Note on the Influence of an Inlet and an Intake upon the Velocity Distribution in the Interior

By

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Abstract: The influences of mouths upon the velocity distribution of a lake or a paddle field assumed to be rectangular are treated. They are remarkable in a paddle field but not in a lake.

Introduction

The velocity distribution in a lake is generally complicated and that of a paddle field under irrigation is of importance in view of its agricultural products. The aim of this note is to estimate the effects of an inlet and an intake upon it. To simplify the problem the following assumptions are made: (1) the shape is rectangular; (2) the water is homogeneous; (3) the bottom of a lake and the boundary walls of a paddle field are taken as free surfaces; (4) the inflow and the outflow are perpendicular to the boundary surface; (5) the inertia terms are small; and (6) the motion is two-dimensional.

The assumption (3) is not always reasonable, but it will not be very important if the scale of the field is large. The last assumption (6) holds in a paddle field. The same method can be applied to the three-dimensional motion and to the inflow or the outflow in an arbitrary direction.

Theoretical Consideration

The equations of motion and continuity are given by

$$A_x \frac{\partial u}{\partial x} + A_y \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial p}{\partial x},$$  \hspace{1cm} (1)

$$A_x \frac{\partial^2 v}{\partial x^2} + A_y \frac{\partial^2 v}{\partial y^2} = \frac{1}{\rho} \frac{\partial p}{\partial y} - g,$$  \hspace{1cm} (2)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$  \hspace{1cm} (3)

Here the $y$ axis is directed downward, $u$, $v$ are the components of the velocity, $A_x$ and $A_y$ the coefficients of horizontal and vertical eddy viscosity respectively, $p$ the pressure and $g$ the acceleration of gravity.

By a cross differentiation the above equations are reduced to
\[ (A_x \frac{\partial^2}{\partial x^2} + A_y \frac{\partial^2}{\partial y^2}) P^2 \psi = 0, \]  

where \( \psi \) is the stream function, and \( u \) and \( v \) are given by

\[ u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}. \]

The boundary conditions are as follows:

\[ \psi = \frac{\partial \psi}{\partial y} = 0 \quad \text{for} \quad y = 0 \quad \text{(the surface)}, \]

\[ \text{and} \quad y = H \quad \text{(the bottom)}, \]

\[ \frac{\partial \psi}{\partial y} = u_0(y) \quad \text{for} \quad x = 0, \quad h_1 < y < h_2, \]

\[ = 0 \quad \text{for} \quad x = 0, \quad 0 < y < h_1, \quad h_2 < y < H, \]

\[ -\frac{\partial \psi}{\partial x} = 0 \quad \text{for} \quad x = 0, \]

\[ \frac{\partial \psi}{\partial y} = u_1(y) \quad \text{for} \quad x = l, \quad h_3 < y < h_4, \]

\[ = 0 \quad \text{for} \quad x = l, \quad 0 < y < h_3, \quad h_4 < y < H, \]

\[ -\frac{\partial \psi}{\partial x} = 0 \quad \text{for} \quad x = l, \]

and

\[ \int_{h_1}^{h_2} u_0(y) \, dy = \int_{h_3}^{h_4} u_1(y) \, dy, \]

as the inflow must be equal to the outflow.

We put the solution of equation (4) in the form:

\[ \psi = \sum s \sin \lambda_s y \left( a_s \cosh \lambda_s x + b_s \sinh \lambda_s x + c_s \cosh r \lambda_s x + d_s \sinh r \lambda_s x \right) + e y, \]

where \( r = \sqrt{\frac{A_y}{A_x}} \), \( \lambda_s = \frac{s \pi}{H} \) and \( s \) is an integer, \( a_s, b_s, c_s, d_s \) and \( e \) are unknown constants.

Now \( u_0(y) \) and \( u_1(y) \) are expanded as follows:

\[ u_0(y) = e + \sum s \frac{P_{1s}}{H} \cos \frac{s \pi y}{H}, \]

\[ u_1(y) = e + \sum s \frac{P_{2s}}{H} \cos \frac{s \pi y}{H}. \]

Here

\[ e = \frac{1}{H} \int_{h_1}^{h_2} u_0 \, dy = \frac{1}{H} \int_{h_3}^{h_4} u_1 \, dy \]

and

\[ P_{1s} = \frac{2}{H} \int_{h_1}^{h_2} u_0 \cos \frac{s \pi y}{H} \, dy, \quad P_{2s} = \frac{2}{H} \int_{h_3}^{h_4} u_1 \cos \frac{s \pi y}{H} \, dy. \]

If \( u_0 \) is a constant, we have

\[ P_{1s} = \frac{2u_0}{s \pi} \left( \sin \frac{s \pi}{H} h_2 - \sin \frac{s \pi h_1}{H} \right). \]
The constants are determined by the boundary conditions:

$$
\begin{align*}
  a_s &= \frac{H}{s\pi} \left\{ P_1(\lambda_d \cosh \lambda_d - \cosh \lambda_d \sinh \lambda_d) - P_2(\lambda_d \cosh \lambda_d - \cosh \lambda_d) \right\}, \\
  b_s &= \frac{H}{s\pi} \left\{ P_1(\lambda_d \cosh \lambda_d - \cosh \lambda_d \sinh \lambda_d) + P_2(\lambda_d \cosh \lambda_d - \cosh \lambda_d) \right\}, \\
  c_s &= \frac{H}{s\pi} P_1 - a_s, \\
  d_s &= -\frac{b_s}{r}, \\
  \Delta &= 2 \cosh \lambda_d \cosh r \lambda_d - 2\left(1 + \frac{1}{r}\right) \sinh \lambda_d \sinh r \lambda_d.
\end{align*}
$$

