Gravitational energy release in an evolving Earth model

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The energy budget of the Earth's core balances the heat lost through cooling with the sum of gravitational, latent heat and radioactive sources (if any). The gravitational and latent heat sources are due to the freezing of core mix onto the surface of the inner core. Gravitational energy is released because the light components of core mix that are released during freezing are buoyant, and rise as they rejoin the fluid core. This source of energy can be regarded as part of the total gravitational energy released as the entire Earth cools and contracts. The main purpose of this paper is to present a new method of evaluating the total energy release. The method is applied to two Earth models. Both show that the gravitational source that stirs the fluid core is less than 30% of the total gravitational energy released through the contraction of the Earth as it cools.
Key words: Earth's core, thermal evolution, inner core growth, core convection, geodynamo.

1. Introduction

Energy and entropy balances are significant in determining the evolution of the Earth and particularly in clarifying its internal structure. They are delicate enough to raise questions that are hard to answer, especially those concerning the growth and age of the solid inner core (SIC). For example, estimates of its age range from less than 1 Ga to 4.6 Ga, the age of the Earth. The search for answers has led several investigators to construct Earth model of various degrees of sophistication; see for example Stacey and Stacey (1999), Labrosse *et al.* (1997, 2001), Labrosse (2003), Gubbins *et al.* (2003, 2004), Nimmo *et al.* (2004). The present paper describes further developments of the model of Roberts *et al.* (2003), which was itself based on the analysis of Braginsky and Roberts (1995), a paper that will be referred to here as 'BR'; see also Braginsky and Roberts (2005). The model employed the Preliminary Reference Earth Model (PREM) of Dziewonski and Anderson (1981) but, in the new models described here, it is slightly modified to use a more up-to-date estimate of the density jump at the inner core boundary (ICB). It also makes use of new estimates of some key parameters that appeared after the Roberts *et al.* (2003) paper was published. That model supposed that the core-mantle boundary (CMB) was fixed. It was therefore concerned with core thermodynamics only. The present models allow for the inward motion of the CMB as the Earth cools and contracts. They therefore include the thermodynamics of the mantle too.

The new models are used for a single purpose: to describe the energy budget of a cooling Earth. A new method of achieving this is demonstrated for a simple system in Section 2. This is generalized to the Earth in Section 3. The rates of gravitational and internal energy loss are calculated for the new models in Section 4. The results depend on the assumed cooling rate, and this also affects the rapidity with which the SIC accretes mass through freezing of the overlying fluid. The concomitant release and ascent of light constituents of core mix is an important source of gravitational power to drive motions in the fluid outer core (FOC). This source is included in the present models and is found to be roughly 30% as large as the gravitational energy release in the cooling and contraction of the entire Earth. Its evaluation and the related question of the entropy budget for the Earth are however topics of subsidiary interest in the present paper, which is only aimed at assessing the global energy budget.

The notation employed in this paper is summarized in Table 1.

2. Orientation

It may be helpful to illustrate the basic physics of the present paper by a simple example. Consider a non-rotating, non-magnetic body $\mathcal{B}$ of self-gravitating homogeneous fluid such as a gas sphere, which is losing heat from its surface Σ at a rate $\mathcal{Q}^q_\Sigma$ greater than the rate $\mathcal{Q}^R$ at which internal sources, such as dissolved radioactivity, can replenish it. If $\mathcal{Q}^q_\Sigma - \mathcal{Q}^R$ is sufficiently large, as we shall suppose it is, heat is carried outwards mainly by convective motions that thoroughly mix $\mathcal{B}$ and in particular homogenize the specific entropy S (see below). The heat equation then becomes

$$\mathcal{M}\widetilde{T}\dot{S} = \mathcal{Q}^R - \mathcal{Q}^q_\Sigma\,, \tag{1}$$

where $\dot{S}(=\partial_t S)$ is the time derivative of S, $\mathcal{M}$ is the mass of $\mathcal{B}$, and $\widetilde{T}$ is the mass-weighted mean temperature, given in terms of the temperature T and the mass density ρ by

$$\mathcal{M}\widetilde{T} = \int_{\mathcal{B}} T\rho \mathrm{dV} = 4\pi\int_0^{\mathcal{R}} T\rho r^2 dr. \tag{2}$$

Table 1. Notation.

Symbol	Meaning
A	Molecular weight
$A_{n,n+1}$	Area of interface $\Sigma_{n,n+1}$, between zones n and $n+1$ in PREM
$\mathcal{A}$	Rates of working ($\mathcal{A}^\xi$ gravitational; $\mathcal{A}^P$ pressure)
α	Expansion coefficients (α, thermal; α^S, entropic; α^ξ, compositional)
$\mathbf{B}$, $\mathbf{B}_c$	magnetic field; convective buoyancy force
C_p	Specific heat at constant pressure
∂_t (or overdot)	Eulerian time derivative ($\partial_t = \partial/\partial t$)
d_t	Lagrangian time derivative ($= \partial_t + \mathbf{u}\cdot\boldsymbol{\nabla}$)
$\overline{d}_{\bar{t}}$	Mean Lagrangian derivative $\partial_{\bar{t}} + V\partial_r$, where $\partial_r = \partial/\partial r$
$\Delta\rho$, $\Delta\xi$	Jump in ρ and ξ at ICB
Δ_{ma}, Δ_2	Solidification parameters
$\mathcal{E}$	Energies ($\mathcal{E}^I$, internal; $\mathcal{E}^g$, gravitational; $\mathcal{E}^K$, kinetic)
ε	Energies per unit mass (ε^I, internal; ε^H, enthalpy; ε^K, kinetic)
$\mathbf{F}^\nu$	Viscous body force per unit volume
g, $\mathbf{g}$, G	Gravitational accelerations; Newton's constant of gravitation
γ	Grüneisen parameter ($= \alpha K_s/\rho C_p$)
h^L, h^N, h^ξ	Latent heat; generalized latent heat; heat of reaction (per unit mass)
K_s	Incompressibility ($= \rho(\partial P/\partial\rho)_{S,\xi}$)
κ	Thermal diffusivity
m	Rate of increase of mass of SIC, per unit area of ICB
μ	Chemical potential
ν	Kinematic viscosity
M, $\mathcal{M}$	Mass, total mass of configuration
ξ	Mass fraction of light component of core mix
Ω	Angular velocity of Earth
P	Pressure
q^R	Internal heat source per unit mass
Q	Power (Q^q, heat flow across surfaces; Q^D, viscous + ohmic dissipation; Q^L & Q^N, latent heat releases; Q^R, internal heat source)
r, $\mathbf{r}$, $\hat{\mathbf{r}}$	Distance from geocenter; radius vector; unit radius vector ($= \mathbf{r}/r$)
$R_{n,n+1}$	Radius of interface $\Sigma_{n,n+1}$ between zones n and $n+1$ in PREM
$\mathcal{R}$, R	Outer radius of configuration; radius of ICB (also $R = R_{12}$)
$r_{\rm FS}$	Rejection coefficient for light material in freezing
ρ	Density
S	Entropy per unit mass
t, $\bar{t}$, t_c	Time, evolutionary time, convective time
T	Temperature
$\mathbf{u}$, V, V_p, V_s	Fluid velocity; mean radial velocity; seismic velocities
U	Gravitational potential ($\mathbf{g} = -\boldsymbol{\nabla}U$)
W	Lagrangian derivative of pressure ($= \overline{d}_{\bar{t}}P$)

Note: Suffixes core, E, CMB, FOC, ICB, m and SIC attached to above quantities refer to entire core (SIC+FOC), entire Earth, core-mantle boundary, fluid outer core, inner core boundary, mantle and solid inner core; tildes denote mass weighted averages.

