Chapter 2

Infinitesimal Strains and Their Accumulations

In this chapter we will study infinitesimal deformation and the rate of deformation. Most geologic structures are the integral of infinitesimal deformations, so that it is important to understand how infinitesimal and finite deformations are related.

2.1 Infinitesimal strain

The lifetime of a deformation zone or an orogenic belt can be over millions of years, during which time rock masses undergo large deformations. Tectonism is fairly slow and the incremental deformation in our usual time-scale is very small. Therefore, the deformations detected by a geodetic survey are usually very small. Geology detects the result of long-term deformations.

Geological measurement is much less accurate than geodetic one. Thus, it is usually difficult for geology to detect such small strains as geodesy detects. However, it is possible in some cases¹ that geologic structures provide clues to quantify those strains. In addition, it is necessary to know the theories of infinitesimal strains to understand the following sections in which the dynamics of tectonic deformatoins is discussed.

2.1.1 Definition

Infinitesimal strain is a very small strain. If a deformation gradient tensor is nearly equal to the identity tensor ($F \approx I$), it is called infinitesimal deformation. Let us first rewrite $F$ with the displacement²,

$$u = x - \xi.$$

¹Note that strain is defined as the change of length compared to the original one. Given a tiny strain, the associated length change can be large enough for detection by geological measurement.

²The famous textbook by Turcotte and Schubert [245] defines the displacement $u = \xi - x$ in order to deal with contraction and compression as positive strain and stress, respectively.
We assume that the deformation and associated displacement are very small. Namely, \( \xi \approx x \). Three components of \( \xi \) are independent of each other, consequently \( \frac{\partial \xi_i}{\partial \xi_j} = \delta_{ij} \). Therefore, we have the deformation gradient tensor

\[
F = \frac{\partial x}{\partial \xi} = \frac{\partial (\xi + u)}{\partial \xi} = I + \frac{\partial u}{\partial \xi} \approx I + \frac{\partial u}{\partial x}.
\]

The last term is a tensor indicating the spatial gradient of displacement, so that we use the expression \( \nabla u = \frac{\partial u}{\partial x} \) for it. Taking the limit \( u \to 0 \), we obtain

\[
F = I + \nabla u,
\]

where the small departure from \( I \) is indicated by \( \nabla u = \frac{\partial u}{\partial x} \). The equation \( F = I \) indicates no deformation. Let us write the difference as \( \delta F \), with which we have the deformation gradient for infinitesimal deformation

\[
F = I + \delta F \quad (\delta F \approx O),
\]

(2.1)

where

\[
\delta F = \nabla u.
\]

In this case,

\[
2G + I = F^T \cdot F = (F + \delta F)^T \cdot (F + \delta F) \approx I + \delta F + \delta F^T.
\]

where the quadratic term \( (\delta F^T \cdot \delta F) \) was neglected. Geometrical linearity applies if this approximation is satisfied.

Rearranging the above equation, we have Green’s strain tensor

\[
G = \frac{1}{2} (\delta F + \delta F^T) = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right],
\]

(2.2)

the components of which are

\[
G_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) = \frac{1}{2} \left( \frac{\partial u_j}{\partial \xi_i} + \frac{\partial u_i}{\partial \xi_j} \right).
\]

(2.3)

Almansi’s strain tensor (Eq. (1.25)) is linearized to be

\[
2A + I = 2I - (F \cdot F^T)^{-1} = 2I - \left[ \left( I + \delta F \right) \cdot \left( I + \delta F^T \right) \right]^{-1}
\]

\[
\approx 2I - \left[ I + (\delta F + \delta F^T) \right]^{-1} \approx I + \delta F + \delta F^T.
\]

Therefore, Green’s and Almansi’s strain tensors are identical \( (G = A) \) for infinitesimal strains. We, accordingly, define the infinitesimal strain tensor,

\[
E = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right] \quad \text{or} \quad E_{ij} = \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right).
\]

(2.4)
2.1. INFINITESIMAL STRAIN

This is often referred to as a strain tensor. Elongation and contraction are indicated by the positive and negative components of this tensor (2.1.3). If we use the opposite sign convention, the infinitesimal strain tensor is written as

\[ \varepsilon = -\frac{1}{2} \left[ \nabla u + (\nabla u)^T \right] \quad \text{or} \quad \varepsilon = -\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \]

It is obvious that the infinitesimal strain tensors \( \mathbf{E} \) and \( \varepsilon \) satisfy the following equations,

\[ \mathbf{E} = \mathbf{E}^T, \quad \varepsilon = \varepsilon^T, \quad \mathbf{E} = -\varepsilon. \quad (2.5) \]

2.1.2 Relationships among strain tensors for infinitesimal deformations

We have seen that the infinitesimal strain tensor \( \mathbf{E} \) was derived from \( \mathbf{G} \) and \( \mathbf{A} \) by geometrical linearization. Then, how is \( \mathbf{E} \) related to other strain tensors? By the linealization, the right and left Cauchy-Green tensors become

\[ \mathbf{U} = \left[ (I + \delta \mathbf{F}) \cdot (I + \delta \mathbf{F}^T) \right]^{1/2} \approx \left( I + \frac{1}{2} \left( \delta \mathbf{F} + \delta \mathbf{F}^T \right) \right)^{1/2} = I + \mathbf{G} = I + \mathbf{E} = \mathbf{V}. \quad (2.6) \]

indicating that both tensors coincide with each other for infinitesimal strains. The orthogonal tensor \( \mathbf{R} \), which is related to the deformation gradient \( \mathbf{F} \) by the equation \( \mathbf{F} = \mathbf{R} \cdot \mathbf{U} \), becomes a tensor approximately equal to the unit tensor:

\[ \mathbf{R} = \mathbf{F} \cdot \mathbf{U}^{-1} = \left( I + \delta \mathbf{F} \right) \cdot \left( I + \frac{1}{2} \left( \delta \mathbf{F} + \delta \mathbf{F}^T \right) \right)^{-1} \approx \left( I + \frac{1}{2} \left( \delta \mathbf{F} - \delta \mathbf{F}^T \right) \right) = I + \mathbf{\Omega}. \quad (2.7) \]

The last term is called the rotation tensor defined by the equation

\[ \mathbf{\Omega} \equiv \frac{1}{2} \left[ \nabla u - (\nabla u)^T \right] \quad \text{or} \quad \Omega_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (2.8) \]

\( \mathbf{E} \) and \( \mathbf{\Omega} \) are the symmetric and antisymmetric parts of \( \nabla u \), respectively. Using Eqs. (2.6) and (2.7), we obtain

\[ \mathbf{U} + \mathbf{R} = 2I + \delta \mathbf{F} = I + \mathbf{F} = 2I + \mathbf{G} + \mathbf{\Omega}. \]

Thus

\[ \mathbf{F} = \mathbf{U} + \mathbf{R} - I = \mathbf{V} + \mathbf{R} - I = I + \mathbf{G} + \mathbf{\Omega}. \quad (2.9) \]

