current velocity at \( y = L_0 \) is larger than that of current width \( L_1 \) in this region for SCW. On the other hand, the variations of \( U_1 \) and \( L_1 \) of SFW are in phase for \( u_{s0}(0) < 0 \) and out of phase for \( u_{s0}(0) > 0 \). The amplitude of the current width of SFW is larger than that of the current velocity at \( y = L_0 \) where \( \kappa U_0 < 2 \) and \( \kappa L_0 \) is not nearly zero.

The distributions of the values of \( \kappa H_s \) and \( \kappa H_\perp \) are shown in Figs. 4(a) and (b). In Fig. 4(b), the contour line of \( \kappa H_\perp = 0 \) is coincident with the line of \( u_{s0}(0) = 0 \) as shown in Appendix C. \( H_s \) is always positive except for \( \kappa L_0 = 0 \) and is larger than \( H_\perp \) almost everywhere. This implies that the amplitude of current depth \( h_{s1} \) of SFW at \( y = 0 \) is smaller than that of SCW. The variations of \( h_{s1}(x, 0, t) \) and \( U_1 \) are in phase for SCW but out of phase for SFW. \( H_s \) and \( H_\perp \) become infinitely large and small, respectively, as \( \kappa L_0 \to \infty \) as shown in Appendix (C). These results suggest that, when the width of a current is very large, SCW and SFW are trapped near the coast and near the surface density front, respectively.

Figure 7 illustrates these wave structures in easy-to-visualize form and includes information on the phase relationships between \( U_1 \), \( L_s \), and \( h_{s1}(x, 0, t) \).

### 3.3. Effect of nonlinearity
When we consider the terms of \( O(\varepsilon) \) and neglect the interaction between SCW and SFW, \( U_1 \) takes the form
\[ U_1 = R_s \phi_s + \frac{\varepsilon}{2} (R_s \cdot \nabla U_0) R_s \phi_s + O(\varepsilon^2). \] (3-8)

Then, \( u_{s1}(x, y, t) \) and \( h_{s1}(x, y, t) \) become
\[ u_{s1} = U_1 \cdot \nabla U_0 u_{s0} + \frac{\varepsilon}{2} (U_1 \cdot \nabla U_0)^2 u_{s0} + O(\varepsilon^2), \] (3-9)
\[ h_{s1} = U_1 \cdot \nabla U_0 h_{s0} + \frac{\varepsilon}{2} (U_1 \cdot \nabla U_0)^2 h_{s0} + O(\varepsilon^2). \] (3-10)

On the other hand, we obtain in place of Eq. (3-3),
Fig. 3. Contours of each component of the right eigenvectors $\mathbf{R}_+(r_{21})$ of the matrix $\mathbf{A}(U_0)$ on the $\kappa L_0 - \kappa U_0$ plane: $r_{21}$ (a), $r_{20}$ (b), $r_{2-1}$ (c), and $r_{2-2}$ (d). The contour lines of $r_{21}=0$ and $r_{2-1}=0$ are coincident with the line of $u_0(0)=0$.

Fig. 4. Contours of the depth amplitudes of the semigeostrophic gravity waves in the upper layer multiplied by $\kappa$ on the $\kappa L_0 - \kappa U_0$ plane: $\kappa H_+$ (a), $\kappa H_-$ (b). The contour line of $\kappa H_-=0$ is coincident with the line of $u_0(0)=0$.

\[ U_{1+} + \mathbf{A}(U_0)U_{1+} + \varepsilon(U_1 + \mathbf{F}_{U_1} \mathbf{A}(U_0))U_{1+} = O(\varepsilon^2). \]  
(3-11)

Now we use a coordinate system moving with the long time scale $\tau$, that is,

\[ \xi = x - \lambda_+(U_0)t, \]
\[ \tau = \varepsilon t. \]  
(3-12)
Then Eq. (3-11) becomes
\[ U_{1z} + U_1 \cdot \nabla_{U_0} A(U_0) U_{1z} = O(\varepsilon), \]  
(3-13)
and the product of Eq. (3-13) with the left eigenvector \( L_z \) of matrix \( A(U_0) \) gives
\[ \phi_x + \alpha_z \phi_y \phi_y = O(\varepsilon), \]  
(3-14)
where
\[ \alpha_z = R_x \cdot \nabla_{U_0} \lambda_z(U_0). \]