The second form for $\mathcal{M}\widetilde{T}$ assumes that $\mathcal{B}$ is spherically symmetric and of radius $\mathcal{R}$. Spherical symmetry requires that the typical convective velocity, $\mathcal{U}_c$, is tiny compared with the free-fall velocity $(g\mathcal{R})^{1/2}$, where $\mathbf{g} = -g(r)\hat{\mathbf{r}}$ is the gravitational acceleration, r being distance from the center of $\mathcal{B}$ and $\hat{\mathbf{r}}$ the radial unit vector; g is given by

$$g = \frac{GM}{r^2}, \quad \text{where} \quad M(r) = 4\pi \int_0^r \rho r^2 dr \tag{3}$$

is the mass contained within the sphere of radius r, so that $\mathcal{M} = M(\mathcal{R})$; here G is Newton's constant of gravitation.

Equation (1) does not contain the rate, $\mathcal{Q}^D$ (> 0), at which heat is produced in $\mathcal{B}$ by the viscous dissipation of kinetic energy. As is generally recognized, $\mathcal{Q}^D$ (> 0) represents recycled energy that, if included in (1), must be accompanied by the rate of working of the buoyancy forces driving the convection; see (15) below. Equation (1) may be re-expressed as

$$\dot{\mathcal{E}}^I + \dot{\mathcal{E}}^g = \mathcal{Q}^R - \mathcal{Q}^q_\Sigma, \tag{4}$$

where $\mathcal{E}^I$ and $\mathcal{E}^g$ are the internal and gravitational energies of $\mathcal{B}$:

$$\mathcal{E}^I = 4\pi \int_0^{\mathcal{R}} \varepsilon^I \rho r^2 dr\,,$$

$$\mathcal{E}^g = \int_{\mathcal{B}} \mathbf{g}\cdot\mathbf{r}\,\rho\,\mathrm{dV} = -4\pi \int_0^{\mathcal{R}} g\rho r^3 dr\,, \tag{5}$$

ε^I being the internal energy per unit mass. By the assumption $\mathcal{U}_c \ll (G\mathcal{M}/\mathcal{R})^{1/2}$ made earlier, $\mathcal{E}^K \ll G\mathcal{M}^2/\mathcal{R} = \mathrm{O}(\mathcal{E}^g)$, so that $\dot{\mathcal{E}}^K$ does not appear in (4). A derivation of (4) from (1) is given below. The expression $(5)_2$ for $\mathcal{E}^g$ is an old one, given for example by Eddington (1926). Two equivalent expressions are

$$\mathcal{E}^g = \tfrac{1}{2}\int_{\mathcal{B}} U\rho\,\mathrm{dV}, \qquad \mathcal{E}^g = -\frac{1}{8\pi G}\int_{\infty} g^2\mathrm{dV}, \tag{6}$$

where U is the gravitational potential ($\mathbf{g} = -\nabla U$) and the ∞ in $(6)_2$ denotes all space; see BR, Appendix B. The fact that there are 3 very different expression all giving the same $\mathcal{E}^g$ shows that strictly it is impossible to define a unique local gravitational energy density and to hold different parts of $\mathcal{B}$ responsible for different parts of $\mathcal{E}^g$. Nevertheless, we shall find it convenient later to regard $\varepsilon^g = \mathbf{g}\cdot\mathbf{r}$ as a gravitational energy density.

It will be supposed that the timescale $\tau_c = \mathcal{R}/\mathcal{U}$ of convective overturning is much less than the evolutionary timescale $\tau_E = \mathcal{E}^g/(\mathcal{Q}^q_\Sigma - \mathcal{Q}^R)$:

$$\epsilon \equiv \tau_c/\tau_E \ll 1. \tag{7}$$

This assumption is basic, and has been implicit in the study of virtually all evolutionary Earth models. Here, as in BR, we formalize the small ϵ theory by making use of the two-timescale method (see e.g., Nayfeh, 1973, Ch. 6). Two time variables are introduced, a fast convective time t_c ($= t$) and a slow evolutionary time $\bar{t} = \epsilon t_c$. From any variable Q, an average $\overline{Q}$ over t_c is introduced, which therefore depends only on $\bar{t}$ (and space variables). The convective part $Q_c = Q - \overline{Q}$ depends on both t_c and $\bar{t}$ but $\partial_t Q_c = \partial Q_c/\partial t_c + \epsilon\partial Q_c/\partial\bar{t}$ is to leading order $\partial Q_c/\partial t_c$ so that, in a first approximation, Q_c depends only parametrically on $\bar{t}$. Our adoption of the two-time scale method is therefore, to the level of approximation in ϵ to which it is taken here, essentially equivalent to the intuitive procedure more usually adopted.

The fluid velocity is separated into its convective and averaged parts as

$$\mathbf{u} = V(r,\bar{t})\hat{\mathbf{r}} + \mathbf{u}_c(\mathbf{r}, t_c, \bar{t}), \tag{8}$$

where $V(r,\bar{t})\hat{\mathbf{r}}$, the convective average of $\mathbf{u}$, describes the contraction $-V$ of $\mathcal{B}$ through cooling. The mass continuity equation,

$$d_t\rho + \rho\nabla\cdot\mathbf{u} = 0, \qquad (d_t = \partial_t + \mathbf{u}\cdot\nabla), \tag{9}$$

when convectively averaged gives

$$\bar{d}_t\overline{\rho} + \overline{\rho}r^{-2}\partial_r(r^2V) = 0, \qquad (\bar{d}_t = \partial_{\bar{t}} + V\partial_r). \tag{10}$$

This shows that, even though $V \ll |\mathbf{u}|$, it is essential to retain V in the averaged equation. The assumption $\mathcal{U}_c \ll (g\mathcal{R})^{1/2}$ made earlier implies that $\rho_c \ll \overline{\rho}$ and the convective part of (9), obtained by subtracting (10) from (9), therefore simplifies at leading order to its anelastic form: $\nabla\cdot(\bar{\rho}\mathbf{u}_c) = 0$; see Section 4 of BR or Braginsky and Roberts (2007).