Since \( \mathbf{G} \) approaches \( \mathbf{E} \) for small strains,

\[ \mathbf{F} = \mathbf{I} + \mathbf{E} + \mathbf{\Omega}. \quad (2.10) \]

For finite strains \( \mathbf{F} = \mathbf{R} \cdot \mathbf{U} \neq \mathbf{U} \cdot \mathbf{R} \), as matrix multiplications are generally non-commutative. By contrast, matrix summations are commutative, so that Eq. (2.10) indicates that the order of rotation \( \mathbf{\Omega} \) and shape-change \( \mathbf{E} \) are commutative for infinitesimal deformations.
2.1.3 Physical interpretation of infinitesimal strain and spin tensors

To interpret the tensors $E$ and $\Omega$, we shall see how a small rectangle element deforms. Let us examine the diagonal components of $E$, first. If displacement is parallel to the coordinate axis $O-1$ (Fig. 2.1(a)),

$$u_2 = u_3 = 0, \quad \frac{\partial u_i}{\partial x_j} = 0 \quad (i = 1, 2, 3; \ j = 2, 3).$$

Therefore, all the components of $E$ vanishes except for $E_{11}$. In this case, neglecting higher-order terms, we have the length change

$$ds - ds_0 \approx \left( u_1 + \frac{\partial u_1}{\partial x_1} dx_1 + dx_1 - u_1 \right) dx_1 = \frac{\partial u_1}{\partial x_1} dx_1 = E_{11} dx_1.$$

Therfore, we obtain $(ds - ds_0)/ds_0 = E_{11}$. Accordingly, the diagonal terms of $E$ represent the elongation along the coordinate axes.

Using the angles $\theta$ and $\psi$ shown in Figs. 2.1(b) and (c), we have

$$2E_{12} = 2E_{21} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \tan \psi + \tan \theta.$$

For infinitesimal deformations, the last two terms of this equation become $\psi + \theta$. These two components indicate the angular decrease of the corner at which two sides of the rectangle originally parallel to the axes $O-1$ and $-2$ meet. The indices of $E_{12}$ and $E_{21}$ correspond to the axes. Consequently, $2E_{12}$ and $2E_{21}$ represent the engineering shear strain of the corner. The term “shear strain” often refers to the diagonal components of $E$, which are half of the engineering shear strains. Comparing Figs. 1.11 and 2.1(b), the parameter $q$ in Eq. (1.13) is related to the displacement gradients as $\frac{\partial u_1}{\partial x_2} = 2q$ and $\frac{\partial u_2}{\partial x_1} = 0$. Therefore, $q$ equals shear strain for infinitesimal simple shears.

The infinitesimal strain tensor is symmetric (Eq. (2.5)), so that it has three real eigenvalues, $E_1$, $E_2$, and $E_3$, and three mutually perpendicular eigenvectors. Consider a rectangular parallelepiped whose edges are parallel to the eigenvectors. The length of the edge parallel to the $i$th eigenvector is elongated by $(1 + E_i)$. Therefore the volume change is given by

$$\frac{V}{V_0} - 1 = (E_1 + 1)(E_2 + 1)(E_3 + 1) - 1 \approx \text{trace } E = E_1,$$

where $V_0$ and $V$ stand for the volume before and after deformation. This quantity is called the dilatation, and is equal to the first basic invariant of the tensor $E$ for infinitesimal strains.

Figure 2.1(d) shows the counterclockwise rotation of a rectangle around the $O-3$ axis. Two sides at the corner were parallel to the coordinate axes before deformation. We are considering very small rotations associated with an infinitesimal deformation, so that the sides that were initially parallel to the $O-1$ and $-2$ axes are rotated by the angles $\partial u_2/\partial x_1$ and $-\partial u_1/\partial x_2$, respectively. If it is a rigid-body rotation, the two quantities are equal. The average of these angles is

$$\Omega_{12} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right).$$
Accordingly, the $ij$th component of $\Omega$ represents the mean rotation about the $k$th coordinate axis, where $i \neq j \neq k$.

Generally an antisymmetric tensor has three independent components. The vector that is made up of these components is called the axial vector of the antisymmetric tensor. Rotation vector $\varpi = (\varpi_1, \varpi_2, \varpi_3)^T$ is defined as the axial vector of the rotation tensor:

$$
\Omega_{ij} = -\epsilon_{ijk} \varpi_k.
$$
are identical:

$$\Omega \cdot a = \varpi \times a. \quad \text{(2.14)}$$

This is a vector quantity, and the first component is

Left-hand side $= -\varpi_3 a_2 + \varpi_2 a_3$,
Right-hand side $= \varpi_2 a_3 - \varpi_3 a_2$.

therefore both sides are identical. The identity of Eq. (2.14) is readily demonstrated for other components. Combining Eqs. (2.8) and (2.12), we obtain

$$\varpi = \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)^T,$$

which is the expression of the rotation vector by means of displacement. It is obvious from the right-hand side that

$$\varpi = \frac{1}{2} (\nabla \times u).$$

### 2.2 Time rate of change

When we describe the temporal change of variables such as the density and velocity, the time rate of change of the variables is important for theoretical considerations. Let us first take a position as a variable. The time rate of change of position is velocity of a particle. Consider that the particle occupied the position $\xi$ at time $t = 0$ and $x(\xi, t)$ at time $t$. The velocity is the time derivative

$$v = v(\xi, t) = \frac{\partial x(\xi, t)}{\partial t}. \quad \text{(2.15)}$$

This has the Lagrangian coordinates ($\xi_1, \xi_2, \xi_3$) and time $t$ as the arguments, so that this is the Lagrangian description of velocity. The function $v(\xi, t)$ describes the velocity change that would be measured by an observer traveling with the particle. On the other hand, there is the Euler description of velocity $v = v(x, t)$ that indicates the velocity change that would be observed at the specific location $x$.

