Statistical Characteristics of Individual Waves in Laboratory Wind Waves

II. Self-Consistent Similarity Regime*

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Abstract: On the basis of data on the statistical characteristics of individual waves in laboratory wind waves reported in part I of this series, a self-consistent similarity regime is found to exist among properties of the individual waves, such as the nondimensional frequency, the wave number, the phase speed, and the steepness. Also, it is shown that forms of past empirical formulas for the development of the peak wave can be derived starting from the 3/2-power law, as an extension of the present laboratory experimental data. In the derivation, only values of the coefficient of the 3/2-power law, and the fraction of momentum transferred from the wind retained by the wind waves, remain on an empirical basis.

1. Introduction

In part I of this series (TOKUDA and TOBA, 1981), statistical characteristics of individual waves in laboratory wind waves were investigated. Here the term "individual waves" expresses actual undulations of the water surface at any instant, in contrast to "component waves" which are derived from Fourier decomposition of surface undulations. This approach has been used in the light of the fact that the physics of wind waves is quite different from a superposition of linear-wave trains owing to their strong nonlinearities. This investigation of individual-wave characteristics is expected to supplement the traditional approach that uses Fourier decomposition of water surface undulations.

It was demonstrated in Part I that the individual wave field under strong coupling with the wind is characterized by a conspicuous similarity structure which manifests itself as a simple spectral form and as a form of statistical law among the nondimensional wave height, the wave period and the phase speed. Theoretical derivation of the similarity structure is difficult. However, to investigate various actual features of the similarity and to construct a self-consistent regime for it, are very useful for the development of an appropriate scheme of wave prediction and for the parameterization of air-sea interactions.

In this article, a self-consistent regime existing among various properties of the individual waves is formulated. Furthermore, it is shown that the empirical formulas obtained in the present experiment can be extended to conditions observed in the sea.

2. Self-consistent similarity regime for individual waves

The 3/2-power law of Part I (TOKUDA and TOBA, 1981: Eq. 4.3) is rewritten as

\[ H^* = B_\sigma \sigma^{2 - 3/2} (\sigma^* \sigma^*) \]

where

\[ H^* = \frac{gH}{u_*^3}, \sigma^* = \frac{u_*^2 \sigma}{g} \text{ and } B_\sigma = (2\pi)^{3/2}B, \]

in which \( H \), \( \sigma \), \( u_* \) and \( g \) are the wave height of individual waves, the angular frequency, the friction velocity of air and the acceleration of gravity, respectively, and \( B = 0.043 \). The subscript \( p \) denotes the characteristics of the spectral peak. The nondimensional phase speed \( C^* \) is defined by

\[ C^* = C/u^* \]
The steepness $\delta$ is

$$\delta = k^* H^*/2\pi$$  \hspace{1cm} (2.3)

where $k^* = u_*^2 k/g$ and $k$ is the wave number. The 3/2-power law (2.1) requires, by combination with (2.3), the following relation between the wave number and the angular frequency

$$k^* = [(2\pi)^{-1/2} B^{-1} \delta]^{\delta^{4/3}}$$  \hspace{1cm} (2.4)

where $\delta$ is unknown at this stage. Combining this with (2.2) and $C = \sigma/k$, we obtain the phase speed

$$C^* = [(2\pi)^{-1/2} B^{-1} \delta]^{-1/2} \sigma^{2/3}$$  \hspace{1cm} (2.5)

or

$$C^* = [(2\pi)^{-1/2} B^{-1} \delta]^{-2/3} k^{4/3}$$  \hspace{1cm} (2.6)

The corresponding equations by the linear theory are

$$k^* = \sigma^{2/3}, \quad C^* = \sigma^{1/3}, \quad C^* = k^{4/3}$$  \hspace{1cm} (2.7)

It was indicated in Part I that the phase speed of individual waves $C^*$ was directly proportional to $k^{4/3}$ when the steepness was assumed as constant from the data, in the main frequency range of the high frequency side. However, in the case of ocean, the phase speed of the main frequency range is believed to be expressed approximately by the linear theory and is very large compared with the surface wind drift. Consequently, we should allow that the steepness is weekly dependent on the wave number or the frequency, in order that the empirical formulas indicated in Part I may be extended to conditions in the sea. We now assume the following expression for $C^*$:

$$C^* = k^{4/3} + u_*^*, \quad u_*^* = u_*/u_0$$  \hspace{1cm} (2.8)

where $u_*$ is the phase speed anomaly defined by

$$u_* = C - C_i, \quad C_i = (g/k)^{1/2}$$  \hspace{1cm} (2.9)