The momentum equation is

$$\rho d_t\mathbf{u} = -\nabla P + \rho\mathbf{g} + \mathbf{F}^\nu, \tag{11}$$

where P is the pressure and $\mathbf{F}^\nu$ is the viscous force per unit volume. Since $V = \mathrm{O}(\epsilon\mathcal{U}_c)$ and $\overline{d}_t = \mathrm{O}(\epsilon d_t)$, it follows that $\overline{\rho}\overline{d}_t V$ makes no contribution to the evolutionary part of (11) at leading order in ϵ so that

$$0 = -\nabla\overline{P} + \overline{\rho}\,\overline{\mathbf{g}}, \quad \text{i.e.,} \quad \partial_r\overline{P} = -\overline{\rho}\,\overline{g}, \tag{12}$$

which shows that the convectively averaged state, also called the *reference state*, is in hydrostatic equilibrium. Solutions to (12) must obey

$$\overline{P}(\mathcal{R}) = 0. \tag{13}$$

Equations (12) and (13) do not suffice to determine the reference state.

The convective part of (11), obtained by subtracting (12) from it, is

$$\overline{\rho}d_t\mathbf{u}_c = \mathbf{B}_c + \mathbf{F}^\nu, \tag{14}$$

where $\mathbf{B}_c \approx -\nabla P_c + \rho_c\overline{\mathbf{g}}$ is the buoyancy force. The convective energy equation is obtained by taking the scalar product of (14) with $\mathbf{u}_c$ and integrating over $\mathcal{B}$. The left-hand side of (14) gives $\dot{\mathcal{E}}^K$ but, since the convection is plausibly turbulent, $\dot{\mathcal{E}}^K$ is small compared with the dissipation rate $\mathcal{Q}^D$, here created by viscosity. When $\dot{\mathcal{E}}^K$ is discarded, the average of the convective energy equation becomes

$$0 = \overline{\mathcal{Q}}^B - \overline{\mathcal{Q}}^D, \quad \text{where} \quad \mathcal{Q}^B = \int_{\mathcal{B}} \mathbf{u}_c\cdot\mathbf{B}_c\,\mathrm{dV} \tag{15}$$

is the rate of working of the buoyancy forces. According to (15), $\overline{\mathcal{Q}}^D$ is directly supplied by the convective buoyancy force. It therefore should not appear in the convective average (1) of the heat equation. For further discussion, see for example Section 7 of BR.

No further details about the nature of the convective motion and its dissipation will be required below, with one exception: we shall assume that, except in boundary layers, $\mathbf{u}_c$ is large enough to homogenize the contents of $\mathcal{B}$ and in particular make $\overline{S}$ almost uniform, independent of r:

$$\overline{S} = \overline{S}(\bar{t}). \tag{16}$$

This requires that the Péclet number, $\mathcal{U}R/\kappa$, of the convection should be large, where κ is the thermal diffusivity. Equation (16) is the thermodynamic input necessary to complete the specification of the reference state. A popular alternative to assumption (16) uses a thermodynamic relation that gives

$$\dot{\overline{S}} = \frac{C_p}{\overline{T}}\overline{d}_{\bar{t}}\overline{T} - \frac{\alpha}{\overline{\rho}}\overline{d}_{\bar{t}}\overline{P}, \tag{17}$$

where C_p is the specific heat at constant pressure and α is the thermal expansion coefficient, both evaluated in the mean state. This is supplemented by *ad hoc* assumptions about the quantities appearing on the right-hand side of (17), e.g., that $\overline{T}^{-1}\overline{d}_{\bar{t}}\overline{T}$ is independent of r. The advantage of the present approach is that it depends only on the r-independence (16) of $\overline{S}$ and this has a convincing theoretical basis: thorough convective mixing. From now on the overbars denoting convective averages, that were implicit in (1) and (4), will usually be omitted; $\overline{d}_{\bar{t}}$ will be denoted by d_t.

We can evaluate $\dot{\mathcal{E}}^g$ and $d_t P$ by first using (3) to rewrite $\mathcal{E}^g$ in $(5)_2$ and P in $(12)_2$ as

$$\mathcal{E}^g = -G\int_0^{\mathcal{M}} \frac{M\,dM}{r}, \qquad P = \frac{G}{4\pi}\int_M^{\mathcal{M}} \frac{M\,dM}{r^4}. \quad (18)$$

Since M and $\mathcal{M}$ do not change following the radial motion $V\ (= d_t r)$ of the fluid, the rates of increase of $\mathcal{E}^g$ and P are

$$\dot{\mathcal{E}}^g = -G\int_0^{\mathcal{M}} \left(-\frac{V}{r^2}\right) M\,dM,$$

$$d_t P = \frac{G}{4\pi}\int_M^{\mathcal{M}} \left(-\frac{4V}{r^5}\right) M\,dM\,,$$

i.e.,

$$\dot{\mathcal{E}}^g = 4\pi\int_0^{\mathcal{R}} Vg\rho r^2 dr, \quad d_t P = -4\int_r^{\mathcal{R}} Vg\rho\frac{dr}{r}. \quad (19)$$

We may now give the promised derivation of (4) which starts from the thermodynamic relation

$$d_t\varepsilon^I = (P/\rho^2)d_t\rho + T\dot{S}\,. \quad (20)$$

By integrating this over $\mathcal{B}$ and using $(10)_2$ we obtain

$$\dot{\mathcal{E}}^I = 4\pi\int_0^{\mathcal{R}} d_t\varepsilon^I \rho r^2 dr = \mathcal{M}\widetilde{T}\dot{S} - 4\pi\int_0^{\mathcal{R}} P\,\partial_r(r^2 V)dr\,.$$

An integration by parts, using (12) and (13), now gives

$$\mathcal{M}\widetilde{T}\dot{S} = \dot{\mathcal{E}}^I + \dot{\mathcal{E}}^g\,, \quad (21)$$

and (4) follows from (1).

This completes the justification of the theory, much of which is regarded as standard for Earth models. We move to a newer development. Defining $W = d_t P$, $(19)_2$ gives

$$\frac{dW}{dr} = \frac{4g\rho}{r}V. \quad (22)$$

From standard thermodynamics, we have

$$\rho^{-1}d\rho = K_s^{-1}dP - \alpha_S dS, \quad (23)$$

where $K_s = \rho(dP/d\rho)_S$ is the incompressibility and $\alpha_S = -\rho^{-1}(\partial\rho/\partial S)_P = \alpha T/C_p$ is the entropic expansion coefficient. In using this to relate W to the motional derivatives of ρ and S, we again make use of $(10)_2$ and (16), obtaining

$$\frac{dV}{dr} = -\frac{2V}{r} - \frac{W}{K_s} + \frac{\gamma\rho T}{K_s}\dot{S}, \quad (24)$$

where $\gamma = \alpha K_s/\rho C_p$ is the Grüneisen parameter. Equations (22) and (24) define a second order, linear, inhomogeneous system for V and W, solutions to which must satisfy the two boundary conditions

$$V(0) = 0, \qquad W(\mathcal{R}) = 0. \quad (25)$$

These suffice to determine $V(r)$ and $W(r)$ from the $\dot{S}$ given by (1). The rate of change in gravitational energy can then be found from $(19)_1$ and the rate of change in internal energy from (21). The required functions, appearing in the coefficients multiplying W, V and $\dot{S}$ on the right-hand sides of (22) and (24), are derived by solving (3), (12) and (13) for the S assigned in (16).