There are Euler and Lagrangian descriptions for the time rate of change of variables. They have the independent variables ($x, t$) and ($\xi, t$), respectively:

Euler or spatial derivative

$$\frac{\partial}{\partial t} \equiv \frac{\partial}{\partial t} \bigg|_{x=\text{const}}, \quad \text{(2.16)}$$

Lagrangian or material derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} \bigg|_{\xi=\text{const}}. \quad \text{(2.17)}$$

and are generally different from each other. The operator $D/Dt$ stands for the Lagrangian derivative, which is also called the material derivative. Material derivative is often denoted by a dot, for example

$$\dot{x} = \frac{Dx}{Dt} = v.$$
2.3. **STRAIN RATE**

If the operand is composed of plural symbols, we put the operand in parentheses and put a dot on the right parenthesis such that

\[ \frac{D F^T}{Dt} = \left( F^T \right). \]

How are the two time derivatives related? Let \( F \) be a scalar function of position and time. The material derivative of the function is

\[ \frac{D F}{Dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ F(x(\xi, t + \Delta t), t + \Delta t) - F(x(\xi, t), t) \right]. \]  

(2.18)

Because of \( x(\xi, t + \Delta t) = x(\xi, t) + v \Delta t + \cdots \), the first-order approximation of the first term in the square brackets above is

\[ F(x(\xi, t + \Delta t), t + \Delta t) \approx F(x(\xi, t) + v \Delta t, t + \Delta t). \]  

(2.19)

Note that the Taylor expansion of a function \( f(x, y) \) is

\[ f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y + \cdots \]

The first-order approximation of the right-hand side of Eq. (2.19) becomes

\[ F(x(\xi, t) + v \Delta t, t + \Delta t) = F(x(\xi, t), t) + \nabla F \cdot (v \Delta t) + \frac{\partial F}{\partial t} \Delta t. \]

Using these approximations we rewrite Eq. (2.18) and get

\[ \frac{D}{Dt} F(x, t) = \left( \frac{\partial}{\partial t} + \sum_i v_i \frac{\partial}{\partial x_i} \right) F(x, t) = \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) F(x, t) \]  

(2.20)

or simply

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + v \cdot \nabla. \]  

(2.21)

This formula is applicable to vectors and tensors. Let \( F = (F_1, F_2, F_3)^T \) be a vector field that depends on time, then it is rewritten by the sum of the products of the scalar functions and base vectors: \( F(x, t) = F_1(x, t) e^{(1)} + F_2(x, t) e^{(2)} + F_3(x, t) e^{(3)}. \) Because of the independency of the base vectors on time, we have \( DF_i e^{(i)}/Dt = e^{(i)} (DF_i/Dt). \) Therefore, the material derivative of a vector is obtained by taking the material derivative of each components. We obtain the material derivative of tensors in the same way. Accordingly, we simply write the material derivatives of the vectors and tensors such that

\[ \frac{D F}{Dt} = \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) F. \]

### 2.3 Strain rate

The velocity of deformation is a basic quantity for the theory of tectonics. Velocity is the change during a very short time \( dt \). The deformation in the infinitesimally short period is an infinitesimal deformation.
2.3.1 Velocity gradient

The time rate of deformation is obtained from the velocity gradient tensor

\[
L \equiv \frac{\partial \mathbf{v}}{\partial \mathbf{x}} = \nabla \mathbf{v} = \left( \frac{\partial v_i}{\partial x_j} \right).
\]

The difference in velocity between two positions that are separated by the distance \(d\mathbf{x}\) is

\[
d\mathbf{v} = \left( \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right) \cdot d\mathbf{x} = L \cdot d\mathbf{x}.
\]

Several important quantities are obtained from the velocity gradient tensor—first, we obtain the material derivative of the deformation gradient

\[
\frac{\mathrm{D}F_{ij}}{\mathrm{D}t} = \frac{\partial \mathbf{v}_i}{\partial x_j} = \frac{\partial \mathbf{v}_i}{\partial \xi_j} = \sum_k \frac{\partial v_i}{\partial x_k} \frac{\partial x_k}{\partial \xi_j}.
\]

\[
\therefore \dot{\mathbf{F}} = L \cdot \mathbf{F}.
\]

The velocity gradient tensor is divided into symmetric and antisymmetric parts

\[
D \equiv \frac{1}{2} (L + L^T) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),
\]

\[
W \equiv \frac{1}{2} (L - L^T) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)
\]

which are called the stretching tensor and spin tensor, respectively. Obviously, we have

\[
L = D + W.
\]

Note that the definition of \(D\) is similar to that of \(E\) (Eq. (2.4)). They have the relation

\[
D = \dot{E}.
\]

This is derived as follows:

\[
\dot{E}_{ij} = \frac{\mathrm{D}}{\mathrm{D}t} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2} \left( \frac{\partial}{\partial x_j} \frac{\mathrm{D}u_i}{\mathrm{D}t} + \frac{\partial}{\partial x_i} \frac{\mathrm{D}u_j}{\mathrm{D}t} \right)
\]

\[
= \frac{1}{2} \left[ \frac{\partial}{\partial x_j} \left( \frac{\mathrm{D}(x_i - \xi_i)}{\mathrm{D}t} \right) + \frac{\partial}{\partial x_i} \left( \frac{\mathrm{D}(x_i - \xi_i)}{\mathrm{D}t} \right) \right] = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = D_{ij}.
\]

The material derivative of infinitesimal strain tensor \(\dot{E}\) has identical components with \(D\), although they have different definitions. The former is referred to as the strain rate tensor. As the name suggests, \(\dot{E}\) indicates the velocity of strain. This is the infinitesimal strain during a short period \(dt\), so that the trace of \(\dot{E}\) represents the velocity of the volume change (2.11). Namely, the equation

\[
\dot{J} = (\text{trace } L) \cdot J = (\nabla \cdot \mathbf{v}) J
\]

(2.28)
is obtained (Exercise 2.2) as the time rate of the Jacobian (Eq. (1.7)), Therefore, if $\nabla \cdot \mathbf{v} = 0$, then we have $J = 0$. This is a differential equation indicating that $J$ has no temporal change. The equation has the solution $J = \text{const.}$ The constant should be 1, otherwise $J \neq 0$ when a volume change occurs. Consequently, a rock mass has no volume change if $\nabla \cdot \mathbf{v} = 0$. If the velocity field satisfies the equation

$$\nabla \cdot \mathbf{v} = 0,$$  \hspace{1cm} (2.29)

the deformation is called incompressible. In that case, density does not change, either. Incompressibility is often assumed in tectonic models, but is not applicable to phenomena with significant density changes associated, for example, with metamorphic reactions from low to high pressure phases.

How fast are actual tectonic movements? A million years is often used as the unit to measure geologic ages. The lifetime of an orogenic belt is on the order of ten to a hundred million years during which rock masses deform by hundreds of %. A million years is approximately equal to $3.16 \times 10^{13}$ s, because a year is approximately 60 seconds $\times$ 60 minutes $\times$ 24 hours $\times$ 365.25 days $= 31,557,600$ s. If a rock mass deforms by 100 % in a million years, the strain rate is $1/(3.16 \times 10^{13}$ s) $\approx 3.17 \times 10^{-14}$ s$^{-1}$.

Tectonics of island arcs is often faster than that of continents [263]. Extensional deformations in the Early Miocene was as fast as $\sim 10^{-13} - 10^{-14}$ s$^{-1}$ in western Northeast Japan (§12.2). The Northeast Japan arc has been subject to compressional tectonics since the Pliocene, and deformation was accelerated in the Pleistocene. The strain rate representative for Northeast Japan has been estimated by several methods. A seismological estimate from moment tensor accumulation (§2.7) indicates the rate to be about $1 \times 10^{-16}$ s$^{-1}$ [258]. This is the rate averaged over a few decades. That over the last $\sim$100 years was determined by triangulation at $(3 - 6) \times 10^{-15}$ s$^{-1}$ [160]. Northeast Japan has undergone compressional deformations, mainly in the Quaternary [202]. Folds and faults have shortened the crust under the arc. They were combined with chronostratigraphy to infer the strain rate averaged over the last 1.3 million years [201]. The result was on the order of $10^{-14}$ s$^{-1}$. The discrepancy between the estimated rates may shed light to non-steady movements.

