On the Free Oscillation of Water in a Lake of Elliptic Boundary*

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Abstract: The fundamental normal mode of the free oscillation of water in a lake of elliptic boundary and of uniform depth is investigated exactly by the use of Mathieu functions and modified Mathieu functions in two cases. The relationship between the frequency of the fundamental normal mode of oscillation and the eccentricity of the elliptic boundary of the lake is shown graphically. Also, an approximate formula for said relationship is derived, which may be adequately used to find the approximate value of the frequency.

1. Introduction

The problem of the free oscillation of water in an elliptic lake of uniform depth has been studied mathematically in detail but rather in a tedious way by JEFFREYS and GOLDSTEIN. More recently, Professor HIDAKA has given very interesting results by means of numerical integration not only for the case of elliptic lake of uniform depth, but also for the cases of various shapes other than ellipse and of nonuniform depth.

In the present paper, the fundamental normal mode of the free oscillation of water in a lake of elliptic boundary and of uniform depth is investigated exactly by using Mathieu functions and modified Mathieu functions in rather a simple manner. We have carried out some numerical calculations for modified Mathieu functions by using its expansion formulae in Bessel functions. In the following two cases, the resulting relationships between the frequency of the fundamental normal mode of oscillation and the eccentricity of the elliptic boundary of the lake are shown graphically. In the first case the vertical displacement of the water surface from its equilibrium position is assumed to be symmetrical about both principal axes of the ellipse, and in the second case the displacement is symmetrical about the major axis but anti-symmetrical about the minor axis.

Though an exact solution in terms of Mathieu function is mathematically elegant, it is rather complicated for a numerical treatment. The approximate formula for the relationship between the frequency and the eccentricity of the boundary ellipse is also investigated. For the first case this relationship was given in the previous paper, and for the second case, the similar relationship has been derived in the present paper. This formula is correct to the fourth power of the eccentricity which may adequately be used for any values of the eccentricity.

It is one of the aim of the present paper to examine the accuracy of the above formulae by comparing these approximate values of $\frac{\sigma a}{c}$ with those values obtained by the exact solution.

2. Fundamental equation and its solution

Let $w$ be the vertical displacement of the free surface of the lake from its equilibrium position, and let the rectangular coordinate-axes $(x, y)$ be

![Fig. 1.](image-url)
taken in the plane of the undisturbed level as shown in Fig. 1. Then, as is well known, \( v \) satisfies the equation:

\[
\frac{\partial^2 v}{\partial t^2} = c^2 \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),
\]

(1)

where \( c = \sqrt{gd} \), \( g \) and \( d \) being respectively the acceleration due to gravity and the uniform depth of the lake.

When the free surface of the lake oscillates in a normal mode, the displacement is of the form

\[
v = \zeta(x, y) \cos \sigma t,
\]

(2)

where \( \zeta \) is a function of \( x, y \) and \( \sigma \) is the circular frequency. If we insert (2) into (1) and write

\[
k = \frac{\sigma}{c},
\]

(3)

we obtain a partial differential equation for \( \zeta \), namely:

\[
\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} + k^2 \zeta = 0,
\]

(4)

Since the boundary of the lake is vertical everywhere, the boundary condition is given by

\[
(\frac{\partial \zeta}{\partial n}) \text{ on the elliptic boundary} = 0,
\]

(5)

where \( \frac{\partial}{\partial n} \) indicates the differentiation along the normal to the vertical boundary wall. Denoting the focal length of the ellipse by \( 2h \), we now introduce the elliptic coordinates \((\xi, \eta)\) defined as:

\[
x + iy = h \cosh(\xi + i\eta).
\]

(6)

Then equation (4) and boundary condition (5) become respectively

\[
\frac{\partial^2 \zeta}{\partial \xi^2} + \frac{\partial^2 \zeta}{\partial \eta^2} + 2k_1^2 (\cosh 2\xi - \cos 2\eta) \zeta = 0,
\]

(7)

\[
(\frac{\partial \zeta}{\partial \xi})_{\xi = \xi_0} = 0,
\]

(8)

where

\[
2k_1 = k h = \frac{\sigma}{c} h,
\]

(9)

and \( \xi = \xi_0 \) signifies the boundary of the ellipse.