The distributions of the values of \( \alpha_z \) on the \( \kappa L_0 - \kappa U_0 \) plane are shown in Figs. 5(a) and (b), and their explicit forms are shown in Appendix (D). Since \( \alpha_z \) is positive everywhere, the SCW tends to steepen towards the downstream side of wave crests for \( \phi, U_1 \) and \( h_{1z}(x, 0, t) \). This tendency is the same as that of the nonlinear Kelvin wave reported by Bennet (1973). \( \alpha_z \) is negative except for the region of \( h_{xy}(0) < 0 \). Therefore, the SFW shows the opposite tendency for \( \phi \). Since there is no mean current which satisfies both \( \alpha_+ = 0 \) and \( \alpha_- = 0 \), the nonlinear semigeostrophic gravity waves must steepen towards either the downstream or upstream side of wave crests. Then, as the waves steepen, the ageostrophic effect becomes important for the evolution of the waves. This effect will be discussed in the next section.

4. Effect of ageostrophic currents \( O(\varepsilon) \)
4.1. Derivation of K-dV equation

In the preceding section, we clarified the relationships between \( U_1, L_z \) and \( h_{1z}(x, 0, t) \) for SCW and SFW, and the nonlinearity of these waves. In this section, we will examine the ageostrophic effect on weak nonlinear semigeostrophic gravity waves. The interaction between SCW and SFW is not considered.

Since the ageostrophic effect is caused by wave motion, we consider the following solutions,
\[ u = u_{00}(y) + \varepsilon \{ u_{10}(x, y, t) + \partial_x^2 u_0(x, y, t) \}, \]
\[ h = h_{00}(y) + \varepsilon \{ h_{10}(x, y, t) + \partial_x^2 h_0(x, y, t) \}, \]
\[ v = \varepsilon \{ v_{10}(x, y, t) + \partial_x^2 v_0(x, y, t) \}. \]

Setting \( \varepsilon = O(\varepsilon) \), Eqs. (2-5), (2-6) and (2-8) become
\[ \begin{aligned}
    u_{211} + u_{10} u_{110} + (u_{00} - 1) u_{110} + \varepsilon \{ u_{11} u_{11}^0 + u_{10} u_{10} + \gamma (u_0 + u_{00} u_{10})
    + (u_{00} - 1) u_0 + h_{10} \} = O(\varepsilon^2),
\end{aligned} \]
(4-1)
\[ \begin{aligned}
    v_{111} + u_{00} v_{110} + u_{10} + h_{10} = O(\varepsilon),
\end{aligned} \]
(4-2)
\[ \begin{aligned}
    v_{110} - u_{00} - \kappa^2 h_0 = O(\varepsilon),
\end{aligned} \]
(4-3)
where \( \gamma = \varepsilon^2 / \varepsilon \). Equations (4-2) and (4-3) imply that the ageostrophic currents are induced by the semigeostrophic currents. Using the fact that \( \partial / \partial x = -\lambda_z \partial / \partial x + O(\varepsilon) \), where we abbreviated \( \lambda_z(U_0) \) to \( \lambda_z \) for convenience, we have the following equations from Eqs. (4-2) and (4-3),
\[ \begin{aligned}
    u_{00} + h_{10} &= \varepsilon (\lambda_z - u_{00}(y)) V_x(y) \phi_{xx} + O(\varepsilon),
\end{aligned} \]
(4-4)
\[ \begin{aligned}
    u_{10} + \kappa^2 h_0 &= V_x(y) \phi_{xx} + O(\varepsilon),
\end{aligned} \]
(4-5)
where
\[ \begin{aligned}
    V_x(y) &= \frac{1}{\kappa^2 h_{00}(y)} \left[ (u_{00}(y) - \lambda_z) R_x \cdot \nabla_{U_0} h_{00} + R_x \cdot \nabla_{U_0} h_{00} \right].
\end{aligned} \]

Fig. 5. Contours of the coefficients of the nonlinear term in Eq. (3-15) on the \( \kappa L_0 - \kappa U_0 \) plane: \( \alpha_+ \) (a), \( \alpha_- \) (b).
Therefore $u_a$ and $h_a$ can be expressed in the following forms,

$$u_a = f_s(y)\phi_{sxx} + O(\varepsilon),$$

$$h_a = g_s(y)\phi_{sxx} + O(\varepsilon).$$

(4-6)