3. Geophysical Generalization

Several additional complications must be faced when applying the ideas of Section 2 to the Earth. The basic assumption, that deviations from adiabaticity are small, is much better satisfied by the FOC than the mantle, but it is nevertheless widely recognized that heat transport by subsolidus convection dominates heat transport by thermal conduction in the mantle, and that this convection maintains adiabaticity (isentropy). According to Schubert *et al.* (2001), the thermal (and interfacial) boundary layers in which thermal conduction is significant or dominates "comprise less than 2% of the volume of the Earth."

We continue to assume that the basic state is spherically symmetric. This requires that the angular velocity, Ω, of the Earth is everywhere small compared with $(g/r)^{1/2}$, that the Lorentz force created by the geomagnetic field $\mathbf{B}$ is sufficiently small, and that the zonal flows, associated with $\mathbf{B}$ and the thermal wind, have a negligible effect. Equation (15) continues to hold but the Joule losses now contribute to, and plausibly dominate, $\mathcal{Q}^D$.

In discussions of the gross thermodynamics of the Earth, greatest interest usually centers on the core because it is recognized that the gravitational energy released in the freezing of the SIC is a thermodynamically efficient way of stirring the FOC and powering the geodynamo (Braginsky, 1963). We at first follow most previous authors by adopting the simplest model of core mix: a two-component alloy of iron and light constituents, the mass fraction of the latter being ξ. There are therefore 3 independent thermodynamic variables (here ρ, S and ξ), so that now $\varepsilon^I = \varepsilon^I(\rho, S, \xi)$. There is also a variable conjugate to ξ, the chemical potential $\mu(\rho, S, \xi)$.

The vigorous convection in the FOC homogenizes ξ as thoroughly as it homogenizes S so that (except in boundary layers) it too is r-independent:

$$\overline{\xi} = \overline{\xi}(\bar{t})\,. \quad (26)$$

We denote ξ and S in the FOC by ξ_2 and S_2, the suffix being chosen to conform with the numbering of the zones in the PREM model, on which the numerical work reported below is based; the suffix $_1$ stands for the SIC, $_2$ for the FOC, $_3$ for the bottom of the mantle, etc. The interface between zone n and zone $n+1$ is denoted by $\Sigma_{n,n+1}$, except that Σ_{12} will often be replaced by 'ICB' and Σ_{23} by 'CMB'; $R_{n,n+1}$ and $A_{n,n+1} = 4\pi R^2_{n,n+1}$ will be the radius and area of $\Sigma_{n,n+1}$ although R_{12} will usually be replaced by R. The value on $\Sigma_{n,n+1}$ of a variable Q, such as P or μ, that is continuous at $\Sigma_{n,n+1}$ will be denoted by $Q_{n,n+1}$. For a variable Q, such as ρ or ξ, that is discontinuous, $[\![Q]\!]_n^{n+1} = Q_{n+1} - Q_n$ will be denoted by ΔQ in the case of the ICB. Zone interfaces move slowly as the Earth evolves but, if no material crosses them, which is a reasonable assumption in the case of the core-mantle boundary (CMB), conservation of mass and continuity of pressure require that solutions to (22) and the generalized (24) satisfy the interface conditions

$$[\![\rho V]\!]_n^{n+1} = 0, \qquad [\![W]\!]_n^{n+1} = 0\,. \quad (27)$$

Strictly, $(27)_1$ does not apply at a phase boundary, because material passes through such a surface as the Earth evolves.

The errors in applying $(27)_1$ at the 410 km and 660 km phase boundaries are so tiny that it is not worth sacrificing simplicity in order to eliminate them. The relative motion between the ICB and core mix through freezing is so significant that, for this phase boundary, $(27)_1$ must be replaced by

$$\rho_1(\dot{R} - V_1) = \rho_2(\dot{R} - V_2) = m \text{ (say)}, \tag{28}$$

where $R(t)$ is the radius of the ICB and $m(t)$ is the rate per unit area at which mass is added to the SIC through freezing. Since $\Delta\rho = [\![\rho]\!]_1^2$ is small compared with ρ_1 and ρ_2 at the ICB, V_1 and V_2 are small also, of order $\dot{R}\Delta\rho/\rho_2 \sim 0.07\dot{R}$. BR took $V_1 \equiv 0$ and $V_{\rm CMB} = 0$. The derivations of (38) and (46) below represent a generalization of their results to cases in which $V_1 \neq 0$ and $V_{\rm CMB} \neq 0$.

It is easy to believe that subsolidus convection will thoroughly mix the SIC too, provided that there are sufficient energy sources to drive it. The possibility that the levels of ^{40}K in the core are non-negligible is increasingly discussed (e.g., Section 5 below), and we suppose here that they suffice to drive convection in the SIC that is strong enough to homogenize S_1 and ξ_1. In contrast, most Earth models exclude inner core convection but nevertheless assume an adiabatic temperature profile in the SIC instead of a slightly sub-melting point gradient. Fortunately the difference is small and has negligible effect on the outcome of the numerical work below. We now replace (24), both for the FOC *and* the SIC, by its generalization for the alloy:

$$\frac{dV}{dr} = -\frac{2V}{r} - \frac{W}{K_s} + \frac{\gamma\rho T}{K_s}\dot{S} + \alpha_\xi\dot{\xi}\,, \tag{29}$$

where $\quad \alpha_\xi = \dfrac{1}{\rho}\left(\dfrac{\partial\rho}{\partial\xi}\right)_{P,S}$

is the compositional expansion coefficient. For want of better information, we shall assume that $\dot{S}_1 = \dot{S}_2$, but $\dot{\xi}_1$ and $\dot{\xi}_2$ are determined by conservation of the mass of the light constituent. In terms of the rejection factor, $r_{\rm FS} = [\![\xi]\!]_1^2/\xi_2$, introduced by BR, we have

$$\frac{\dot{\xi}_1}{\xi_1} = \frac{\dot{\xi}_2}{\xi_2} = \frac{A_{12}r_{\rm FS}}{(1-r_{\rm FS})\mathcal{M}_1 + \mathcal{M}_2}m. \tag{30}$$

In determining the reference state in the core, $(12)_1$ is augmented by another consequence of the homogeneity of S and ξ: in both FOC and SIC,

$$\frac{d\mu}{dr} = -\alpha_\xi g = -\alpha_\xi\frac{dU}{dr}\,. \tag{31}$$

We shall assume below that α_ξ is a constant. Then $(31)_1$ gives

$$\mu = \mu_0 - \alpha_\xi U\,, \tag{32}$$

where μ_0 is a constant.