Considering those normal modes of oscillation in which the vertical displacement of the water surface from its equilibrium position is symmetrical about both the major and minor axes of the ellipse, the appropriate particular solution of the equation (7) can be written

\[
\zeta = C \epsilon_{2m}(\xi, q) \epsilon_{2m}(\eta, q),
\]

(m=0, 1, 2, 3, ......)

(10)

where \( \epsilon_{2m}(\eta, q) \) and \( \epsilon_{2m}(\xi, q) \) are Mathieu and modified Mathieu functions of \( m \) th order respectively, and

\[
q = k_1^2 \frac{1}{4} k^2 h^2 = \frac{1}{4} \frac{\sigma^2}{c^2} h^2.
\]

(11)

For simplicity we restrict our attention to the case when the displacement \( \zeta \) is given by

\[
\zeta = C_0 \epsilon_{20}(\xi, q) \epsilon_{20}(\eta, q),
\]

(12)

where \( C_0 \) is an arbitrary constant. Substituting from (12) into (8) we have

\[
C \epsilon'_{20}(\xi_0, q) = 0.
\]

(13)

If we obtain the numerical values of the positive root \( q \) of this equation (13), the corresponding numerical value of \( \frac{\sigma a}{c} \) can be obtained from the relation:

\[
\frac{\sigma a}{c} = 2 \sqrt{\frac{q}{\xi_0}},
\]

(14)

which follows immediately from (11) considering \( h = a \xi \), where \( a \) and \( \xi \) are a half-length of the major axis and the eccentricity of the ellipse respectively. As for the elliptic boundary of each lake \( \xi_0 \) is fixed, the smallest positive root of the equation (13) will give the frequency of the fundamental normal mode of oscillation of this lake with which we are now concerned.

In the second case, if we take those normal modes of oscillation in which the displacement is symmetrical about the major axis but anti-symmetrical about the minor axis, the admissible particular solution can be given by
\[ \zeta = c_{2m+1}(\xi, q) \, ce_{2m+1}(\eta, q). \quad (m=0,1,2,3, \ldots) \]  

As in the symmetrical mode, we have investigated the simple case when \( \zeta \) is expressed by

\[ \zeta = C_1 ce_{1}(\xi, q) ce_{1}(\eta, q), \]  

where \( C_1 \) is an arbitrary constant as in the first case.

Then evidently, the value of \( \frac{\sigma a}{c} \) in the present case can also be obtained by (14), using the numerical value of the smallest positive root \( q \) of the following equation:

\[ Ce_1(\xi_0, q) = 0. \]  

(17)

3. Numerical results

The writers have obtained the smallest positive root of the equation

\[ Ce_0(\xi_0, q) = 0, \]

in three cases in which the values of \( \frac{a}{b} \) are equal to \( \frac{5}{4}, \frac{2}{1} \), and \( \frac{3}{1} \) respectively. To this end, the values of \( Ce_0(\xi_0, q) \) for different values of \( q \) have been calculated and plotting these values of \( Ce_0(\xi_0, q) \) against \( q \) the searching root have been determined graphically. The results are shown in Table 1. It may be added here that for the numerical calculations of the modified Mathieu functions \( Ce_0(\xi_0, q) \), use has been made of the following product series expression because of their rapid convergency:

\[ Ce_0(\xi_0, q) = \frac{P_0}{A_0^{(0)}} \sum_{\nu=0}^{\infty} (-1)^{\nu} A_0^{(\nu)} J_\nu(v_1)J_\nu(v_2), \]

where

\[ v_1 = \sqrt{q} \, e^{-\xi_0}, \quad v_2 = \sqrt{q} \, e^{\xi_0}, \]

\[ P_0 = \frac{ce_0(0, q)ce_0\left(\frac{\pi}{2}, q\right)}{A_0^{(0)}} , \]

\( \xi_0 \) being connected with the value of \( \frac{a}{b} \) as:

\[ \cosh \xi_0 = \frac{a}{\sqrt{a^2 - b^2}} = \frac{1}{\varepsilon} . \]

Substituting those values of \( q \) into (14), we obtain the probable values of \( \frac{\sigma a}{c} \) for the fundamental normal mode of oscillation as shown in Table 1 below.