On the other hand, since $u(y_0,0) = -\kappa h_0 v = 0$ at $y = 0$ and $y = L_0$, from Eq. (4-1), we have the following condition

$$u u_{xt} + u u_{tt} + h_{xt} = -\frac{1}{\varepsilon^2}[u u_{tt} + u u_{tt} + h_{xx}]$$

$$-\frac{1}{\varepsilon}[u u_{tt} + v_{tt} + O(\varepsilon)] = 0, \quad L_0.$$

(4-7)

Substitution of Eqs. (3-8) — (3-10) and (4-6) into Eq. (4-7) yields

$$\phi_\varepsilon + \alpha_s\phi_s + \varepsilon[\alpha_s\phi_s + \gamma\mu_s\phi_{sxx}] = O(\varepsilon^2),$$

(4-8)

where

$$\mu_s = \frac{(U_0 - \lambda_s)f_{s}(L_0) + g_s(L_0)}{r_{s1} - r_{s2}}$$

$$= \frac{(u_{s0}(0) - \lambda_s)f_{s}(0) + g_s(0)}{r_{s1} - r_{s2}} = \frac{\kappa U_0 \sinh \kappa L_0 - \cosh \kappa L_0}{r_{s1} - r_{s2}}.$$  

(4-9)

The coefficients $\mu_s$ can be obtained from Eqs. (4-4) — (4-6) and (4-9) as follows. Using Eq. (4-6), Eqs. (4-4) and (4-5) become

$$f_s + g_s = (\lambda_s - u_{s0}(y)) V_s(y),$$

(4-10)

$$f_s + g_s = V_s(y).$$

(4-11)

Therefore we get

$$g_s - \kappa^2 g_s = [\lambda_s - u_{s0}] V_s - V_s.$$  

(4-12)

Because $h_a$ vanishes at $y = L$, $h_a(L_0)$ must be of $O(\varepsilon)$, that is,

$$g_s(L_0) = 0.$$  

(4-13)

The general solution of Eq. (4-12) is

$$g_s(y) = g_{s*}(y) + C_1 \cosh \phi(L_0 - y)$$

$$+ C_2 \sinh \phi(L_0 - y),$$

(4-14)

where

$$g_{s*}(y) = -\frac{1}{2} \int_{L_0}^{y} (u_{s0}(y) - \lambda_s) V_s(\eta)$$

$$+ \frac{1}{2\kappa} V_s(\eta)(e^{-\kappa(y - \eta)} - e^{\kappa(y - \eta)})d\eta.$$  

(4-15)

Using Eqs. (4-14) and (4-15), Eq. (4-10) yields

$$f_s(y) = f_{s*}(y) + \kappa C_1 \sinh \phi(L_0 - y)$$

$$+ \kappa C_2 \cosh \phi(L_0 - y),$$

(4-16)

where

$$f_{s*}(y) = -\frac{\kappa}{2} \int_{L_0}^{y} (u_{s0}(y) - \lambda_s) V_s(\eta)$$

$$+ \frac{1}{2} \int_{L_0}^{y} V_s(\eta)(e^{-\kappa(y - \eta)} + e^{\kappa(y - \eta)})d\eta.$$  

(4-17)

The boundary conditions, Eqs. (4-9) and (4-13), imply that $C_1 = 0$ and

$$C_2 = \frac{[u_{s0}(0) - \lambda_s]f_{s*}(0) + g_{s*}(0)](r_{s1} - r_{s2})}{\kappa(r_{s1} - r_{s2})(U_0 - u_{s0}(0)) \cosh \kappa L_0}$$

$$+ [\kappa^2 U_0 (U_0 - \lambda_s) r_{s1} - r_{s1} + r_{s2}] \sinh \kappa L_0.$$  

So that

$$\mu_s = \frac{\kappa(U_0 - \lambda_s)[(u_{s0}(0) - \lambda_s)f_{s*}(0) + g_{s*}(0)]}{\kappa(r_{s1} - r_{s2})(U_0 - u_{s0}(0)) \cosh \kappa L_0}$$

$$+ [\kappa^2 U_0 (U_0 - \lambda_s) r_{s1} - r_{s1} + r_{s2}] \sinh \kappa L_0.$$  

(4-18)

The explicit forms of $f_{s*}(0)$ and $g_{s*}(0)$ can be evaluated as shown in Appendix (E).

