A Complementary Note on the Diffusion of the Seaward River Flow off the Mouth

By

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Abstract: The rotation of the earth causes the seaward movement of less dense river water upon leaving the mouth of river to deflect to the right side of its initial direction in the northern hemisphere and to the left side in the southern hemisphere.

Introduction

It has been indicated that, so far as is concerned so slow two-dimensional homopycnal seaward flow of river water (inflow equally dense) that the inertia terms are neglected, the effect of the earth’s rotation appears through the meridional variation in the Coriolis parameter only, which tends to deflect the flow to the left and this effect is negligibly small (Takano, 1954).

In some cases, however, the inflow of river water is found to be deflected to the right in the northern hemisphere (for instance, Bates, 1953). The present note deals, in a currentless, tideless and wave-free basin, with the slow hypopycnal inflow (inflow less dense) subject to the Coriolis forces which has not been discussed in the previous note.

Effect of the Coriolis forces

As in the previous note, the equation of motion and the equation of continuity are approximately given by

\[
A_h \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u + \frac{\partial}{\partial z} \left( A_r \frac{\partial u}{\partial z} \right) + \rho f v = \frac{\partial \rho}{\partial x} \tag{1}
\]

\[
A_h \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) v + \frac{\partial}{\partial z} \left( A_r \frac{\partial v}{\partial z} \right) - \rho f u = \frac{\partial \rho}{\partial y} \tag{2}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3}
\]

Here, the x-axis is taken to be perpendicular to the straight shore line considered as the y-axis, the origin on the center of the mouth \((-1<y<1)\) the z-axis vertically downward (Fig. 1), \(u, v, w\) the three components of the velocity, \(A_h\) and \(A_r\) the horizontal and vertical eddy viscosities, \(f\) the Coriolis parameter, \(\rho\) the pressure and \(\rho\) the density.

![Fig. 1]

On assuming that the stress vanishes at both the surface and the bottom, integrations from the surface \(z=-\xi\) down to the bottom \(z=d, d\) is assumed to be constant) of the above equations yield

\[
A_h \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) M_x + f M_y = \frac{\partial P}{\partial x}, \tag{4}
\]

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\[ A_s \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) M_y = \frac{-\partial P}{\partial y}, \quad (5) \]
\[ \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0, \quad (6) \]
where \( M_x = \int_{-\zeta}^{\zeta} u dz, \quad M_y = \int_{-\zeta}^{\zeta} v dz \)
and \( P = \int_{-\zeta}^{\zeta} p dz. \)

Introducing the stream function \( \psi \) so that
\[ M_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad M_y = -\frac{\partial \psi}{\partial x} \quad \text{and putting} \quad \xi = \frac{x}{l}, \quad \eta = \frac{y}{l}, \]
we have
\[ \psi = \frac{2}{\pi} \int_{0}^{\infty} dx \int_{0}^{l} M_0 \cos \alpha \lambda \frac{\cos \alpha \lambda}{\alpha} \frac{\sin \alpha \lambda e^{-\alpha \lambda}}{(\alpha \lambda + 1) d\lambda} + \frac{4A_h}{\pi} \int_{0}^{\infty} dx \int_{0}^{l} M_0 \cos \alpha \lambda \cos \alpha \lambda \frac{\sin \alpha \lambda e^{-\alpha \lambda}}{(\alpha \lambda + 1) d\lambda} \]
\[ = \frac{M_0}{\pi} \left\{ -\frac{4A_h}{\pi} \frac{(1+\eta) \tan^{-1} \eta}{\xi} - (1-\eta) \tan^{-1} \frac{1-\eta}{\xi} \right\} + \frac{4A_h}{l} \frac{\xi^2 - \eta^2 + 1}{\{\xi^2 + (1-\eta)^2\} \{\xi^2 + (1+\eta)^2\}} \right\} \cdot \frac{\xi}{(\xi^2 + (1-\eta)^2)^{1/2}} \right\}. \quad (11) \]