Stable continents are not rigid, but are very slowly deforming. As an example, earthquakes rarely occur in Fenoscandia. The stress field responsible for such seismic activity is consistent with the combination of plate boundary forces with the plate flexure accompanied by Holocene glacial rebound [229]. The average strain rate of cratonic lithosphere is on the order of $10^{-15}$ s$^{-1}$ [37]. With a strain rate below the order of, say, $10^{-17}$, the region is too slow, even for geology, to detect and may seem rigid.

### 2.4 Pile up of infinitesimal deformations

Most tectonic structures are the result of finite deformations that are an accumulation of infinitesimal deformations. Therefore, it is important for tectonics how finite deformations are related to infinitesimal deformations.
2.4.1 Reference configuration

We have considered deformation by comparing the configurations that were represented by position vectors $\xi$ and $x$ at the time 0 and $t$. However, the rate of deformation changes with time. The rate is the infinitesimal deformation within an infinitesimal time interval $dt$, and the reference configuration for the infinitesimal deformation is renewed every moment. Accordingly, consider the times $t = 0, t$ and $\tau$, where $|t - \tau| = dt$. Assuming the configuration at time $t$ be the reference, we have

$$dx(\tau) = F_\tau(t) \cdot dx(t),$$

where $dx(\tau)$ is the infinitesimal vector at the time $\tau$ and $F_\tau(t)$ is the deformation gradient tensor at the time $\tau$ with the reference configuration at the time $t$. Thus,

$$dv(t) = \frac{\partial dx(\tau)}{\partial \tau} \bigg|_{\tau=t} = \frac{\partial F_\tau(t)}{\partial \tau} \bigg|_{\tau=t} \cdot dx(t) = \dot{F}_i(t) \cdot dx(t).$$

Note that

$$F_0(\tau)_{ij} = \frac{\partial x(\tau)_i}{\partial x(0)_j} = \sum_k \frac{\partial x(\tau)_i}{\partial x(t)_k} \frac{\partial x(t)_k}{\partial x(0)_j}.$$  

Namely, $F_0(\tau) = F_i(\tau) \cdot F_0(t)$, where $F_0(t)$ is the deformation gradient tensor at time $t$ with the reference one at the time 0. Velocity gradient at time $t$ is, therefore,

$$L(t) = \dot{F}_i(t) = \frac{\partial F_i(t)}{\partial \tau} \bigg|_{\tau=t} = \frac{\partial [F_0(\tau) \cdot F_0(t)^{-1}]}{\partial \tau} \bigg|_{\tau=t} = F_0(t) \cdot F_0(t)^{-1}. \quad (2.30)$$

This is identical to Eq. (2.23), but its reference is indicated. Using the polar decomposition theorem $F_i(\tau) = R_i(\tau) \cdot U_i(\tau)$, the above equation becomes

$$L(t) = \frac{\partial F_i(t)}{\partial \tau} \bigg|_{\tau=t} = [R_i(\tau) \cdot U_i(t) + R_i(\tau) \cdot U_i(\tau)] \bigg|_{\tau=t} = R_i(t) + U_i(t).$$

Similarly, we obtain $L(t) = \dot{R}_i(t) + \dot{U}_i(t)$. Comparing these and $L = D + W$,

$$D(t) = U_i(t) = V_i(t) \quad (2.31)$$
$$W(t) = R_i(t). \quad (2.32)$$

These equations relate stretching and spin tensors with the time rate of the tensors indicating finite strain and rotation within a moment at time $t$.

However, it should be noted that the relations (Eqs. (2.31) and (2.31)) do not hold if the configuration at time 0 is chosen for the reference. This is demonstrated as follows. Substituting $F_0(t) = R_0(t) \cdot U_0(t)$ into Eq. (2.30),

$$L(t) = [R_0(t) \cdot U_0(t) + R_0(t) \cdot U_0(t)] \cdot U_0(t)^{-1} \cdot R_0(t)^T$$
$$= \dot{R}_0(t) \cdot R_0(t)^T + R_0(t) \cdot \dot{U}_0(t) \cdot U_0(t) \cdot R_0(t)^T. \quad (2.33)$$
2.4. PILE UP OF INFINITESIMAL DEFORMATIONS

On the other hand, taking the material derivative of both sides of $R \cdot R^T = I$, we have

$$\dot{R}_0(t) \cdot R_0(t)^T + R_0(t) \cdot \dot{R}_0(t)^T = 0. \quad (2.34)$$

Combining Eqs. (2.33) and (2.34), we obtain the equations

$$D(t) = \frac{1}{2} R_0(t) \cdot \left[ \dot{U}_0(t) \cdot U_0(t)^{-1} + U_0(t)^{-1} \cdot \dot{U}_0(t) \right] R_0(t)^T, \quad (2.35)$$

$$W(t) = \dot{R}_0(t) \cdot R_0(t)^T + \frac{1}{2} R_0(t) \cdot \left[ \dot{U}_0(t) \cdot U_0(t)^{-1} - U_0(t)^{-1} \cdot \dot{U}_0(t) \right] \cdot R_0(t)^T \quad (2.36)$$

which indicates the relationship between instantaneous rates at time $t$ (left-hand side) and finite deformation and rotation (right-hand side) since the time 0. Note that $W$ is no longer equal to the material derivative of $R$ indicated by Eq. (2.32). In the case of $t \to 0$, both $R_0(t)$ and $U_0(t)$ approaches $I$, hence, Eq. (2.36) reduces to Eq. (2.32).

2.4.2 Pure shear rift

Continental rifts have wide variations: there are, for example, wide and narrow rifts and symmetric as well as asymmetric ones. If the lithospheric thinning is roughly approximated as a pure shear on the vertical section across the rift zone, it is called a pure shear rift (Fig. 2.2). In contrast, if extension occurs by displacement along a normal detachment fault that extends completely through the crust, it is called a simple shear rift. The North Sea was a rift zone that was active before ocean floor spreading began at the Mid Atlantic ridge.

