Water Exchange between Adjacent Vortices under an Additional Oscillatory Flow*

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Abstract: It is shown that the coupling effect of the steady vortices and the Eulerian oscillatory flow yields the 8-shaped Lagrangean motion through which adjacent vortices inter-communicates, inducing water exchange between them. The water exchange coefficient is fairly large. This coupling effect is considered to play an important role in the water exchange across the narrow strait which is accompanied with a strong tidal current and a pair of tidal residual circulations.

1. Introduction

Let us consider the water exchange between two adjacent vortices in the horizontally two-dimensional plane.

An example of the vortices is the tidal residual circulations which are studied by Yanagi (1977) and Oonishi (1977). They showed that a tidal current induces two steady vortices with opposite signs near the narrow strait. The magnitude of the velocity of the vortex is of the same order of the tidal oscillatory current. Therefore, it will be concluded that the long term water movement is essentially controlled by the steady current of the vortices. On the other hand, it is obvious that the steady vortices are effective on the water exchange between two adjacent vortices by themselves. Adding the oscillatory tidal motion, however, it is possible to explain the water exchange among two vortices over the time scale longer than the period of the oscillatory flow. Such a co-existence of the steady flow and the oscillatory flow is not unusual in the actual sea. Because there is no Eulerian mean velocity across the vortices, we are concerned with the Lagrangean particle motion.

The relationship between Eulerian and Lagrangean drift velocities when oscillatory currents dominate has been given by Longuet-Higgins (1953) as

\[ \bar{u}_L = \bar{u} + \left( \int u' \ dt \cdot \bar{v} \right) \]

(1)

where \( \bar{u}, u' \) and \( \bar{u}_L \) are the velocities of the mean flow, the oscillatory flow and the mean Lagrangean drift, respectively, and the over-bar denotes the time average over one oscillation cycle. Equation (1) expresses that the Lagrangean drift velocity is the sum of the Eulerian mean velocity and the Stokes’ drift due to the oscillatory current. However, this equation is invalid in the situation when the magnitude of the velocity of the steady vortices is of the same order of the oscillatory current.

Besides the Stokes’ drift, there is a possibility that the coupling occurs among the steady vortices and the oscillatory motion. The aim of the present paper is to show explicitly the water exchange associated with such a coupling. For clarity’s sake, the following examinations are confined to the cases in which the Stokes’ drift vanishes.

2. Equations

Let us suppose the following situation.
1) The steady flow is horizontally two-dimensional.
2) The steady flow is non-divergent.
3) The steady flow has a pattern of succeeding vortices.
4) The oscillatory flow has no Stokes’ drift by itself.
5) The steady and the oscillatory flows have the velocities of the same order of magnitude.

A simple example of the Eulerian velocity field satisfying the above conditions is
\[
\begin{align*}
    u^*(x^*, y^*, t^*) &= a^* l^* \sin k^* x^* \cos l^* y^* \\
    &\quad + b^* \cos \omega^* t^* \\
    v^*(x^*, y^*, t^*) &= -a^* k^* \cos k^* x^* \sin l^* y^*
\end{align*}
\]  
(2)  

Symbols \((x, y), t, (u, v)\) denote the horizontal coordinates, time and the components of the velocity, respectively. A symbol \(*\) denotes the dimensional variables.

Then, Lagrange equations determining the particle position \((X^*, Y^*)\) at time \(t^*\) is given by

\[
\begin{align*}
    \frac{dX^*}{dt^*} &= u^*(X^*, Y^*, t^*) \\
    \frac{dY^*}{dt^*} &= v^*(X^*, Y^*, t^*)
\end{align*}
\]  
(3)  

Now, (2) and (3) are nondimensionalized, yielding the following equations.

\[
\begin{align*}
    \frac{dX}{dt} &= A \sin \pi X \cos \pi Y + 2\pi B \cos 2\pi t \\
    \frac{dY}{dt} &= -A \cos \pi X \sin \pi Y
\end{align*}
\]  
(4)  

where

\[
\begin{align*}
    X^* &= \frac{\pi}{k^*} X \\
    Y^* &= \frac{\pi}{l^*} Y \\
    t^* &= \frac{2\pi}{\omega^*} t
\end{align*}
\]  
(5)  

It should be noted that the period of the oscillation and the half wave length of the steady flow (the size of one vortex) are used to measure time and distance, respectively.