The quantity $C_i$ represents the phase speed of the irrotational wave motion approximated by the linear wave, and $u_*$ is interpreted as representing an overall effect of the rotational part of the special ordered motion accompanying the individual waves; consequently, we may call this value the effective wind drift. If the phase speed of Stokes wave of finite amplitude is used for $C_i$, the values of $u_*$ become smaller by about 15%, but the following qualitative arguments remain unaltered, and we adopt the linear phase speed for simplicity. The experiment shows that $u_*^*$ has an approximately constant equilibrium value, which is independent

\[\text{Fig. 1. Phase speed of the individual waves at the spectral peak as a function of frequency and wave number for eight fetches and three wind conditions (see Tokuda and Toba, 1981: Table 1). The dotted line indicates the dispersion relation for linear water waves, and the solid lines Eq. (2.8) with $u_*^* = 0.206$ in (b), and converted by (2.4) in (a). The value of $u_*^*$ is obtained from Fig. 2.}\]
of the wind speed or the degree of wind-wave development. The data are shown in Fig. 1 for peak waves for three wind conditions and for eight fetches, where the solid curve in the right part represents (2.8) with \( u_0^* = 0.206 \). In the left part of Fig. 1, the solid line is entered by its conversion through (2.4), by use of \( B \) and \( \delta \) determined later in this paper. Data points for very short fetches deviate from the solid curves. This is due to the fact that the equilibrium value of \( u_0^* \) is not reached at very short fetches. These conditions are clearly shown in the upper part of Fig. 2. This situation will be discussed below.

We define another quantity \( u_\theta^* \) by

\[
\nu_\theta^* \equiv u_\theta / u_0, \quad u_\theta = \frac{H}{\alpha} \sqrt{\frac{g}{4}}
\]  (2.10)

where \( u_\theta \) expresses the mass transport velocity of individual waves when they are approximated by second order Stokes waves. This value is interpreted as representing the irrotational part of the motion of the individual waves. It was already pointed out by TOBA (1978, 1979) that if we assumed dimensionally

\[
u_0^* \alpha \nu_0 \alpha u_0 \tag{2.11}\]

and combined this with (2.10), the 3/2-power law (2.1) could be immediately obtained. The above relation is supported experimentally by the fact that \( u_0^* \) has a similar trend as \( u_1^* \) in Fig. 2. They attain the following equilibrium values at a fetch around \( F = 3.5 \) m:

\[
u_0^* = 0.114 \pm 0.007 \tag{2.12}
\]

\[
u_1^* = 0.206 \pm 0.017 \tag{2.13}
\]

The second term on the right-hand side of these equations represents the standard deviation of the data of Fig. 2. The values of \( u_1^* \) are rather scattered since they are residual values defined by (2.9). From the combination of (2.10) and (2.11), the value of \( B \) is given by

\[
B = 2(2\pi)^{-1/2} u_\theta^*^{1/2} \tag{2.14}
\]

The use of \( u_\theta \) of (2.12) in (2.14) gives

\[
B = 0.0429 \pm 0.0013 \tag{2.15}
\]

This value is in good agreement with the experimental value of \( B = 0.043 \pm 0.002 \) determined from the values of \( B \) for \( F > 3.5 \) m in Table 2 of Part I. These values of \( B \) were obtained from the slope of the \( H^* - T^* \) diagram, and are plotted in the lower part of Fig. 2.

Fig. 2 demonstrates that values of \( u_0^* \), \( u_\theta^* \), and \( B \) behave in a similar fashion to one another with changes in fetch. These values should be zero at \( F = 0 \) i.e. at the wall, and they actually attain equilibrium values at \( F \approx 3.5 \) m which are independent of the wind and of the degree of wind-wave development. Fig. 3 also shows that (2.8) with a constant \( u_1^* \) can be applied not only to the peak waves but also to the individual waves with any wave number.

PLANT and WRIGHT (1979, 1980) have recently presented phase speed data of component waves in laboratory wind waves, measured with microwave scattering techniques. Their data shows phase speed trends similar to our data for individual waves. From their data \( u_\theta^* \) is

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**Fig. 2.** Nondimensional effective wind drift \( u_\theta^* \), mass transport velocity of individual waves \( u_\theta^* \), and the coefficient of the 3/2-power law \( B \) (from TOKUDA and TOBA, 1981: Table 2) as a function of fetch. The values increase with the fetch to attain their equilibrium values; Solid lines are best fit lines.
estimated as around 0.16.