The motion of the ICB is determined by the melting temperature, $T_m(P,\xi)$, of core mix. The condition $T_2(R) = T_m(R)$ holds continuously on the ICB as it moves outwards, so that $\mathcal{D}T_2/\mathcal{D}t = \mathcal{D}T_m/\mathcal{D}t$ where $\mathcal{D}/\mathcal{D}t = d_t + (\dot{R} - V_2)\partial_r = d_t + (m/\rho_2)\partial_r$ is the derivative following the motion of the boundary. This condition may be written in a form similar to (6.37) of BR:

$$\frac{1}{\rho_2 R}\left(m - \frac{W_2}{g}\right) = -r_S\dot{S}_2 - r_\xi\dot{\xi}_2, \quad \text{at} \quad r = R, \tag{33}$$

where

$$r_S = \frac{1}{\Delta_{ma}C_p},\ r_\xi = -\frac{1}{\Delta_{ma}T}\left[\frac{h_\xi}{C_p} + \left(\frac{\partial T_m}{\partial\xi}\right)_P\right], \tag{34}$$

$$\Delta_{ma} = \frac{\gamma\rho g R}{K_s}\left[\frac{(\partial T_m/\partial P)_\xi}{(\partial T/\partial P)_{S\xi}} - 1\right],\ h_\xi = T\left(\frac{\partial S}{\partial\xi}\right)_{PT}. \tag{35}$$

Equation (33) provides a link between $\dot{S}_2$ and m which is needed in Section 4, but some of the parameters in (34) and (35) are poorly known, particularly r_ξ which involves both the heat of reaction h_ξ and the depression of the freezing point of iron through the "impurity" ξ, both of which are rather uncertain.

The heat equation for the mantle analogous to (1) is

$$-\mathcal{Q}_{\rm m}^S \equiv \mathcal{M}_{\rm m}\widetilde{T}_{\rm m}\dot{S}_{\rm m} = \mathcal{Q}_{\rm m}^R + \mathcal{Q}_{\rm CMB}^q - \mathcal{Q}_E^q, \tag{36}$$

where $\mathcal{Q}_E^q$ is the heat flow into the lithosphere and $\mathcal{Q}_{\rm CMB}^q$ is the heat flow from the core to the mantle. The generalization of (1) for the core (FOC+SIC) is derived in Section 7 of BR and leads to their (7.23), which is written here as

$$\int_{\rm core}(T\dot{S} + \mu\dot{\xi})\rho\,{\rm dV} = \mathcal{Q}_{\rm core}^R - \mathcal{Q}_{\rm CMB}^q + \mathcal{Q}^L, \tag{37}$$

where $\mathcal{Q}^L = h_L A_{12} m$ is the rate of latent heat release by freezing at the ICB, defined by $h_L = [\![\varepsilon^H]\!]_1^2$ where ε^H is the specific enthalpy. The left-hand side of (37) may also be written as $\mathcal{M}_1(\widetilde{T}_1\dot{S}_1 + \widetilde{\mu}_1\dot{\xi}_1) + \mathcal{M}_2(\widetilde{T}_2\dot{S}_2 + \widetilde{\mu}_2\dot{\xi}_2)$, where as in Section 2 the tildes denote mass-weighted averages. BR express (37) differently. They take $h_N = T_{12}[\![S]\!]_1^2$ as the effective latent heat per unit mass; since μ is continuous at a phase boundary, $h_N = h_L - \mu_{12}[\![\xi]\!]_1^2$. By (30), $Q^N = h_N A_{12} m = Q^L - \mu_{12}(\mathcal{M}_1\dot{\xi}_1 + \mathcal{M}_2\dot{\xi}_2)$, so that (37) may be written alternatively as

$$-Q_{\rm core}^S \equiv \mathcal{M}_{\rm core}\widetilde{T}_{\rm core}\dot{S}_{\rm core} = \mathcal{Q}_{\rm core}^R - \mathcal{Q}_{\rm CMB}^q + \mathcal{A}^\xi + \mathcal{Q}^N, \tag{38}$$

where

$$\mathcal{A}^\xi = \int_{\rm core}(\mu_{\rm ICB} - \mu)\dot{\xi}\rho\,{\rm dV}\,. \tag{39}$$

This form has the advantage of being independent of the energy level μ_0. Equation (32) implies

$$\mathcal{A}^\xi = 4\pi\alpha_\xi\left(\dot{\xi}_1\int_0^R(U - U_{\rm ICB})\rho r^2 dr + \dot{\xi}_2\int_R^{R_{\rm CMB}}(U - U_{\rm ICB})\rho r^2 dr\right). \tag{40}$$

For the mantle, the alternative form of (36) analogous to (4) is

$$\dot{\mathcal{E}}_{\rm m}^I + \dot{\mathcal{E}}_{\rm m}^g = \mathcal{Q}_{\rm m}^R + \mathcal{Q}_{\rm CMB}^q - \mathcal{Q}_E^q - \mathcal{A}_{\rm CMB}^P\,, \tag{41}$$

where $\mathcal{A}^P_{\rm CMB}$ is the rate at which energy is lost to the mantle and gained by the core through the contraction of the Earth:

$$\mathcal{A}^P_{\rm CMB} = -A_{\rm CMB} P_{\rm CMB} \dot{R}_{\rm CMB} \,. \quad (42)$$

For the core, the equation analogous to (4) is slightly complicated by the fact that $\dot{R}$ does not coincide with V_1 or V_2 at the ICB, so that

$$\dot{\mathcal{E}}^I_1 = m A_{12} \varepsilon^I_1 + \int_{\rm SIC} d_t \varepsilon^I_1 \, \rho_1 \, {\rm dV}$$

and similarly for $\dot{\mathcal{E}}^I_2$. This leads by the generalization

$$d_t \varepsilon^I = (P/\rho^2) d_t \rho + T\dot{S} + \mu\dot{\xi}, \quad (43)$$

of (20) to

$$\dot{\mathcal{E}}^I_1 + \dot{\mathcal{E}}^I_2 = -m A_{12} [\![\varepsilon^I]\!]^2_1 + \int_{\rm core} [(P/\rho^2) d_t\rho + T\dot{S} + \mu\dot{\xi}]\rho \, {\rm dV}. \quad (44)$$

Using the continuity equation and integration by parts, the first term in the integral can be re-expressed as

$$-\int_{\rm core} P \nabla\cdot\mathbf{V} \, {\rm dV} = A_{12} P_{\rm ICB} [\![V]\!]^2_1 + \mathcal{A}^P_{\rm CMB} + \int_{\rm core} \mathbf{V}\cdot\nabla P \, {\rm dV}$$

so that, since (30) implies $[\![V]\!]^2_1 = -m[\![\rho^{-1}]\!]^2_1$, we have by (44)

$$\dot{\mathcal{E}}^I_1 + \dot{\mathcal{E}}^I_2 = -Q^L + \mathcal{A}^P_{\rm CMB} + \int_{\rm core} (T\dot{S} + \mu\dot{\xi})\rho \, {\rm dV} \,. \quad (45)$$

This may be combined with (37) to give

$$\dot{\mathcal{E}}^I_{\rm core} + \dot{\mathcal{E}}^g_{\rm core} = \mathcal{Q}^R_{\rm core} - \mathcal{Q}^q_{\rm CMB} + \mathcal{A}^P_{\rm CMB}. \quad (46)$$