<table>
<thead>
<tr>
<th>( a/b )</th>
<th>( v )</th>
<th>( q )</th>
<th>( \frac{\sigma a}{c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5/4</td>
<td>0.600</td>
<td>1.785</td>
<td>4.45</td>
</tr>
<tr>
<td>2/1</td>
<td>0.866</td>
<td>8.57</td>
<td>6.76</td>
</tr>
<tr>
<td>3/1</td>
<td>0.943</td>
<td>21.8</td>
<td>9.90</td>
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</table>

The curve of \( \frac{\sigma a}{c} \) plotted against the eccentricity \( \varepsilon \) is shown by a thick line curve in Fig. 2, where is also shown the well-known value of \( \frac{\sigma a}{c} \) equal to 3.8318 for the symmetrical fundamental
normal mode of oscillation of water surface in an exactly circular lake \((\varepsilon=0)\) of radius \(a\).

In the previous paper, for the symmetrical case one of the authors has obtained an expansion formula for the relationship between the frequency and the eccentricity of the elliptic boundary. The result is

\[
\frac{\sigma a}{c} = 3.8318 + 0.9580\varepsilon^2 + 1.4778\varepsilon^4 + 1.5479\varepsilon^6 + \cdots. \tag{18}
\]

The values calculated by using this expansion formula are also shown by a thin line curve in Fig. 2.

It will be seen that this expansion formula (18) is particularly useful for calculating the numerical values of \(\frac{\sigma a}{c}\) in case when the lake is nearly circular.

In case when the displacement is symmetrical about the major axis but anti-symmetrical about the minor axis, the numerical value of the root of the equation (17) should be obtained for some values of \(\xi\).

We have calculated the smallest positive root of this equation in three cases in which \(\frac{a}{b}\) are equal to \(\frac{5}{4}\), \(\frac{5}{3}\) and \(\frac{3}{1}\) respectively by the same manner as in the symmetrical case.

Here, use has been made of the formula

\[
Ce_1(\xi, q) = \frac{P_1}{A_1^{(2)}} \sum_{r=0}^{\infty} (-1)^r A_1^{(2)} \left[ J_r(v_1)J_{r+1}(v_2) + J_{r+1}(v_1)J_r(v_2) \right],
\]

where

\[
v_1 = \sqrt{q} e^{-\xi}, \quad v_2 = \sqrt{q} e^{\xi}, \quad P_1 = \frac{c e_1(0, q)c e_1(\pi/2, q)}{\sqrt{q} A_1^{(2)}}.
\]

Thus, substituting these values of \(q\) into (14) we obtain the probable values of \(\frac{\sigma a}{c}\) as set out in Table 2, where is also shown the well known value of \(\frac{\sigma a}{c}\) equal to 1.841 in a circular lake \((\varepsilon=0)\). It is obvious therefrom that the eccentricity has little influence on the frequency of the slowest longitudinal mode.

<table>
<thead>
<tr>
<th>(a/b)</th>
<th>(\varepsilon)</th>
<th>(q)</th>
<th>(\frac{\sigma a}{c})</th>
<th>(\frac{\sigma a}{c})</th>
<th>(\frac{\sigma a}{c})</th>
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<tr>
<td>1</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>1.841</td>
<td>1.841</td>
</tr>
<tr>
<td>5/4</td>
<td>0.600</td>
<td>0.313</td>
<td>1.361</td>
<td>1.856</td>
<td>1.856</td>
</tr>
<tr>
<td>5/3</td>
<td>0.800</td>
<td>0.560</td>
<td>1.877</td>
<td>1.865</td>
<td>1.865</td>
</tr>
<tr>
<td>3/1</td>
<td>0.943</td>
<td>0.787</td>
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<tr>
<td>0</td>
<td>1.000</td>
<td>1.000</td>
<td>1.887</td>
<td>1.891</td>
<td>1.891</td>
</tr>
</tbody>
</table>

The curve representing this relationship is plotted in Fig. 3.