When the existence of the upper homogeneous layer consisting principally of river water is assumed and an appropriate vertical density distribution in the lower layer is specified, the integrated pressure \( P \) is generally expressed as a function of the thickness \( h \) of the upper layer as well as the surface elevation \( \xi \). Moreover, if the horizontal pressure gradient vanishes at the bottom or a deep level, the above two variables are not independent and \( p \) becomes a function of \( h \) only. For example, the Reid's model (Reid, 1948) gives
\[ \frac{\partial \zeta}{\partial x} = \frac{2\Delta \rho}{\rho_0} \frac{\partial \psi}{\partial x}, \quad \frac{\partial \zeta}{\partial y} = \frac{2\Delta \rho}{\rho_0} \frac{\partial \psi}{\partial y}, \]
\[ \frac{\partial P}{\partial x} = \frac{5gh \Delta \rho}{2} \frac{\partial \psi}{\partial x}, \quad \frac{\partial P}{\partial y} = \frac{5gh \Delta \rho}{2} \frac{\partial \psi}{\partial y}, \quad (12) \]
where \( \rho_0 \) is the density in the upper homogeneous layer and \( \Delta \rho \) the difference in density between this layer and a sufficiently deep level; while from a density model:
\[ \rho = \rho_0 \quad \text{in the upper layer} \quad (-\xi \leq z \leq h), \]
\[ \rho = \rho_0 + \frac{\Delta \rho}{d} \frac{z-h}{d-h} \quad \text{in the lower layer} \quad (h \leq z \leq d), \]
it follows that
\[ \frac{\partial \zeta}{\partial x} = \frac{\Delta \rho}{\rho_0} \frac{\partial \psi}{\partial x}, \quad \frac{\partial \zeta}{\partial y} = \frac{\Delta \rho}{\rho_0} \frac{\partial \psi}{\partial y}, \]
\[ \frac{\partial P}{\partial x} = \frac{\Delta \rho}{6} \frac{\partial \psi}{\partial x}, \quad \frac{\partial P}{\partial y} = \frac{\Delta \rho}{6} \frac{\partial \psi}{\partial y}. \quad (13) \]

The second term in the bracket of the solution (11) is symmetric with respect to the x-axis, vanishing on a hyperbola \( \eta^2 - \xi^2 = 1 \) and being positive inside \( \eta^2 - \xi^2 = 1 \) and a segment \( x=0, -l \leq y \leq l \). The first term
\[ -\frac{4A_h}{\pi} \frac{(1+\eta) \tan^{-1} \eta}{\xi} - (1-\eta) \tan^{-1} \frac{1-\eta}{\xi} \]
is negative for \( \eta > 0 \) and positive \( \eta < 0 \) in the northern hemisphere. As the integrated pressure may be considered as representative of the thickness of the upper homogeneous layer, the first term modifies the solution derived from the neglect of the Coriolis force so that the homogeneous layer deepens at the right side and becomes shallow at the left side; in other words, the hypopycnic inflow is deflected to the right in the northern hemisphere. In the figure
2 is illustrated this effect in the cases of
\[ R \left( = \frac{f l^2}{A_b} \right) = \frac{1}{500}, \frac{2}{500}, \frac{4}{500}, \frac{8}{500}, \frac{16}{500} \]
and \[ \frac{32}{500}. \]

\[ \frac{\partial P}{\partial x} = \frac{\rho_0}{2} \frac{\partial h^2}{\partial x}, \frac{\partial P}{\partial y} = \frac{\rho_0}{2} \frac{\partial h^2}{\partial y}, \]  \( (14) \)

where \( \rho_0 \) is the density in the upper homogeneous layer.

**Conclusions**

As long as the hypopycnal inflow is slow enough to neglect the inertia terms, the vertical distribution in density can be appropriately specified and the horizontal pressure gradient vanishes at the bottom or a sufficiently deep level, the Coriolis force deflects the seaward flow toward the right and induces a water level sloping from the right-hand bank to the left-hand at the mouth in the northern hemisphere. These results are in agreement with observations.

Actually, since the volume transport \( M_0 \) of the inflow at the mouth is not uniform but more concentrated along the central line of river than in the vicinity of the bank and inertia terms are not always negligible, the seaward flow is less diffused and less deflected than that indicated above.

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**References**

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