Then, by the neglect of small quantities the stream function is given by

$$
\psi = \sum_s \frac{H}{s\pi} \sin \frac{s\pi}{H} \int \left[ \frac{(1-r)\left(P_1 e^{\frac{s\pi}{H} \tau} + P_2 e^{\frac{s\pi}{H} \tau \left(1-\tau\right)}\right)}{2-r} + \frac{1}{r} \left(P_1 e^{\frac{s\pi}{H} \tau} + P_2 e^{\frac{s\pi}{H} \tau \left(1-\tau\right)}\right) + \frac{y}{r} \int_{h_0}^{h_0} \omega dy \right]. \quad (15)
$$

In the case of a paddle field, its length and its width are comparable, so it is more reasonable to take $A_x$ equal to $A_y$ because both $A_x$ and $A_y$ are horizontal eddy viscosity, and to replace the boundary conditions (7) and (9) by

$$
\frac{\partial^2 \psi}{\partial x^2} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = l, \quad (16)
$$
corresponding to those for the other two boundary walls.

In the same way we have

$$
\psi = \sum_s \sin \lambda_d \eta \left( (a_s x + b_s) \cosh \lambda_d x + (c_s x + d_s) \sinh \lambda_d x \right) + e y, \quad (17)
$$

where

$$
\begin{align*}
  a_s &= -\frac{1}{2 \sinh \lambda_d} (P_1 \cosh \lambda_d - P_2), \\
  b_s &= P_{11}/\lambda_d, \\
  c_s &= -\frac{P_{11}}{2}, \\
  d_s &= \frac{1}{\lambda_d \sinh \lambda_d} \left( P_1 \left( \frac{\lambda_d^2}{2 \sinh \lambda_d} + \cosh \lambda_d \right) + P_2 \left( P_2 \left( \frac{\lambda_d^2}{2 \sinh \lambda_d} \cosh \lambda_d \right) \right) \right), \\
  e &= \frac{1}{H} \int_{h_0}^{h_0} \omega dy.
\end{align*}
$$

If the small quantities are neglected, the stream function becomes
\[ \psi = \sum \sin \frac{s \pi y}{H} \left[ \left( \frac{x}{2} + \frac{H}{s \pi} \right) P_{2s} e^{-i \frac{r}{H} (x^2 - y^2)} - \left( \frac{x}{2} - \frac{H}{s \pi} - \frac{l}{2} \right) P_{2s} e^{i \frac{r}{H} (x^2 - y^2)} \right] + \frac{\gamma}{H} \int_{h_2}^{h_1} \psi_1 dy. \]  

We shall give some examples:

1. \( h_1 = 0, \ h_2 = \frac{H}{10}, \ h_3 = 0, \ h_4 = \frac{H}{10}, \ l = 200H, \ u_0 = u_1 = \text{const}, \ r = \frac{1}{20}. \)

2. \( h_1 = 0, \ h_2 = \frac{H}{10}, \ h_3 = \frac{9H}{10}, \ h_4 = H, \ l = 200H, \ u_0 = u_1 = \text{const}. \ r = \frac{1}{20}. \)

3. \( h_1 = 0, \ h_2 = \frac{H}{10}, \ l = 200H, \ r = \frac{1}{20}, \ u_0 = \frac{3}{2} \left( \frac{H}{10} \right)^2 \left( \frac{H}{10} \right)^2 - y^2. \)

4. \( h_1 = 0, \ h_2 = \frac{H}{20}, \ h_3 = 0, \ h_4 = \frac{H}{20}, \ l = H, \ u_0 = u_1 = \text{const}. \)

5. \( h_1 = 0, \ h_2 = \frac{H}{20}, \ h_3 = \frac{9H}{20}, \ h_4 = H, \ l = H, \ u_0 = u_1 = \text{const}. \)

of which the results are shown in figs. 1—3.

Conclusions

The above results lead us to the following conclusions:

1. In the cases (1)—(3), the initial velocity falls out rapidly and twenty times the depth off the mouth, the velocity is nearly uniform from the surface down to the bottom, consequently, the inlet has its influence only in its immediate neighborhood. Then near the intake we have a profile just the same as in the case (1), and just the inverse profile in the case (2). Of course the actual bottom is not free but it is inferred that the velocity gradient vanishes rapidly except in the layer close to the bottom on which the bottom friction has influence.

2. In fig. 2 the dashed lines coincide with the full lines except for the close neighborhood of the inlet. This means that the velocity distribution at the inlet is not important, provided the mass transport is a constant. In other word, the influence of the mouth is not significant.

3. The initial velocity falls out more rapidly if \( l \) becomes smaller according to the equation (15); because the decay of the initial velocity depends principally on \( r \), \( r \) is proportional to \( A_e^{-1/2} \) and \( A_e \) increases with \( l \).

4. The paddle field will not be irrigated in some parts if its inlet and intake are disposed at certain positions.

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Fig. 1. The velocity distributions at \( x=0, \ H, \ 2H, \ 5H \) and \( 20H \). The full lines represent the results of case (1) and (2), and the dashed lines the results of case (3).

Fig. 2. The stream lines for the case (4). The half part of a field is shown. unit: \( H\nu_0/20 \).

Fig. 3. The stream lines for the case (5). The half part of a field is shown. unit: \( H\nu_0/20 \).