4. An expansion formula for \(\frac{\sigma a}{c}\) in powers of \(\varepsilon\)

In this section we shall proceed to obtain an expansion formula for \(\frac{\sigma a}{c}\) in \(\varepsilon\) for anti-symmetrical mode in case when \(\varepsilon\) becomes small.

When an elliptic boundary differs only slightly from a circular one its eccentricity \(\varepsilon\) is very small. Accordingly \(q\) is also very small as will be seen from the relation \(q = \frac{1}{4} \left(\frac{\sigma a}{c}\right)^3 \varepsilon^2\) which follows immediately from (14). In such a case, \(\frac{\sigma a}{c}\) can be expanded in a power series of \(\varepsilon\).

We have obtained such an expansion formula for \(\frac{\sigma a}{c}\) correct to the order of \(\varepsilon^4\) when the
vertical displacement of water $\zeta$ can be expressed by (16). If we use the following series expression (19):

$$C e_{1}(\xi, q) = \frac{c e_{1}^{2}(\pi - q)}{\sqrt{2} A_{1}^{(1)}} \sum_{r=0}^{\infty} (-1)^{r} A_{r+1}^{(1)} J_{2 \kappa_{1}}(2 \kappa_{1} \cosh \xi_{0}),$$

(19)

the boundary condition (17) leads to

$$\sum_{r=0}^{\infty} (-1)^{r} A_{r+1}^{(1)} J_{r+1}(2 \kappa_{1} \cosh \xi_{0}) = 0.$$  

(20)

Since

$$2 \kappa_{1} = kh, \quad h = \sqrt{a^{2} - b^{2}} = a \varepsilon,$$

and

$$\cosh \xi_{0} = \frac{a}{\sqrt{a^{2} - b^{2}}} = \frac{1}{\varepsilon},$$

we get

$$2 \kappa_{1} \cosh \xi_{0} = k a = \frac{\sigma a}{c}.$$  

Then (20) becomes

$$\sum_{r=0}^{\infty} (-1)^{r} A_{r+1}^{(1)} J_{r+1}(\frac{\sigma a}{c}) = 0,$$  

(21)

where

$$A_{0}^{(1)} = \left\{ \begin{aligned}
&\frac{1}{2} \varepsilon^{2} + \frac{1}{1024} \varepsilon^{4} \\
&+ \frac{1}{98304} \varepsilon^{5} + \cdots \end{aligned} \right\} A_{1}^{(1)},$$

$$A_{1}^{(1)} = \left\{ \begin{aligned}
&\frac{1}{3072} \varepsilon^{4} + \frac{1}{73728} \varepsilon^{6} + \cdots \end{aligned} \right\} A_{2}^{(1)},$$

$$A_{2}^{(1)} = \left\{ \begin{aligned}
&\frac{1}{589824} \varepsilon^{5} + \cdots \end{aligned} \right\} A_{3}^{(1)}.$$

Now, let us suppose that we may take

$$\frac{\sigma a}{c} = u_{0} + u_{1} \varepsilon^{2} + u_{2} \varepsilon^{4} + \cdots,$$  

(22)

then the left hand side of (21) can be expanded in a power series of $\varepsilon^{2}$. Neglecting all the terms of order higher than the fourth in $\varepsilon$, we obtain the following three equations from the condition that (21) must hold independently of the values of $\varepsilon$, that is

$$J_{1}^{1}(u_{0}) = 0,$$  

(23)

$$u_{1} J_{2}^{1}(u_{0}) + \frac{1}{32} u_{0}^{3} J_{3}^{1}(u_{0}) = 0,$$  

(24)

$$u_{2} J_{3}^{1}(u_{0}) + \frac{1}{32} u_{0}^{3} J_{4}^{1}(u_{0})$$

$$+ \frac{1}{1024} u_{0}^{3} \{ J_{2}^{1}(u_{0}) + \frac{1}{3} J_{3}^{1}(u_{0}) \} = 0.$$  

(25)

By these equations we can determine all the required coefficients in (22). It is well known, the smallest root of (23) is

$$u_{0} = 1.841,$$  

(26)

this being the value of $\frac{\sigma a}{c}$ for the anti-symmetrical fundamental normal mode of oscillation of water surface in an exactly circular lake of radius $a$.