Now we use the coordinate system represented by Eq. (3-12). Then Eq. (4-8) becomes

$$\phi_{s*} + \alpha_s \phi_s \phi_{s*} + \gamma \mu_s \phi_{sxxx} = O(\varepsilon).$$

Thus the semigeostrophic gravity waves of finite small amplitude satisfy the Kortweg-de Vries (K-dV) equation. Zabusky and Kruskal (1965), and others, showed that a long wave governed by this equation develops into a train of solitary waves. This solitary wave solution is

$$\phi_s = \phi_0 + \delta \sech \left[ \left( \frac{\alpha_s \phi_{s*}}{12\gamma \mu_s} \right)^{1/2} (\xi - (\phi_0 + \lambda_s) t) \right],$$

where

$$\lambda_s = \frac{\alpha_s}{3} \phi_0.$$  

For $\alpha_s \mu_s^{\gamma - 1} > 0$, $\phi$ is positive, and for $\alpha_s \mu_s^{\gamma - 1}$
<0, \phi is negative. Since \phi_\infty is an arbitrary constant, we take \phi_\infty to be zero in the following discussion.

4.2. Discussion of the solitary wave solutions

The distributions of the values of \nu_\pm are shown in Figs. 6(a) and (b). The value \nu_+ is positive everywhere. This implies that the larger the wave number is, the smaller the propagation speed of the SCW is. On the other hand, as \nu_- is negative everywhere, the SFW shows the opposite tendency. It is thought that this contrasting behavior results from the opposing directions of propagation of the two waves: the SCW propagates in the positive \kappa direction and the SFW propagates in the negative \kappa direction relative to the current. The reason that the contour lines of \nu_\pm resemble those of \nu_{\pm2} is that dispersion arises from \nu_{\pm1} which equals \partial L_\pm/\partial t at \gamma = L_0.

At the limit of \kappa U_0 = 1 and \kappa L_0 \to \infty, the SCW becomes a Kelvin wave without mean currents, and \nu_+ = 0. This Kelvin wave corresponds to that in Bennet (1973) and the wave steepens and may be break as time goes by. On the other hand, the nonlinearity of SFW vanishes (\alpha \to 0) at this limit, as can easily be proved but is not shown here. Thus, SFW becomes a long frontal wave with an isolated geostrophic surface front. Such a wave has recently been investigated by Paldor (1983).

Schematic views of these semigeostrophic solitary waves are illustrated in Figs. 7(a) and (b). The solitary SCW is divided into two types (I and II in Fig. 7a) by the line \nu_{\pm 0} = 0. For \nu_{\pm 0} > 0, \lambda_1, U_1, L_1 and \nu_{\pm 1}(x, 0, t) are positive (region I). For \nu_{\pm 0} < 0, L_1 is negative and the other parameters are positive (region II). The solitary SFW is divided into three types (I, II and III in Fig. 7b) by the line \nu_{\pm 0} = 0 and the line \nu_{\pm 0} = 0 on the \kappa L_0 - \kappa U_0 plane. For \nu_{\pm 0} > 0, \nu_{\pm 1}(x, 0, t) and L_1 are positive and the other parameters are negative (region I).
Where \( u_{00}(0) < 0 \) and \( h_{00}(0) \) is not nearly zero, \( U_1 \) and \( L_1 \) are positive and \( h_{11}(x, 0, t) \) and \( \lambda_s \) are negative (region II). For \( h_{00}(0) \geq 0, h_{11}(x, 0, t) \) is positive and the other parameters are negative (region III).

5. Conclusions and oceanic applications

We have shown that semigeostrophic currents which have finite and uniform potential vorticity and are bounded away from a coastline by a surface density front have two kinds of waves. One of them (SCW) is regarded as the generalized Kelvin wave and the other (SFW) is the wave associated with the surface density front. These waves have a dispersive nature and satisfy the K-dV equation when the weak nonlinearity and ageostrophic effect are taken into consideration.