Equation (2.8) has a form of current-transport of linear waves similar to PLANT and WRIGHT (1979, 1980). However, it should be noted here that, contrary to what one might think, this speed of transport can not be arbitrarily selected, but is determined by the coupling of the wind and the individual waves itself. This explains the constant value of \( u_s^* \).

The steepness \( \delta \) in (2.5) is determined by the use of (2.8) in (2.6):

\[
\delta = (2\pi)^{1/2} B (k^* - 1/6 + u_s^* k^{1/3})^{-1/2}
\]

The value of \( \delta \) estimated from (2.16) by use of \( B = 0.0429 \) and \( u_s^* = 0.206 \) is shown by a solid curve in Fig. 4, and the measured values for the peak waves are also given. The measured values approach the solid curve as the fetch approaches 3.5 m.

It should be noted that \( \delta \) is a very weak function of \( k^* \) in (2.16). The phase speed of the peak wave asymptotically approaches the value for linear waves as the wave number of the dominant waves decreases, since \( u_s^* \) is constant and the 2nd term of (2.8) becomes small compared with the 1st term.

3. Development of the peak wave with fetch

By virtue of similarity, the growing wind-wave field can be represented by the properties of the wave with the peak frequency. In this section, forms of empirical formulas so far proposed for the development of the peak wave in the fetch limited case are derived as an extension of the short-fetch results. We start from the
The 3/2-power law and use the linear wave approximation for the peak wave. Hereafter we omit the subscript \( p \). From (2.1) it follows that

\[
d \ln H^*/d \ln T^* = 3/2
\]  (3.1)

The equation between wave energy \( E \) and wave momentum \( M \) for the irrotational part of the motion

\[
E = CM
\]  (3.2)

is written in a differential form,

\[
dE = CdM + M dC
\]  (3.3)

Using \( E \sim H^{**} \) (cf. Eq. 4.7 of Part I: TUKUDA and TOBA, 1981) and the linear wave approximation \( C \sim T^{**} \), together with (3.1), we obtain

\[
CdE/(EdC) = 2d \ln H^*/d \ln T^* = 3
\]  (3.4)

From this the ratio of the two terms of (3.3) becomes

\[
CdM/(MdC) = CdE/(EdC) = 1 = 2
\]  (3.5)

Consequently, (3.3) becomes simply

\[
dE = (3/2) CdM
\]  (3.6)

On the other hand, with the group velocity \( C_g \) and time \( t \), we have

\[
\frac{dE}{dt} = \frac{\partial E}{\partial t} + \frac{\partial}{\partial F}(C_g E)
\]  (3.7)

For steady state

\[
\frac{dE}{dt} = \frac{d}{dF}(C_g E) = \frac{1}{2} \left[ E \frac{dC}{dF} + C \frac{dE}{dF} \right]
\]  (3.8)

The increment of the wind wave energy \( dE \) includes the term representing the increment of wave height \( (CdE) \) and that of wave length or period \( (EdC) \). Equations (3.4) and (3.5) express that the 3/2-power law imposes a condition between them, giving a simple relation (3.6). Using (3.4) in (3.8) we obtain

\[
\frac{dE}{dt} = \frac{2}{3} C \frac{dE}{dF}
\]  (3.9)

Together with (3.6), we arrive at

\[
\frac{dE}{dF} = \frac{9}{4} \frac{dM}{dt}
\]  (3.10)

If we use the momentum retention factor \( G \), or the factor of momentum transferred from the wind retained by wind waves:

\[
G \equiv \frac{1}{\tau} \frac{dM}{dt}
\]  (3.11)

where \( \tau \) is the sea surface wind stress, (3.10) becomes

\[
\frac{dE}{dF} = \frac{9}{4} \tau G
\]  (3.12)

In a nondimensional form, it follows that

\[
\frac{dE^*}{dF^*} = \frac{9}{4} \left( \frac{\rho_a}{\rho} \right) G \sim 2.7 \times 10^{-3} G
\]  (3.13)

where \( \rho_a \) and \( \rho \) are the density of air and water,

\[
F^* \equiv gF/\mu_s^2, \quad E^* \equiv g^2 E/(\rho g \mu_s^4)
\]

and \( \tau = \rho_s \mu_s^2 \).