Strain rate of pure shear In order for pure-shear rifts, we shall derive the expression for the strain rate of coaxial deformation that has a deformation gradient in the form of Eq. (1.14). The velocity gradient tensor is derived using Eq. (2.23):

$$L = F \cdot F^{-1} = \begin{pmatrix} \dot{X} & 0 & 0 \\ 0 & \dot{Y} & 0 \\ 0 & 0 & \dot{Z} \end{pmatrix} \begin{pmatrix} 1/X & 0 & 0 \\ 0 & 1/Y & 0 \\ 0 & 0 & 1/Z \end{pmatrix} = \begin{pmatrix} \dot{X}/X & 0 & 0 \\ 0 & \dot{Y}/Y & 0 \\ 0 & 0 & \dot{Z}/Z \end{pmatrix}. \quad (2.23)$$

Figure 2.2: Schematic cross-section of a pure shear rift, the central part of which is undergone pure shear.
All the non-diagonal components are zero so that the spin tensor $W$ vanishes for coaxial deformat-
ations\(^3\). The stretching tensor, on the other hand, is identical to the velocity gradient tensor

$$D = L = \begin{pmatrix}
\dot{X}/X & 0 & 0 \\
0 & \dot{Y}/Y & 0 \\
0 & 0 & \dot{Z}/Z
\end{pmatrix}. \quad (2.37)$$

Coaxial, incompressible, plane strain is a pure shear. Taking the $Y$-axis perpendicular to the plane,
the plane strain condition results in $\dot{Y}/Y = 0$. Volume conservation is represented by trace $L = 0$.
Therefore, we have $\dot{X}/X = -\dot{Z}/Z$ for pure shear. In this case, the strain rate is

$$\dot{E} = D = \begin{pmatrix}
\dot{X}/X & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\dot{X}/X
\end{pmatrix}. \quad (2.38)$$

The horizontal stretch of the lithosphere compared to the pre-rift stage is usually referred to as
the $\beta$ factor in theoretical models of continental rifting. Taking the $x$ and $z$ axes in horizontal and
vertical directions (Fig. 2.2), Eq. (2.38) is rewritten to

$$\dot{E} = D = \begin{pmatrix}
\dot{\beta}/\beta & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -\dot{\beta}/\beta
\end{pmatrix}. \quad (2.39)$$

Based on this equation, we are able to calculate the finite deformation of a pure shear rift from the
temporal variation of the rate of strain. It should be noted that if the rate is constant, the $\beta$ factor
increases non-linearly (Exercise 2.3). Quantitative stratigraphy (§3.10) allows us to infer the history
of lithospheric thinning, from which the temporal variation of the rate of strain can be determined
(§3.15).

### 2.5 Rotation

Tectonic rotations are observable from geological objects including geologic structures and paleo-
magnetic deflections. A train of twisted en echelon veins (Fig. 1.13) is an example of the former.
Paleomagnetism is often used to study large-scale tectonic rotations. Accordingly, let us study how
rotation is related to a velocity field.

#### 2.5.1 Angular velocity

Consider an angular velocity vector $w = we$, where $w$ is the angular velocity and $e$ is the unit vector
parallel to the rotation axis pointing as a right-handed screw. If the initial point of a position vector
$a$ is on the axis of rotation, the radius of rotation of the end point is $|a| \sin \theta$, where $\theta$ is the angle
between $e$ and $a$ (Fig. 2.3(a)) The angle satisfies the equation, $e \times a = |e| |a| \sin \theta = |a| \sin \theta$,

\(^3\)A spin tensor characterizes rotational motion so that the velocity field of coaxial deformation is irrotational.
2.5. ROTATION

Figure 2.3: (a) Rotation corresponding to the angular velocity \( \mathbf{w} = \mathbf{w} \mathbf{e} \), where \( \mathbf{e} \) is the unit vector parallel to the rotation axis. (b) Angular velocity is an infinitesimal rotation in an infinitesimal time interval. Infinitesimal rotation can be resolved to the successive rotations around the coordinate axes. The components of \( \mathbf{w} \) indicate the rotation angles.

which equals the radius of rotation of the end point. In addition, the vector \( \mathbf{e} \times \mathbf{a} \) points the direction that the end point is orbiting. Since the angular velocity is \( \mathbf{w} \), the orbiting velocity of the point is \( \mathbf{w} \mathbf{e} \times \mathbf{a} = \mathbf{w} \times \mathbf{a} \). Similar to Eqs. (2.13) and (2.14), the spin tensor \( \mathbf{W} \) and its axial vector \( \mathbf{w} \) have the following identities:

\[
W_{ij} = -\epsilon_{ijk} w_k, \tag{2.40}
\]

\[
\mathbf{W} \cdot \mathbf{a} = \mathbf{w} \times \mathbf{a}. \tag{2.41}
\]

where \( \mathbf{a} \) is an arbitrary vector.

We have seen that for rigid-body rotation, an angular velocity vector represents both the rotation axis and angular velocity. The components of \( \mathbf{w} \) indicates the rotation around the \( i \)th coordinate axis (Fig. 2.3(b)). It is discussed in the following subsection how a velocity field is related to the angular velocity.

2.5.2 Physical interpretation of spin tensor

Vorticity  Given a velocity field, the vector

\[
\mathbf{\omega} = \nabla \times \mathbf{v} = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right)^T \tag{2.42}
\]
CHAPTER 2. INFINITESIMAL STRAINS AND THEIR ACCUMULATIONS

Figure 2.4: Physical interpretation of the additive decomposition \( L = D + W \). An infinitesimal deformation in an infinitesimal time interval can be resolved to \( D \) and \( W \), indicating the coaxial strain and rotation, respectively. The sides of this rectangular solid is parallel to the principal strain axes. \( D_i \) stands for the \( i \)th principal strain. \( W \) indicates the infinitesimal rotation, the angular velocity of which is \( w \).

is known as the vorticity. From Eqs. (2.25) and (2.42) the components of \( W \) are obviously

\[
W = \begin{pmatrix}
0 & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \\
\frac{1}{2} \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) & 0 & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \\
\frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) & 0
\end{pmatrix} = \begin{pmatrix}
0 & -\frac{1}{2} \omega_3 & \frac{1}{2} \omega_2 \\
\frac{1}{2} \omega_3 & 0 & -\frac{1}{2} \omega_1 \\
-\frac{1}{2} \omega_2 & \frac{1}{2} \omega_1 & 0
\end{pmatrix} \quad (2.43)
\]

Thus, the vorticity is twice the vector axial vector of \( W \). Let \( w \) be the axial vector, then

\[
\omega = 2w. \quad (2.44)
\]

According to the discussion on the rotation tensor and rotation vector in §2.1.3, we see that \( w \) represents angular velocity. The vector points the direction of rotation axis like a right-hand screw, and \(|w|\) indicates the scalar angular velocity. Since a deformation within an infinitesimal time interval is also infinitesimal, strain and rotation are additive and commutative (Fig. 2.4).