The fact that these waves satisfy the K-dV equation implies that the changes in mass transport of density-driven coastal boundary currents on the upstream side propagate downstream while they are developing into wave-like disturbances. Hanawa (1983), on the basis of the data analysis of sea surface temperatures monitored by a car ferry operating along the Sanriku coast, reported that changes in width of the Tsugaru warm current propagate downstream as wave-like disturbances after an increase in mass transport in the Tsugaru Strait. This observational result is in accord with the present theory. It seems that the semigeostrophic gravity waves play an important role in the propagation of changes in mass transport of coastal density currents.

Kubokawa and Hanawa (1984) observed waves whose propagation speed is close to that of SFW in laboratory experiments in which a density current intrudes along a vertical boundary of a rotating fluid. These waves were generated at the leading edge of the density current. Kubokawa and Hanawa also showed that most of the experimental results of intrusion of a density current in a rotating fluid are interpretable in the context of semigeostrophic gravity wave.

Recently, Maxworthy (1983) investigated experimentally the "solitary internal Kelvin wave". We may be able to regard his "second solitary wave" as the solitary SFW described in this paper, though his experimental conditions were slightly different from the situation in our theoretical model.

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References


Appendix

(A) Values of \( \lambda_s \) at various limits

\[ A_{ij}(i, j=1, 2) \] can be rewritten using \( u_s(x, 0, t) \) as follows,
Semigeostrophic Gravity Waves and the Intrusion of a Density Current 1.

\[ A_{11} = \frac{\cosh \kappa L - 1}{\frac{\kappa U \sinh \kappa L}{\kappa U \sinh \kappa L}} u_0(x, 0, t) + U, \quad (A-1) \]

\[ A_{12} = \frac{\kappa U \sinh \kappa L - \cosh \kappa L + 1}{\kappa U \sinh \kappa L} u_0(x, 0, t), \quad (A-2) \]

\[ A_{21} = \frac{\cosh \kappa L - 1}{\kappa U \sinh \kappa L} u_0(x, 0, t), \quad (A-3) \]

\[ A_{22} = A_{12}. \quad (A-4) \]

Therefore, \( A_{11} = U \) and \( A_{12} = A_{21} = A_{22} = 0 \) can be obtained for \( u_0(x, 0, t) = 0 \). Substituting the above results into Eq. (3-2), we obtain \( \lambda_+ = U \) and \( \lambda_- = 0 \) for \( u_0(x, 0, t) = 0 \).

A.2. Values of \( \lambda_\pm \) as \( \kappa L \to \infty \) and \( \kappa U \geq 1 \)

When \( \kappa L \geq 1 \) and \( \kappa U \geq 1 \), \( A_{ij}(i, j = 1, 2) \) can be rewritten as follows,

\[ A_{11} = \frac{1}{2\kappa^2 U} [\kappa U - 1] e^{\kappa U} + \frac{1}{\kappa^2 U} \left[ 1 - \kappa U + \kappa^2 U^2 \right], \quad (A-5) \]

\[ A_{12} = \frac{1}{2\kappa^2 U} [\kappa U - 1]^2 e^{\kappa U} + \frac{1}{\kappa U} [\kappa U - 1], \quad (A-6) \]

\[ A_{21} = \frac{1}{2\kappa^2 U} [\kappa U - 1] (e^{\kappa U} - 2), \quad (A-7) \]

\[ A_{22} = A_{12}. \quad (A-8) \]

Substituting the above results into Eq. (3-2), we have

\[ \lambda_+ = \frac{1}{2\kappa} \left[ \kappa U - 1 \right] e^{\kappa U} + \frac{1}{\kappa} \left[ \kappa U - 1 + 1 \right], \quad (A-9) \]

\[ \lambda_- = \frac{1}{2\kappa} \left[ \kappa U - 1 \right]. \]

(B) Explicit forms of \( r_{+1}, r_{+2}, r_{-1} \) and \( r_{-2} \)

\[ r_{+1} = \frac{1}{\sqrt{1 + \left( \frac{A_{21}(U_0)}{A_{22}(U_0) - \lambda_+} \right)^2}}, \quad (B-1) \]

\[ r_{+2} = \frac{A_{21}(U_0)}{(A_{22}(U_0) - \lambda_+) \sqrt{1 + \left( \frac{A_{21}(U_0)}{A_{22}(U_0) - \lambda_+} \right)^2}}, \quad (B-2) \]

\[ r_{-1} = -\frac{A_{12}(U_0)}{(A_{11}(U_0) - \lambda_-) \sqrt{1 + \left( \frac{A_{12}(U_0)}{A_{11}(U_0) - \lambda_-} \right)^2}}, \quad (B-3) \]

\[ r_{-2} = \frac{1}{\sqrt{1 + \left( \frac{A_{12}(U_0)}{A_{11}(U_0) - \lambda_-} \right)^2}}, \quad (B-4) \]

where \( \lambda_+ \) and \( \lambda_- \) mean \( \lambda_+(U_0) \) and \( \lambda_-(U_0) \), respectively.