The same form of growth equation of wind waves was proposed by TOBA (1978):

\[
\frac{dE^*}{dF^*} = 4.3 \times 10^{-4} G,
\]

\[
G = 0.062 \left[ 1 - \mathrm{erf} \left( 0.12 E^{1/3} \right) \right]
\]  (3.14)

where

\[
\mathrm{erf} \,(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp \left( -z^2 \right) d\zeta
\]

He derived this form by applying the 3/2-power law and \( dF = C_d \, dt \) to an empirical fetch graph formula by WILSON (1965) obtained for long fetches. The first equation of (3.14) has the same form as (3.13), though it was derived in a different manner. The coefficient has a slightly different value. However, exact coincidence including the value of the coefficient may not be expected owing to the many approximations used, including the representative wave approximation.

If \( G \) is assumed as constant for small fetches, (3.13) is written as

\[
E^* = 2.7 \times 10^{-3} GF^*
\]  (3.15)

If the maximum value of \( G \) for \( G_0 = 0.062 \), obtained for \( E^* = 0 \) in the second equation of (3.14), is used, it follows that

\[
E^* = 1.7 \times 10^{-4} F^*
\]  (3.16)
corresponding to the form of the JONSWAP formula (Hasselmann et al., 1973):

\[ E^* = 1.3 \times 10^{-4} F^* \quad (3.17) \]

If we convert \( E^* \) to \( H^* \) by

\[ E^* = H^* S / S \quad (3.18) \]

with \( S \) as a numerical factor, it follows from (3.16) that

\[ H^* = B_1 F^{n_1 / 2} \quad (3.19) \]

This form corresponds to a formula by Phillips (1977) where \( B_1 = 0.035 \). If we use a value for \( S \) in (3.18) of 8 (corresponding to Toba's 1978 field data), 10 (corresponding to the present experiment data) and 16 (corresponding to the value given by Longuet-Higgins, 1952, for narrow-spectrum waves), \( B_1 \) in (3.19) becomes 0.038, 0.042 and 0.054, respectively. In these cases the values of \( B_1 \) approximately coincide with \( B \) of the 3/2-power law, although whether this is so by necessity is not yet known. Combination of (3.19) with the definition of steepness (2.3) gives

\[ k_p^* = 2\pi B_1^{-1} \delta_p F^{n_1 - 1 / 2} \quad (3.20) \]

Combination of (3.19) with (2.1) gives

\[ \sigma_p^* = 2\pi (B / B_1)^{2 / 3} F^{n_1 - 1 / 3} \quad (3.21) \]

This formula coincides with various past empirical formulas. The JONSWAP formula corresponds to \( B / B_1 = 1.1 \) in (3.21), Mitsuyasu's (1968) formula has \( B / B_1 = 1.0 \). The above mentioned value of \( B_1 = 0.042 \) and \( B = 0.043 \) for the present experiment gives \( B / B_1 = 1.0 \), and the value of \( B_1 = 0.054 \) and \( B = 0.062 \) by Toba (1978) gives \( B / B_1 = 1.1 \). Thus, all of these values indicate that \( B \approx B_1 \). A combination of (3.16) and (3.21) gives

\[ E^* = 0.045 \sigma_p^{n_1} \quad (3.22) \]

corresponding approximately to a formula given by Toba (1978).

The data of the present experiment are shown in Fig. 5. The lines are drawn with values of \( B = B_1 = 0.043 \); and for (3.20), \( \delta_p \) of (2.16) is used. Data points for smaller fetches deviate for the reason discussed in the last section, but all points approach the lines with increasing fetch.

It is concluded from this paper that various empirical formulas so far proposed for the development of peak waves can be derived on the premise of a 3/2-power law as a continuation of the small-fetch laboratory wind waves. In this derivation, only values of \( B \) and \( G \) remain on an empirical basis.

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References


実験室の風波における個々波の統計的な特性
II. 自己斎一な相似体制

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要旨: Part I に報告した実験室の風波における個々波の統計的な特性に関するデータに基づき、無次元の周波数、波数、位相速度、波形勾配などの個々波の特性の関の、

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自己矛盾のない関係を得た。また、3/2 乗則を前提にしたとき、スペクトルピークの波の発達に関する過去の実験法則の形は、実験室の風波の延長として書き出すことができることを示した。その導出においては、3/2 乗則の係数 B と、風から波に入れて風波に残る運動量の割合 G だけがまだ経験的な根拠に頼っている。