**Rigid-body rotation** Let us demonstrate the correspondence between the vorticity and angular velocity for the case of rigid-body rotation. Given a constant \( a \), the velocity field \( v = (-ax_2, ax_1, 0)^T \) represents a rigid-body rotation about the third coordinate axis. This is proved by the observation firstly that \( v \cdot x = 0 \) and, secondly, that \(|v| = \sqrt{a^2x_2^2 + a^2x_1^2} \propto |x| \). The coefficient \( a \) is the angular velocity. Using the identity \( \partial x_i / \partial x_j = \delta_{ij} \), we have the vorticity of this velocity field \( \omega = (0, 0, 2a)^T \). The magnitude is twice the angular velocity, so that the proof has been established.
2.5.3 Spin within shear zone

Physical interpretation of the additive decomposition of velocity gradient tensor $L = D + W$ may be obtained in an analysis of simple shear deformations. Rigid-body rotation within a shear zone is considered assuming a simple shear deformation, where the velocity field is $\mathbf{v} = (2\dot{q}x_2, 0, 0)^T$ and $\dot{q} = \text{const}$. The velocity gradient is

$$ L = \left( \frac{\partial v_i}{\partial x_j} \right) = \begin{pmatrix} 0 & 2\dot{q} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, $$

which is decomposed into symmetric and antisymmetric parts:

$$ D = \dot{q} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad W = \dot{q} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.45) $$

$D$ and $W$ represent shape-change and rotation, both of which do not vanish but occur simultaneously for simple shear (Eq. (2.45)). The characteristic equation of $D$ is

$$ \begin{vmatrix} -\lambda & \dot{q} & 0 \\ \dot{q} & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + \dot{q}^2 \lambda = -\dot{q} \lambda (\dot{q} - \dot{q}) (\dot{q} + \dot{q}) = 0. $$

The eigenvalues are, therefore, $-\dot{q}$, 0 and $\dot{q}$. $\dot{E}$ has the same eigenvalues, because $D = \dot{E}$.

Suppose a square that is deformed to a parallelogram by the instantaneous simple shear, the time interval of which is $\Delta t$. Instantaneous strain is infinitesimal. The order of strain and rotation are commutative for infinitesimal deformations, so that we consider the deformation as a two-step operation, strain followed by rotation (Fig. 2.5). The strain in this velocity field has the principal axes inclined at $45^\circ$ to the flow direction (see §1.7). The rate of strain $\dot{E}$ has the eigenvalues $\dot{q}$, 0 and $-\dot{q}$, so that the diagonals of the square have undergone strain $\pm 2\dot{q} \Delta t$ within the interval $\Delta t$. If the square ABCD in Fig. 2.5 has sides with a length $2a$, the diagonal OA and OB has the length $\sqrt{2}a$. The square, first, becomes the parallelogram $A'B'C'D'$. OA and OB are elongated to OA' and OB', respectively. Both AA' and BB' have an equal length $\sqrt{2}a$. The inclination of the parallelogram is represented by the angle between AB and A'B'. The angle is very small for $\Delta t \rightarrow 0$, so that the tangent of the angle is approximated by the angle itself. It is seen in Fig. 2.5(a) that the tangent equals $2a \dot{q} \Delta t / 2a = \dot{q} \Delta t$. The inclined parallelogram $A'B'C'D'$ is rotated by the angle. The result is a parallelogram $A''B''C''D''$ whose two sides are parallel to the flow direction. The angle of rigid-body rotation is $\dot{q} \Delta t$. It follows that the angular velocity is $\dot{q}$.

The same quantity is obtained from a spin tensor in Eq. (2.45). We obtain the vorticity $\omega = (0, 0, -2\dot{q})^T$, where the minus sign indicates that the direction of rotation in this case is the counterclockwise relative to the third coordinate axis. The axial vector of $W$ is $w = (0, 0, -\dot{q})^T$, which is identical with the angular velocity of the parallelogram in Fig. 2.5. Therefore, the axial
Figure 2.5: Instantaneous simple shear of a square (gray) as the successive shape-change and rigid-body rotation. (a) The instantaneous strain by the effect of $\mathbf{D}$ transforms the square ABCD into the parallelogram shown by solid line A'B'C'D'. The principal axes of this strain are inclined at 45° to the flow direction. The diagonals OA and OB of the square becomes OA' and OB', respectively. (b) Rigid-body rotation of the parallelogram A'B'C'D' to A''B''C''D'”, completing the simple shear.

vector and spin tensor indicate instantaneous rotation in the simple shear, though both rotation and shape change occur.

It should be noted that the three quantities $W$, $w$ and $\omega$ indicate instantaneous spin within a deforming material, whereas the orthogonal tensor $\mathbf{R}$ indicates the finite rotation that is the integral of the spin. For the case of simple shear with the constant $\dot{q}$, the spin tensor is a constant tensor (Eq. (2.45)). Equation (2.32) seems to indicate that $\mathbf{R}$ is proportional to time for a constant spin. However, this is not always true. In this case, Eq. (2.36) rather than Eq. (2.32). $\mathbf{R}$ converges to 90° rotation for the simple shear, even though the spin is constant.

2.6 Paleomagnetic rotation

2.6.1 Paleomagnetic rotation determined by vorticity

We have seen in the previous section that rotation in deforming media is correlated with vorticity. So, the rotation of a tectonic plate is often thought to indicate the magnitude of vorticity in the viscous layer beneath the plate. In that case, if small, rigid, tectonic blocks are floating on a viscous fluid that is subject to shearing flow, the blocks can be used as indicators of the vorticity field in the fluid. This further allows us to estimate the total displacement across the shear zone from the paleomagnetic rotation of the blocks in the zone. Let $h$ be the width of the shear zone in the viscous layer, and $v_0$ be
2.6. PALEOMAGNETIC ROTATION

Figure 2.6: Rotating blocks in a deforming zone between microplates A and B. Plate B moves at a velocity $v_0$ with respect to microplate A, so that the width of the zone $w(t)$ increases. Block rotation within the zone is controlled by the tangential component $v_x$.

The relative velocity across the zone. The displacement during a time interval $t$ is $d = v_0 t$. If simple shear is proceeding in the zone, the velocity field is represented by $v = (v_0 y / h, 0, 0)$, provided that the first coordinate axis is parallel to the movement and $y$ is the length across the zone. Vorticity is, therefore, $|\omega| = |\nabla \times v| = v_0 / h$, and the angular velocity of the blocks is $v_0 / 2h$. If $h$ and $v_0 t / 2h$ is known from geological survey and paleomagnetism, respectively, we are able to estimate the total displacement of the shear zone.

Consider a deformation zone between microplate A and B that is moving with a relative velocity $v_0$ (Fig. 2.6). The microplates are thought to be rigid. If the deformation is homogeneous and plane strain on the horizontal plane, then the velocity field in the zone is

$$v = \left( \frac{y}{w} v_0 \cos \theta, v_0 \sin \theta, 0 \right)^T,$$

where the angle $\theta$, width of zone $w$ and coordinates $O-xyz$ in Fig. 2.6 are used. The vorticity is

$$\omega = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)^T = \left( 0, 0, -\frac{v_0 \cos \theta}{w} \right)^T.$$

This indicates that $\omega$ does not depend on the velocity component normal to the zone $v_y$. Therefore, paleomagnetic rotation in the deforming zone only indicates the tangential movement. This is convenient to estimate the tangential displacement.