(C) Values of \( r_{+1}, r_{+2} \) and \( H_\pm \) at various limits

C.1. Values of \( r_{+1}, r_{+2} \) and \( H_\pm \) for \( u_{x0}(0) = 0 \)

\( A_{11} = U_0 \) and \( A_{12} = A_{21} = A_{22} = 0 \) can be obtained for \( u_{x0}(0) = 0 \) as shown in Appendix (A).

Substituting the above results into the explicit forms of \( r_{+1}, r_{+2} \) and \( H_\pm \), and using Eq. (2-9), we obtain

\[ r_{+1} = 1, \quad r_{+2} = 0, \quad r_{-1} = 0, \quad r_{-2} = 1, \quad H_+ = \frac{1}{\kappa} \sinh \kappa L_0 \] and \( H_- = 0 \), for \( u_{x0}(0) = 0 \).

C.2. Values of \( r_{+1}, r_{+2} \) and \( H_\pm \) as \( \kappa L \to \infty \) and \( \kappa U_0 \geq 1 \)

Substituting Eqs. (A-5)–(A-9) into Eqs. (B-1)–(B-4), we have

\[ r_{+1} = \begin{cases} 1, & \text{for } \kappa U_0 = 1, \\ \frac{1}{\sqrt{2}}, & \text{for } \kappa U_0 > 1, \end{cases} \]

\[ r_{+2} = \begin{cases} 0, & \text{for } \kappa U_0 = 1, \\ \frac{1}{\sqrt{2}}, & \text{for } \kappa U_0 > 1, \end{cases} \]

\[ r_{-1} = -\frac{\kappa U_0 - 1}{\sqrt{(\kappa U_0 - 1)^2 + 1}}, \]

\[ r_{-2} = \frac{1}{\sqrt{(\kappa U_0 - 1)^2 + 1}}. \]

Substituting the above results into Eq. (3-7), the values of \( H_\pm \) become at this limit,

\[ H_+ = \frac{1}{2\kappa} e^{\kappa U_0}, \quad \text{for } \kappa U_0 = 1, \]

\[ H_- = 0, \quad \text{for } \kappa U_0 > 1, \]

(D) Explicit forms of \( \alpha_\pm \)

\[ \alpha_\pm = \frac{1}{2} \left( 1 \mp \sqrt{1 - \left( \frac{A_{11}(U_0) + A_{12}(U_0)}{A_{11}(U_0) + A_{12}(U_0)} \right)^2} \right) \left( r_{+1} \frac{\partial}{\partial U_0} + r_{+2} \frac{\partial}{\partial L_0} \right) \left( A_{11}(U_0) + A_{12}(U_0) \right) \]

\[ = \frac{1}{\sqrt{1 - 4A_{11}(U_0) A_{12}(U_0) / (A_{11}(U_0) + A_{12}(U_0))^2}} \left( r_{+1} \frac{\partial}{\partial U_0} \right) \]
\[ + r z^2 \frac{\partial}{\partial L_0} \]_{A_{12}(U_0)U_0 = \frac{2A_{12}(U_0)U_0}{(A_{11}(U_0)+A_{12}(U_0))^2}} \]
\[
( r z^2 \frac{\partial}{\partial U_0} + r z^2 \frac{\partial}{\partial L_0})(A_{11}(U_0) + A_{12}(U_0)) \]
\]
where
\[
\frac{\partial}{\partial U_0}(A_{11}(U_0) + A_{12}(U_0)) = \cosh \kappa L_0 + 1,
\]
\[
\frac{\partial}{\partial U_0} A_{12}(U_0) U_0 = \frac{1}{\kappa \sinh \kappa L_0} [\kappa U_0 \sinh 2\kappa L_0 - \cosh 2\kappa L_0 + \cosh \kappa L_0],
\]
\[
\frac{\partial}{\partial L_0} A_{12}(U_0) U_0 = -\frac{\cos \kappa L_0}{\sinh \kappa L_0} [\kappa U_0 \sinh 2\kappa L_0 + A_{12}(U_0) + \kappa U_0 (\cosh 2\kappa L_0 + \cosh \kappa L_0)],
\]
\[
\frac{\partial}{\partial L_0} A_{12}(U_0) U_0 = -\frac{\cos \kappa L_0}{\sinh \kappa L_0} [\kappa U_0 \sinh 2\kappa L_0 + A_{12}(U_0) + \kappa U_0 (\cosh 2\kappa L_0 + \cosh \kappa L_0)],
\]
\[
(1 + \frac{1}{2} \kappa U_0^2) \cosh 2\kappa L_0 - 2 \kappa U_0 \sinh 2\kappa L_0 - \cos \kappa L_0 + \kappa U_0 \sinh \kappa L_0].
\]