In the first place, let us examine two simple cases. When \(B=0\), the water particle moves along with one of isopleths of the stream function

\[
\psi = \frac{A}{\pi} \sin \pi x \sin \pi y
\]  
(6)  

The parameter \(A\) is inversely proportional to the nondimensional time for the particle to make one circuit in the steady vortices. The larger \(A\) means the stronger velocity in the vortex. The lines \(x=0, \pm 1, \pm 2, \ldots\) and \(y=0, \pm 1, \pm 2, \ldots\) are the virtual barriers in the sense that the water particle cannot pass through them.

When \(A=0\), the solution of (4) is

\[
\begin{align*}
    \left[ X^* \right]_{t=0}^{\infty} &= B \sin 2\pi t \\
    \left[ Y^* \right]_{t=0}^{\infty} &= 0
\end{align*}
\]  
(7)  

The parameter \(B\) is the nondimensional size of the half of the excursion due to the oscillatory flow. The larger \(B\) means the stronger oscillatory velocity.

In general cases that both \(A\) and \(B\) have non-zero values, only the lines \(y=0, \pm 1, \pm 2, \ldots\) are “barriers”. Therefore, the following considerations can be generally restrained in the region bounded by two lines \(y=0\) and \(y=1\).

Nonlinear equations (4) are integrated with time by the following method.

\[
\begin{align*}
    \frac{X^{n+1} - X^n}{\Delta t} &= \frac{A}{2} \sin \pi X^n \cos \pi Y^n \\
    &\quad + \frac{A}{2} \sin \pi X^{n+1} \cos \pi Y^{n+1} + \pi B \\
    &\quad \times \cos 2\pi n dt + \pi B \cos 2\pi (n+1) dt \\
    \frac{Y^{n+1} - Y^n}{\Delta t} &= -\frac{A}{2} \cos \pi X^n \sin n Y^n \\
    &\quad - \frac{A}{2} \cos \pi X^{n+1} \sin \pi Y^{n+1}
\end{align*}
\]  
(8)  

where \(X^n\) and \(Y^n\) denote \(X\) and \(Y\) at \(t=n \Delta t\), respectively. Equations (8) are solved to obtain \(X^{n+1}\) and \(Y^{n+1}\) from given \(X^n\) and \(Y^n\), or to obtain \(X^n\) and \(Y^n\) from given \(X^{n+1}\) and \(Y^{n+1}\). The successions of these procedures gives the forward or the backward time integration for an arbitrary duration.

3. Results

To demonstrate the actual orbit of the water particle, two cases are shown in Fig. 1. The parameters are taken as \(A=1\), \(B=0.2\). Orbit \(O_1\) and \(O_2\) are of the particles which pass the points \(x=2, y=0.5\) at \(t=-0.2\) and \(x=0, y=0.5\) at \(t=0\), respectively. It should be noted that the figure shows only fractions of the orbits which correspond to the duration from \(t=-4\) to \(t=4\). These two orbits represent two types of the particle motions. The former type of motion is nearly trapped in one cell of the vortices. The latter particle intercommunicates between two cells making a 8-shaped orbital motion. The latter type of motion induces the active water exchange between two adjacent cells.
Fig. 1. Two types of the orbital motions. Orbits $O_1$ and $O_2$ are of the particles which pass the points $x=2$, $y=0.5$ at $t=0.2$ and $x=0$, $y=0.5$ at $t=0$, respectively. The parameters are $A=1$ and $B=0.2$.

Fig. 2. The location of the exchanged water. The waters which are in the regions $R_1$, $R_2$ and $R_3$ at $t=0$ (the upper figure) appears in the regions $R'_1$, $R'_2$ and $R'_3$ at $t=1$ (the lower figure), respectively. The water exchange is defined by the sum of the volumes of $R_1$ and $R_3$, which is equal to the volume $R_2$. The parameters are the same as those in Fig. 1.

Table 1. The exchange coefficient.