The power balance for the entire Earth, obtained from (36) and (38), is

$$\mathcal{M}_{\rm m} \widetilde{T}_{\rm m} \dot{S}_{\rm m} + \mathcal{M}_{\rm core} \widetilde{T}_{\rm core} \dot{S}_{\rm core} = \mathcal{Q}^R_{\rm E} - \mathcal{Q}^q_{\rm E} + \mathcal{A}^\xi + Q^N. \quad (47)$$

Equations (41) and (46) give the form of analogous to (4):

$$\dot{\mathcal{E}}^I_{\rm E} + \dot{\mathcal{E}}^g_{\rm E} = \mathcal{Q}^R_{\rm E} - \mathcal{Q}^q_{\rm E}. \quad (48)$$

The contraction of the Earth on cooling, and particularly the change in volume of core mix as it solidifies at the ICB, necessarily implies that V is negative everywhere, and in particular $V_2(R_{\rm CMB}) = \dot{R}_{\rm CMB} < 0$. Several Earth models assume however that $R_{\rm CMB}$ is fixed. It will be found below that the gravitational energy release caused by the inward motion of the mantle is quite large. It is usually either ignored, or it is regarded as irrelevant to core dynamics, a point of view that clearly has merit: when the energetics of the core alone is under scrutiny, only $\mathcal{A}^\xi$ and $\dot{\mathcal{E}}^g_{\rm core}$ appear in (38) and (46).

It may be seen from (39) that $\mathcal{A}^\xi$, like Q^N, enters (38) because of the changing chemical state of the core; no such term appears in the heat equation (1) for the homogeneous fluid analyzed in Section 2. Although $\mathcal{A}^\xi$ is part of $\dot{\mathcal{E}}^I$, (40) shows that it is intimately linked to the gravitational energy released by the redistribution of mass resulting from freezing at the ICB; see Appendix B of BR. For this reason, it is described here, below and in the abstract as gravitational power that helps to stir the core. Its high thermodynamic efficiency makes it a significant source for the geodynamo, but that topic is beyond the scope of this paper.

4. Application

The theory of Section 3 requires an Earth model. The ones used below are based quite closely on one of the two models developed by Roberts *et al.* (2003). It is therefore necessary here only to describe how that model was modified by information that became available after the publication of that paper.

Although the models are based on PREM, the density of the SIC was artificially increased by 1.7% so that the density jump $\Delta\rho$ at the ICB became 814 kg m^{-3}, consistent with the recent estimate of 820±60 kg m^{-3} of Masters and Gubbins (2003). The agreement between the incompressibilities determined seismically ($K_s = \rho(V_p^2 - \frac{4}{3}V_s^2)$) and hydrostatically ($K_h = -g\rho^2/\rho'$) remained good. The mean density of the FOC is unaffected and at 10900 kg m^{-3} it is low compared with Masters and Gubbins's value of 11160±60 kg m^{-3}. The mass of the inner core increased from 0.97×10^{24} kg to 1.00×10^{24} kg.

Apart from these altered ρ, our 'Model 1' is closely based on Roberts *et al.* (2003). Our values for T, α, γ and the isothermal incompressibility $K_T = C_v K_s / C_P$ on the ICB and our value for the FOC's mean molecular weight $\overline{A}_2$ (= 49.9, slightly larger than the value of 48.1 often adopted) gave a solidification shrinkage of 1.1875×10^{-5} m^3 kg^{-1} corresponding to a density jump $\Delta\rho_S = 187$ kg m^{-3} at the ICB, leaving $\Delta\rho_\xi = 627$ kg m^{-3} as the density jump created by the compositional difference. The latent heat release Q^N, calculated as in Roberts *et al.* (2003), is 0.69×10^6 J kg^{-1}, which may be compared with the value of 0.87×10^6 J kg^{-1} given by Alfè *et al.* (2002b). Our model also gave $\alpha_\xi = 0.6$, $\xi_1 = 7.6\%$, $\xi_2 = 15.9\%$, $\overline{A}_1 = 52.9$ and $r_{\rm FS} = 0.52$, assuming sulfur as the light alloying component of core mix; the results for silicon are not very different. We took the heat of reaction h_ξ as -1.6×10^7 J kg^{-1} and $dT_m/d\xi = -300\,{\rm K}/\Delta\xi$. Following BR, we assumed $T_{\rm ICB} = 5100$ K and obtained $T_{\rm CMB} = 3821$ K and a central temperature $T(0)$ of 5282 K; the mass weighted temperatures, $\widetilde{T}$, of the SIC and FOC were respectively 5173 K and 4400 K. The significant, but uncertain solidification parameters Δ_{ma} and Δ_2, which were 0.02 and 0.05 for BR's model, were here 0.04 and 0.10. Taking the thermal conductivity at the top of the core to be 43 W m^{-1} K^{-1}, the adiabatic heat flow there was $\mathcal{Q}^{\rm ad}_{\rm CMB} = 5.8$ TW. This is less than $\mathcal{Q}^q_{\rm CMB}$ if $\dot{R} > 7 \times 10^{-12}$ m s^{-1}, approximately.

The model employed recent estimates of the Grüneisen parameter in the core, and used the form given in Eq. (49) of Stacey (2005). More specifically, we took

$$\gamma = 1.8345 \exp[0.31909(6562.54/\rho)^{3.709613} - 1], \quad (49)$$

in which ρ is in kg m^{-3}. This gives values only a few percent different from those in Table 7 of Stacey and Davis (2004). For the mantle, we adopted values of γ given in Stacey (1992) and other values needed (e.g., of α) from Schubert *et al.* (2001). The temperature obtained at the bottom of the mantle (above the edge of the thermal boundary layer) was 3091 K, so that $[\![T]\!]^2_3 = 730$ K, which lies with in the range 440 K – 1100 K given in Section 4.9 of Schubert *et al.* (2001) for the jump in T across the D$''$ layer. We also obtained $\widetilde{T}_{\rm m} = 3337$ K. In (36) we took $\mathcal{Q}^q_{\rm E} = 36.9$ TW and $\mathcal{Q}^R_{\rm m} = \mathcal{M}_{\rm m} q^R_{\rm m} = 29.8$ TW, where

Table 2. Model results.