(E) **Explicit forms of** $f_{\#}(0)$ **and** $g_{\#}(0)$

The integration of Eqs. (4-15) and (4-17) can be carried out by introducing a new variable $Y = \exp(\kappa y)$. When $f_{\#}(0)$ and $g_{\#}(0)$ are written as
\[
f_{\#}(0) = k F_{\#1},
g_{\#}(0) = F_{\#2},
\]
$F_{\#}(i=1, 2)$ takes the following form
\[
F_{\#1} = -\frac{1}{2\kappa^2} \left[ \frac{A_{\#1}}{3a} (1 - Y_0) + \frac{1}{2a} (B_{\#1} - B_{\#1}) - \frac{A_{\#1}}{a} \right] (1 - Y_0) + \frac{1}{a} C_{\#1} - C_{\#1} + (-1)^i A_{\#1} - \frac{1}{a} (B_{\#1} - B_{\#1}) + \frac{1}{a} \left[ \frac{A_{\#1}}{a} - b \right] A_{\#1} (1 - Y_0) + \frac{1}{\kappa U_0} [P_{\#1} - P_{\#1} + (-1)^i (C_{\#1} + C_{\#1}) + \frac{b}{a} Y_0^{-2} \left[ C_{\#1} + (-1)^i A_{\#1} - \frac{1}{a} (B_{\#1} - B_{\#1}) + \frac{1}{a} a \right] A_{\#1} + b \right] Y_0^{-1} \left[ D_{\#1} - D_{\#1} \right] \]
\[
- \frac{A_{\#1}}{a} \right] (1 - Y_0) + \frac{1}{a} C_{\#1} - C_{\#1} + (-1)^i A_{\#1} - \frac{1}{a} (B_{\#1} - B_{\#1}) + \frac{1}{a} \left[ \frac{A_{\#1}}{a} - b \right] A_{\#1} (1 - Y_0) + \frac{1}{\kappa U_0} [P_{\#1} - P_{\#1} + (-1)^i (C_{\#1} + C_{\#1}) + \frac{b}{a} Y_0^{-2} \left[ C_{\#1} + (-1)^i A_{\#1} - \frac{1}{a} (B_{\#1} - B_{\#1}) + \frac{1}{a} a \right] A_{\#1} + b \right] Y_0^{-1} \left[ D_{\#1} - D_{\#1} \right] \]
\[
\]
半地衡性重力波とその岸に沿う密度流の侵入への適用

1. 半地衡性重力波

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要旨: ポテンシャル渦度が有限でかつ空間的に一様な、海面上に密度前線をもつ沿岸密度流に伴い生じる半地衡性重力波（Semigeostrophic gravity wave）について調べた。その結果、沿岸密度流には2種類の半地衡性重力波が伴うことが判った。本論文では、この2種類の波動を半地衡性沿岸波（SCW）および半地衡性前線波（SFW）と名付けた。SCW はある極限でケルヴィン波に一致する波動であり、SFW は前線の存在に本質的に依存する波動である。前者は岸での上層の厚さと前線での岸に沿う方向の密度変動として主に現われ、後者は沿岸の幅の変動として主に現われる。また、これらの波動は弱非線形性と非地衡性を考慮すると Kortweg-de Vries 方程式に支配されることを示した。このことは、沿岸密度流の局所的な変動が波動状態を形成させることを示唆している。

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