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<td>0.028</td>
<td>0.063</td>
<td>0.109</td>
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Fig. 3. The exchange coefficient.
Now, let us define the term "water exchange" quantitatively. Equation (7) shows that if \( A = 0 \), the positions of the water particles at \( t = 0, 1/2, 1, 3/2, \ldots \) are the positions averaged with time. On the line \( x = 0 \), the water moves towards right at \( t = 0, 1, 2, \ldots \) and towards left at \( t = 1/2, 3/2, \ldots \). Considering these facts, we define the water exchange in one cycle of the oscillation between two adjacent cells \( C_0 = [-1 \leq x < 0, 0 < y < 1] \) and \( C_1 = [0 \leq x < 1, 0 < y < 1] \) by the volume of water which is in \( C_0 \) at \( t = 0 \) and is in the region \( x > 0 \) at \( t = 1 \). It should be noted that because of the symmetry, this volume is equal to that of the water which is in \( C_1 \) at \( t = 1/2 \) and is in the region \( x < 0 \) at \( t = 3/2 \).

Figure 2 shows the location of the exchanged water at \( t = 0 \) (thick hatch) and at \( t = 1 \) (thin hatch) for \( A = 1, B = 0.2 \). It shows that the coupling of two kinds of flows makes the water exchange.

Finally, the water exchange coefficient is defined by the above volume divided by the volume \( B \) which intercommunicates between two cells due to the oscillatory flow. The coefficient for various combinations of \( A \) and \( B \) are shown in Table I and in Fig. 3. We see that in the range of \( 0 < A < 1 \) and \( 0 < B < 0.5 \), the stronger the steady flow is (the larger \( A \)) and the weaker the oscillatory flow (the smaller \( B \)) is, the larger the exchange coefficient is.

4. Discussions

A linear case gives an insight to the phenomenon. In a simple case, the equation corresponding to (4) are

\[
\begin{align*}
\frac{dX}{dt} &= \pi A X + 2\pi B \cos 2\pi t \\
\frac{dY}{dt} &= -\pi A Y
\end{align*}
\]  

where the stream function of the steady flow is

\[ \psi = \pi A x y \]  

This case may be interpreted as a linear approximation of the vortices near the corner of \( x = 0 \), \( y = 0 \).

The solution of (9) is given by

\[ X = \frac{4B \sin 2\pi t - 2AB(\cos 2\pi t - e^{\pi i t})}{4 + A^2} + \left[ X_0 - \frac{1}{4 + A^2} (4B \sin 2\pi t_0 - 2AB) e^{\pi i t - \pi i t_0} \right] e^{\pi i t - \pi i t_0} \]

\[ Y = Y_0 e^{-\pi i (t - t_0)} \]

where \( (X_0, Y_0) \) is the particle position at \( t = t_0 \). From (11), the particle displacement from \( t = t_0 \) to \( t = t_0 + 1 \) is given as

\[ X - X_0 = X_0 (e^{\pi i t} - 1) + \frac{2AB \sin 2\pi t_0 - 4B \sin 2\pi t_0}{4 + A^2} (e^{\pi i t} - 1) \]

\[ Y - Y_0 = Y_0 (e^{-\pi i t} - 1) \]

In the terms of the right hand sides of (12), the first terms express the drift by the steady flow itself. The second term of the first equation expresses the drift due to the coupling of the steady and the oscillatory flows. The result means the followings.

1) The drift due to the coupling effect depends on the initial time when the particle position is labelled. It suggests that the term "the Lagrangean mean drift velocity" is misleading in this case.

2) If the particle position is labelled at \( t = 0, 1, 2, \ldots \) or \( t = 1/2, 3/2, \ldots \), as was done in the previous section, and if the magnitude \( A \) of the steady flow is weak enough, the drift in one cycle of the oscillation is proportional to \( A^2 B \). This dependence could be an index of the magnitude of the coupling effect, and suggests that the exchange coefficient would be proportional to \( A^2 \). Figure 3 confirms this rough estimation.

The results in the previous section show that the water exchange coefficient is fairly large. This fact suggests that the coupling of the oscillatory and the mean currents plays an essential role in the water exchange across the narrow strait where a strong tidal current and a pair of tidal residual circulations co-exist.

References


付加振動流の存在下での隣接する渦の間の海水交換

大西行雄*, 国司秀明**

要旨: 定常的な渦とオイラー的な振動流のカップリングによって, 隣接する渦の間の海水交換を引き起こす。

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うな, 8 の字型のラグランジュ粒子運動が可能であることが示された。交換係数はかなり大きく, このカップリングは強い潮流と1組の潮汐残留循環を伴う狭い海峡域での海水交換に対して重要な役割を演じると考えられる。