We again assume a simple shear. If the viscous layer is made of a power-low fluid (see §10.8), the width of the shear zone is inversely correlated to the power-law index of the fluid. Accordingly, Sonder et al. [222] argue the rheology of the viscous layer under rotating tectonic blocks on the basis of paleomagnetic declinations around a strike-slip fault. The width may be indicated by the zone where the declination (therefore vorticity) shows an anomaly (Fig. 2.7).
2.6.2 Paleomagnetic rotation determined by vorticity and deformation

We have seen in the previous subsection that vorticity in a viscous layer beneath a rigid tectonic block controls the rotation of the blocks, where the vorticity does not depend on stretching tensor $D$, instantaneous deformation within the fluid. The angular velocity of the blocks is related to the vorticity by Eq. (2.44).

Secular tectonic rotation about a vertical axis is revealed by paleomagnetism and other techniques. Accordingly, let us consider paleomagnetic rotations. The rotation is sometimes argued in relation to Eq. (2.44) or to bookshelf faulting [139]. The activity of a fault system is said to be bookshelf faulting if the activity is accompanied by the rotation of fault slices just like the rotation of books when they fall in a bookshelf. If the angular velocity of fault slices, $w$, is given, the relative velocity of the blocks A and B in Fig. 2.8 is written as $v = (w \sin \theta, w \cos \theta)^T$.

Clockwise and counterclockwise rotation of the carapace of tectonic blocks are related to dextral and sinistral shears, respectively, when they are interpreted either with the vorticity of underlying viscous fluid or bookshelf faulting. The following argument shows that the correspondence between...
2.6. PALEOMAGNETIC ROTATION

Figure 2.9: (a) Fault systems (solid lines) produced by sand-box experiments after Cobbold et al. [41]. Broken gray lines are strain markers that initially made a grid pattern at the surface of sand 20 cm thick. The box was square, 20 by 20 cm, before strain. (b) Stretch and engineering shear strain along the x-axis for the four cases in (a).

the senses of rotation and shear has exceptions [111].

The first example came from sand-box experiments [41]. Since the shallow levels of the crust have fractures, and the fractures are thought to affect the mode of deformation of the crust. Therefore, sand is often used for an analog medium for the crust.

Figure 2.9(a) shows fault systems produced in a sand box. The box was subject to various strains, the parameters of which are shown in Fig. 2.9(b). The box was only extended along the x-axis by about 50% for the case “A”, where a conjugate fault system was created. Marker lines showed that the deformation was accompanied by little systematic rotation of the fault blocks. In experiment “B”, an engineering shear strain by the small amount of 0.2 was added to the case of “A”. The result was that both sinistral and dextral faults appeared with prominent bookshelf faulting in the central area. The overall shear sense of the sand box is sinistral for case “B.” However, there are domains of bookshelf faulting both with clockwise and counterclockwise vertical-axis rotations. The domains encompass fault slices with sinistral and dextral movements, respectively, between the slices. A larger sinistral shear was applied to the box in experiment “C”, resulting in the dominant dextral faults associated with the clockwise rotation of fault blocks. A simple shear was applied to the box in case “D”, where the schema of bookshelf faulting depicted in Fig. 2.8 applies.

The representative rate of simple shear and stretch for the sand-box experiments are indicated by

\[ L^{(1)} = \begin{pmatrix} 0 & 2u \\ 0 & 0 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}. \]
respectively. And, the representative velocity gradient is

\[
L = L^{(1)} + L^{(2)} = \begin{pmatrix} v & 2u \\ 0 & 0 \end{pmatrix}.
\]

Therefore, we have

\[
D = \begin{pmatrix} v & u \\ u & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}.
\]

Hence, the difference between the results of cases “C” and “D” in Fig. 2.9 indicates that \(D\) can affect the rotation of fault blocks, even if \(W\) is constant.

It is beyond our present scope to present a physical explanation for the variety of rotations. Instead, let us study the rotation of a single rigid body surrounded by moving viscous fluid, as it is an intricate problem to deal with plural bodies in that situation [275]. For additional simplicity, we assume Newtonian fluid (§9.1). In general, the rotation of such a rigid particle is affected by the rate of translational movement, strain rate, and vorticity of the surrounding fluid. For example, a windmill rotates by the translational motion of the atmosphere. Inversely, a propeller transforms rotation to wind. However, it is known that translation and rotation are independent for axisymmetric bodies. Propellers do not have the symmetry, so they are capable of transformation. Further, the rotation of bodies with spherical symmetry, or a disk for the two-dimensional case, is independent from \(D\) of the surrounding fluid, but depends only on \(W\) of the fluid.

To see this, suppose an ellipsoidal rigid body in a simple shearing viscous fluid (Fig. 2.10). The body represents shapes without spherical symmetry. Simple shear within a very short time is small, so that the instantaneous strain axes for the simple shearing fluid intersect at 45° with the flowing orientation. In case of a spherical body, the pressing and stretching effects along the axes have the opposite sense of torque on the body with equal magnitudes (Fig. 2.10(b)). The result is that the instantaneous strain of the surrounding fluid, \(D\), does not affect the rotation of the sphere, but the component, \(W\), controls the rotation. By contrast, torques exerted by the effects have different magnitudes on the surface of the ellipsoidal body (Fig. 2.10(c)), so that both \(D\) and \(W\) affect the rotation. Here, we neglect the interaction and deformation of blocks. Such a simple model does not have reality, but gives a hint to consider the implication of the rotation of tectonic blocks.

For further simplicity, let us consider a two-dimensional problem. Suppose an orogenic belt with parallel borders, and the rigid terrains on both sides of this zone are going away from each other with a constant relative velocity. The orientation of their motion is indicated by the angle \(\theta\) (Fig. 2.11). Their motion gives rise to the dextral shear and stretching\(^5\) in the belt, which is affected by the dextral and sinistral simple shear zones for \(\theta = 0\) and 180°, respectively. A rift zone without strike-slip movement is represented by \(\theta = 90°\). Transtensional tectonics is represented by \(0 < \theta < 90°\) and \(90 < \theta < 180°\). Transpressional tectonics is indicated by the angles in the other two quadrants, \(-180 < \theta < -90°\) and \(-90 < \theta < 0°\).

\(^5\)The term dextral “transtension” applies to such a case.
2.6. PALEOMAGNETIC ROTATION

Figure 2.10: Rotation of spherical and ellipsoidal bodies in a simple shearing viscous fluid. (a) Flow pattern far from the bodies. (b) Instantaneous flow in the fluid (gray bold lines) observed from the spherical body is a pure shear with the strain axes intersecting at 45° with the far-field simple shearing motion of the fluid. The instantaneous pure shear causes torque (thin arrows) at the surface of the body. (c) Torque by the same flow at the surface of an ellipsoidal body.