Quantity	Model				Unit
	1a	1b	1c	2	
m	8	12	16	7.35	10^{-8} kg m^{-2} s^{-1}
$\dot{R}$	6.2	9.2	12.3	5.7	10^{-12} m s^{-1}
$\mathcal{Q}^q_{\rm CMB}$	5.20	7.80	10.39	7.80	TW
$\dot{\xi}_{\rm FOC}$	6.6	9.9	13.2	2.8	10^{-20} s^{-1}
$\mathcal{A}^\xi$	0.72	1.08	1.43	0.50	TW
$\mathcal{Q}^{\rm N}$	0.99	1.49	1.98	1.20	TW
$\dot{S}_{\rm core}$	–4.0	–6.1	–8.1	–3.7	10^{-16} W kg^{-1} K^{-1}
$\dot{S}_{\rm m}$	–1.44	+0.49	+2.42	–1.46	10^{-16} W kg^{-1} K^{-1}
$\mathcal{Q}^S_{\rm core}$	3.49	5.23	6.98	3.47	TW
$\mathcal{Q}^S_{\rm E}$	5.42	4.57	3.71	5.43	TW
$\dot{R}_{\rm CMB}$	–3.1	–4.7	–6.2	–2.9	10^{-14} m s^{-1}
$\mathcal{A}^P_{\rm CMB}$	0.64	0.96	1.28	0.59	TW
$\dot{\mathcal{E}}^I_{\rm E}$	–4.59	–3.32	–2.06	–4.80	TW
$\dot{\mathcal{E}}^g_{\rm E}$	–2.54	–3.81	–5.08	–2.33	TW
$\dot{\mathcal{E}}^I_{\rm core}$	–3.31	–4.96	–6.61	–3.43	TW
$\dot{\mathcal{E}}^g_{\rm core}$	–1.25	–1.87	–2.50	–1.15	TW

$q^R_{\rm m} = 7.38 \times 10^{-12}$ W kg^{-1}; see Schubert *et al.* (2001), p. 129.

Results from integrating (22) and (29) for the core and (22) and (24) for the mantle, subject to conditions (25), (27) and (28), are shown in columns 2–4 of Table 2, for the 3 values of m given on its first line under the heading 'Model 1'. The required $\dot{S}$ for the core was obtained from (33), it being assumed that $\dot{S}_1 = \dot{S}_2$ $(= \dot{S}_{\rm core})$; $\dot{\xi}_1$ and $\dot{\xi}_2$ then followed from (30) and $\mathcal{A}^\xi$ from (40). From the value of $\mathcal{Q}^q_{\rm CMB}$ derived from (38), $\dot{S}_{\rm m}$ was obtained from (36). Table 2 shows that a large m leads to a positive $\dot{S}_{\rm m}$, i.e., a mantle that heats up! This is an unavoidable consequence of assuming that $\mathcal{Q}^q_{\rm E}$ (= 36.9 TW) and $\mathcal{Q}^R_{\rm m}$ (= 28.6 TW) are known. Discussions of the mantle's heat balance, such as that in Section 4.1.5 of Schubert *et al.* (2001), often omit the input, $\mathcal{Q}^q_{\rm CMB}$, of heat from the core but estimate that roughly 20% of $\mathcal{Q}^q_{\rm E}$ is due to cooling. Then (36) gives $\dot{S}_{\rm m} = -(\mathcal{Q}^q_{\rm E} - \mathcal{Q}^R_{\rm m})/\mathcal{M}_{\rm m}\widetilde{T}_{\rm m} = -0.2\mathcal{Q}^q_{\rm E}/\mathcal{M}_{\rm m}\widetilde{T}_{\rm m} = -5.5 \times 10^{-16}$ W kg^{-1}K^{-1}. Evidently, if $\mathcal{Q}^q_{\rm CMB}$ is large enough and is included in (36), a positive and therefore unacceptable $\dot{S}_{\rm m}$ must inevitably result. This difficulty is discussed further below.

It was noted in passing that several of the evolutionary quantities arising in this discussion are proportional to m, or equivalently $\dot{R}$. An apparent exception, because of the W_2 term in (33), is $\dot{S}_2$. Observing that W_2 is rather small, BR ignored it, leading to their result $\dot{S}_2 = -\Delta_2 C_P \dot{R}/R$, according to $(30)_2$ and (33). We did not ignore W_2 in constructing Table 1 but found that it is less than 5% of mg. The rate of increase of P, and in particular of P_2, in a contracting Earth is proportional to m, as therefore is W_2, as well as all other terms in (33). This linearity is evident in Table 2 and, following the usual practice (see for example BR, Labrosse, 2003; Gubbins *et al.*, 2003, 2004; Nimmo *et al.*, 2004), we write

$$\mathcal{Q}^S_{\rm core} = C^S_{\rm core}\dot{R},\ \mathcal{A}^\xi = C^\xi \dot{R},\ \mathcal{Q}^N = C^N \dot{R},\ \dot{R}_{\rm CMB} = -\lambda \dot{R}, \quad (50)$$

so that

$$\mathcal{Q}^q_{\rm CMB} = \mathcal{Q}^R_{\rm core} + C^{\rm total}_{\rm core}\dot{R},\ \text{where}\ C^{\rm total}_{\rm core} = C^S_{\rm core} + C^\xi + C^N. \quad (51)$$

For the model considered here

$$(C^S_{\rm core}, C^\xi, C^N, C^{\rm total}_{\rm core}) = (5.66, 1.16, 1.61, 8.43) \times 10^{23}\ \text{J m}^{-1}, \quad (52)$$

and λ is approximately $\frac{1}{2}$%. This value of λ typifies the error created in lines 2–10 of Table 2 if it is assumed (as is often done) that $\dot{R}_{\rm CMB} = 0$. The entries on the last 5 lines would however obviously be drastically affected. Writing, as in (53),

$$\dot{\mathcal{E}}^g_{\rm core} = -C^g_{\rm core}\dot{R},\ \dot{\mathcal{E}}^g_{\rm E} = -C^g_{\rm E}\dot{R},\ \mathcal{A}^P_{\rm CMB} = C^P_{\rm CMB}\dot{R}, \quad (53)$$

the present model gives

$$(C^g_{\rm core}, C^g_{\rm E}, C^P_{\rm CMB}) = (2.03, 4.12, 1.04) \times 10^{23}\ \text{J m}^{-1}. \quad (54)$$

The values derived for $\dot{\mathcal{E}}^g_{\rm E}$ and $\dot{\mathcal{E}}^I_{\rm E}$ are shown in Table 2. It is apparent from the Table that $\dot{\mathcal{E}}^g_E$, $\dot{\mathcal{E}}^g_{\rm core}$ and $\mathcal{A}^P_{\rm CMB}$ are quite large.

Some results for a second model are shown in column 5 of Table 2 under the heading 'Model 2'. This model makes closer contact than Model 1 with the results of *ab initio* calculations made by the group at University College London, which led them to a model core composed of Fe, O and S (or Si). The percentages of these elements given by Alfè *et al.* (2002a) were apparently influenced by the value of $\Delta\rho$ that was preferred before the paper by Masters and Gubbins (2003) appeared, and we made a small adjustment accordingly. By adopting molar percentages of 5% S and 10% O in the FOC and 2.5% S and 0% O in the SIC, and by taking the density of liquid iron at the ICB to be 12968 kg m^{-3} and that of solid iron 1.7% greater, we obtained $\rho_1 = 12980$ kg m^{-3} and $\rho_2 = 12166$ kg m^{-3}, consistent with the values assumed for Model 1. This implied $\xi_1 = 2.5\%$, $\xi_2 = 6.3\%$ and $r_{FS} = 0.61$. The density jump $\Delta\rho$ at the ICB divided into $\Delta\rho_S = 239$ kg m^{-3} and $\Delta\rho_\xi = 575$ kg m^{-3}. The mean atomic weight of the FOC is $\underline{A}_2 = 43$, which is much smaller than the value usually adopted. Also consistent with the recommendations of Alfè *et al.* (2002a), we assumed that $T_{ICB} = 5600$ K, the depression of the melting point caused by the admixture of light elements being 600 K. The heat of reaction, h^ξ, was taken as -2.8×10^7 J kg^{-1}. The adiabatic heat flow at the CMB was found to be 7.02 TW. The mantle was described in the same way as in Model 1.