If we simplify the other coefficients $u_{1}$ and $u_{2}$ determined from the foregoing equations (24) and (25) by using the recurrence formulae for Bessel functions, it is easily found:

$$u_{1} = \frac{1}{8} \frac{u_{0}^{3} - 3}{u_{0}^{3} - 1} u_{0},$$  

(27)

$$u_{2} = \frac{u_{0}}{256(u_{0}^{3} - 1)^{2}} (u_{0}^{8} + 13u_{0}^{6} - 83u_{0}^{4} + 115u_{0}^{2} - 62).$$  

(28)

Thus, we have finally

$$\frac{\sigma a}{c} = u_{0} \left\{ 1 + \frac{1}{8} \frac{u_{0}^{3} - 3}{u_{0}^{3} - 1} \varepsilon^{2} + \frac{1}{256} \frac{u_{0}^{13} + 13u_{0}^{11} - 83u_{0}^{9} + 115u_{0}^{7} - 62}{(u_{0}^{3} - 1)^{2}} \varepsilon^{4} + \ldots \right\}$$

$$= 1.841 + 0.0375 \varepsilon^{2} + 0.00644 \varepsilon^{4} + \ldots.$$  

(29)

The values of $\left( \frac{\sigma a}{c} \right)_{\varepsilon}$ calculated by this formula are shown in Table 2. By comparing these values with those obtained by the exact solution
it seems that this expansion formula really gives good results for any values of $\varepsilon$ less than unity.

If we introduce in our expansion formula the following values $d=27$ meters, $a=30500$ meters and $\varepsilon=\frac{\sqrt{2}}{2}$ which were used by Mr. TAKAYA in the case of the Osaka Bay we get 104.9 minutes as the numerical value of the period of the fundamental normal mode of longitudinal oscillation in the Osaka Bay. This result is nearly equal to Mr. Takaya's 109.7 minutes.

Meanwhile by using Rayleigh's method Lamb obtained the following approximate formula (30) for the case when we are now concerned,

$$\frac{\sigma a}{c} = \sqrt{\frac{6(3a^2+b^2)}{5a^2+2b^2}}. \quad (30)$$

This formula is also useful to get the approximate value of $\frac{\sigma a}{c}$ for values of $\frac{b}{a}$ less than unity. For the comparison the results calculated by using this relation are given in column 6 of Table 2. This approximate estimate is in excess, in accordance with a general principle.

5. Summary

The fundamental normal mode of the free oscillation of water in a lake of elliptic boundary and of uniform depth is investigated exactly by the use of Mathieu functions and modified Mathieu functions in two cases. In the first case the vertical displacement of the water surface from its equilibrium position is to be symmetrical about principal axes. In the second case the displacement of the water surface is to be symmetrical about the major axis but anti-symmetrical about the minor axis. In these two cases, carrying out detailed numerical calculations the relationship between $\frac{\sigma a}{c}$ and $\varepsilon$ is shown graphically where $\sigma$ is the circular frequency, $a$ a half length of the major axis of the ellipse, $\varepsilon$

the eccentricity and $c=\sqrt{gd}$, $g$ and $d$ being respectively the acceleration due to gravity and the uniform depth of the lake.

It has been well illustrated that for symmetrical mode an expansion formula for $\frac{\sigma a}{c}$ (18) derived in the previous paper is particularly useful in case when an elliptic boundary differs only slightly from an exact circle. For anti-symmetrical mode a similar expansion formula seems to give fairly good results for any values of $\varepsilon$ less than unity. In this case it is also known that the eccentricity has little influence on the frequency.

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References