Despite all these differences, Models 1 and 2 behave similarly. For both, the main effect of the enhanced $\Delta\rho$ is to increase the significance of compositional buoyancy, this being a little more pronounced for Model 2 than for Model 1, the compositional coefficient of expansion being greater for Model 2 ($\alpha_\xi = 0.98$). The results for the models shown in Table 2 differ strongly because $Q^R_{\text{core}} = 0$ for Model 1 but $Q^R_{\text{core}} = 2.63$ TW for Model 2, Q^R_{m} being reduced by the same amount. The much smaller solidification rate m for Model 2 can therefore maintain the same CMB heat flow, Q^q_{CMB}, as Model 1b. (In calculations not presented here, it was found that, if $Q^R_{\text{core}} = 2.63$ TW for Model 1, the same Q^q_{CMB} is obtained for $m = 7.96 \times 10^{-8}$ kg m^{-2} s^{-1}, and that other results are not very different from those shown for Model 2 in Table 2.) It is striking how the addition of 2.63 TW of core heat diminishes the rate at which the ICB advances by nearly 40%. If it is assumed that the volume of the SIC has always increased at the same rate, the age $\tau_{\text{SIC}} = R/3\dot{R}$ of the inner core is lengthened from 1.4 Ga to 2.3 Ga. For a recent discussion of the effects of internal heat sources, see Nimmo *et al.* (2004).

5. Conclusions

The idea that potassium is present in the core and that the ^{40}K it contains is significant for core energetics is an old one that was generally abandoned in the belief that equilibrium liquid silicate/liquid metal partioning forbids K from entering the core. This meant however that, if as is widely believed the Earth has a broadly chondritic composition, it is necessary to account for its depletion in K. One favored explanation is that K, being a moderately volatile element, evaporated and was blown away by the solar wind early in the evolution of the solar system. This however is not completely convincing since it fails to explain why other elements, even more volatile than K, are not more depleted. Calderwood (2000, 2001) recently revived interest in the topic by arguing that the missing K is in the core after all. He proposed that, during the formation of the Earth, K would dissolve in Ni and descend with it into the core; see Section 6 of Roberts *et al.* (2003) for further details. It is argued there that as much as 2.77×10^{21} kg of K exists in the core, so that it currently contains 3.23×10^{19} kg of ^{40}K. Taking $q^R_m = 3.5 \times 10^{-9}$ W kg^{-1} for K (Stacey 1977), heat is released at the rate $\mathcal{Q}^R_{\text{core}} = 9.7$ TW. This may be compared with 8.41 TW given by Stacey (1977). The case for significant levels of K in the core was greatly strenthened by the spectacular findings of Gessmann and Wood (2002) who made it clear that K dissolves in the metallic phase if S is also present. They also argued that the ^{40}K heat production would be less than 1.7 TW, which may be compared with the estimate of Murthy *et al.* (2003) of less than 1 TW. How much of this heat would be generated in the SIC depends on the rejection factor. If r_{FS} is small, the larger estimates of 9.7 TW would require a heat flux from the SIC of about 0.5 TW which is approximately the adiabatic heat flow in the models described in Section 4, making a convecting, isentropic SIC seem possible. In Model 2 however, we have assumed, to ease comparisons with a model of Nimmo *et al.* (2004), that the K abundence is only 400 ppm.

Belief in a radioactive core has been strengthened by a recent argument in favor of $\mathcal{Q}^q_{\text{CMB}} = 13 \pm 4$ TW (Lay *et al.*, 2006). Such a large value of $\mathcal{Q}^q_{\text{CMB}}$ raises even more strongly the difficulty with the mantle's heat budget described above. This is eased somewhat when it is recalled that the estimate $q^R_{\text{m}} = 7.38 \times 10^{-12}$ W kg^{-1} employed above was derived on the assumption that *all* radioactivity in the Earth resides in the mantle and crust. If one argues instead that a 9.7 TW source in the core should no longer be attributed to the mantle, q^R_{m} is reduced by almost $\frac{1}{3}$ to $q^R_{\text{m}} = 4.96 \times 10^{-12}$ W kg^{-1}. Such an idea is perhaps more clearly expressed by combining (36) and (38) as

$$\mathcal{Q}^S_{\text{m}} = \mathcal{Q}^q_{\text{E}} - \mathcal{Q}^R_{\text{E}} - C^{\text{total}}_{\text{core}}\dot{R}, \text{ where } \mathcal{Q}^R_{\text{E}} = \mathcal{Q}^R_{\text{m}} + \mathcal{Q}^R_{\text{core}} \quad (55)$$

is the total radioactive source in mantle and core irrespective of where it is situated. If we assume $\mathcal{Q}^R_{\text{E}} = 29.8$ TW as before and $C^{\text{total}}_{\text{core}} = 8.4 \times 10^{23}$ J m^{-1}, then $\mathcal{Q}^S_{\text{m}} > 0$ implies $\dot{R} < 9 \times 10^{-12}$ m s^{-1}, according to Model 1. If this upper bound is assumed to give a constant volumetric growth rate in time, the inner core must have an age τ_{SIC} greater than 1.5 Ga. Also the heat flow $\mathcal{Q}^q_{\text{CMB}}$ from the core cannot exceed $\mathcal{Q}^R_{\text{core}}$ by more than 7.4 TW, i.e., $\mathcal{Q}^g_{\text{CMB}} < 17$ TW for $Q^R_{\text{core}} = 9.7$ TW; this is close to the upper bound in the estimate of Lay *et al.* (2006). (As a point of comparison, we may note that the numerical geodynamo simulation of Glatzmaier and Roberts (1996, 1997) assumed that $\mathcal{Q}^R_{\text{core}} = 0$ and $\mathcal{Q}^q_{\text{CMB}} = 7.2$ TW, and led them to $\dot{R} = 10^{-11}$ m s^{-1}, corresponding to $\tau_{\text{SIC}} = R/3\dot{R} = 1.3$ Ga.)

This paper has exposed once again the difficulties inherent in deriving a satisfactory energy budget for the core. This however was not its principal objective. The main aim was to derive reasonably accurate estimates for the rates of release of gravitational energy in the contraction of the Earth, and the associated loss of internal energy. These are quite large, as can be seen from the last 5 lines of Table 1. For example, $\mathcal{A}^\xi$, which is the rate of gravitational energy release associated with the repartitioning of core constituents between the inner and outer cores and is important for core dynamics, is less than 30% of the total gravitational energy released during the cooling of the Earth.

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