Suppose that there is a small, elliptical, rigid, tectonic block in the zone. The long and short principal radii of the body are $a$ and $b$, respectively, and the aspect ratio is $k = b/a \leq 1$. Let $\phi$ be the orientation of the long axis of the ellipse. The tectonic block is rotated by the deformation of the surrounding rocks.

It is known that a circular block ($k = 1$) rotates with the angular velocity that is equal to half the vorticity ($\dot{\phi} = \omega/2$) of the orogenic belt (§2.6). In contrast, the angular velocity $\dot{\phi}$ of an elliptical block oscillates [67, 111] with increasing $\phi$ as

$$\dot{\phi} = -w \left[ \left( \frac{1 - k^2}{1 + k^2} \right) \cos \theta \cos 2\phi + \tan \theta \sin 2\phi \right] - 1,$$

(2.46)

where $w$ is a reference angular velocity. Substituting $k = 1$ into Eq. (2.46), it is obvious that $w$ equals the angular velocity of a spherical block, determined solely by the vorticity. The content of the brackets is rewritten as

$$\left( \frac{1 - k^2}{1 + k^2} \right) \frac{\cos \theta \cos 2\phi + \sin \theta \sin 2\phi}{\cos \theta} - 1 = \left( \frac{1 - k^2}{1 + k^2} \right) \frac{\cos(\theta - 2\phi)}{\cos \theta} - 1.$$

(2.47)

Therefore, the amplitude of the oscillation depends not only on the vorticity but also on $k$, shape of the block, and on $\theta$.

Equation (2.46) shows orbits in the phase space, $O-\phi\dot{\phi}$, representing the temporal change of the orientation of the block. The oscillation occurs for blocks with $k < 1$. If $k$ is lower than the threshold, the block rotate oppositely, $\dot{\phi} < 0$. For example, a fat block with a larger $k$ rotates counterclockwise in a sinistral transtensional zone with $\theta = 140^\circ$ (Fig. 2.11). However, a thin block with $k$ smaller than the threshold rotates clockwise for a range bounded by the points that is indicated by open circles in Fig. 2.11. The points represents stable and unstable singular points, and the orientation of the thin block approaches the value indicated by the left stable point. The result
Figure 2.11: Phase space $O-\phi \dot{\phi}$ for the rotation of an elliptical rigid block in a sinistral transtensional zone with $\theta = 140^\circ$.

is that thin blocks stop rotation and are oriented at a constant angle to the deforming zone, though the surrounding zone continues deformation.

This example shows that paleomagnetic rotations are not enough to determine the sense of shear for a deforming zone. We have to know the shape of the blocks, from which paleomagnetic data are obtained, and the strain rate normal to the zone. The shape may be understood by detailed geological mapping of the blocks and by paleomagnetic data with spatially high resolution. The strain rate may be determined by an analysis of the geological structures.

Based on these observations, let us look at the paleomagnetic rotation in Northeast Japan associated with the opening of the Japan Sea backarc basin (Fig. 2.12). While the Northeast Japan arc drifted southward from the eastern margin of Eurasia in the Early Miocene time, extensional tectonics prevailed in the arc [263]. Figure 2.12(a) shows the paleomagnetic directions in NE Japan. It is important that the rotation in the forearc ceased 1–2 million years earlier than that in the backarc [84]. Figure 2.12(b) shows the movement of the arc inferred from the synthesis of onshore and offshore geological and geophysical data around Japan. Northeast Japan was separated from Shikote Alin, and drifted southward. Therefore, the overall sense of shear between Shikote Alin and the Northeast Japan forearc was dextral. Therefore, the model of bookshelf faulting suggest that clockwise rotations were dominant in the backarc. However, the observed paleomagnetic rotations have mostly the opposite sense (Fig. 2.12(a)). The paleomagnetic directions are also opposite to those predicted from the vorticity under the backarc associated with the dextral movement of the arc. This discrepancy is probably due to the association of strike-slip movement and extensional deformations (Fig. 2.12(c)). When the arc migrated, it was broken into blocks by transtensional tectonics [263].
2.7 Deformation of rock mass by the activity of minor faults

A rock mass encompassing many small-scale faults deforms by the activity of the faults (Fig. 1.17). Displacements by the movement of a fault are discontinuous at the fault surface. However, continuum mechanics applies to the description of the gross deformation of the mass, as the fault displacements are much smaller than the dimension of the mass and the number of faults is large. The deformation caused by a single fault is very small, so that infinitesimal deformation is considered for the mass by integrating the activity of the faults. Specifically, we are able to estimate $\mathbf{E}$ and $\mathbf{O}$ for the mass from the data of the minor faults.\footnote{See [155] for further reading.}

To see this, we assumed a rock mass with volume $V$ has minor faults whose displacements are very small compared to the size of the mass, $\ell$. Let the vector $\mathbf{n}$ be the unit normal to a fault surface, and $\mathbf{b}$ be a unit vector parallel to the fault movement (Fig. 2.13). The initial and terminal of the position vector $\mathbf{x}$ is placed on the opposite sides of the fault. The displacement of the terminal due to the fault movement is indicated by the vector $\mathbf{u}$. The portion of the rock mass depicted by dashed lines in Fig. 2.13 is broken into blocks by the fault. However, since we assume a very small displacement of the fault, $d$, the deformation of the portion can be approximated by a simple shear with the engineering shear strain $\gamma$. Therefore, we obtain $d = |\mathbf{u}| = \gamma (\mathbf{x} \cdot \mathbf{n})$, and the displacement gradient is

$$\nabla \mathbf{u} = \nabla [\gamma (\mathbf{x} \cdot \mathbf{n}) \mathbf{b}]. \quad (2.48)$$

Each fault has its own parameters $\mathbf{b}$, $\mathbf{n}$ and $\gamma$. If these parameters are spatially constant, they can be moved out of the operand of the differentiation in Eq. (2.48). Therefore, we have

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \gamma b_i \sum_k x_k n_k \right) = \gamma b_i \sum_k n_k \frac{\partial x_k}{\partial x_j}$$

Note that $x_1$, $x_2$ and $x_3$ are free variables and independent of each other. Hence, $\partial x_k / \partial x_j = \delta_{kj}$ and $\partial u_i / \partial x_j = \gamma b_i n_j$. The product $b_i n_j$ is a second-order tensor, so that we write

$$\mathbf{M}_a \equiv b_i n_j. \quad (2.49)$$

This is called an asymmetric moment tensor. This bears the information on the orientation of the fault plane and slip direction. Finally, we have the deformation gradient of the rock mass due to this fault is

$$\mathbf{F} = \mathbf{I} + \gamma \mathbf{M}_a. \quad (2.50)$$

Our next task is to estimate $\gamma$ from the parameters that we can obtain from outcrops. If the cross-sectional area of the rock mass is $S$, we have $V = S (\mathbf{x} \cdot \mathbf{n})$. Hence, we have $\gamma = d / (\mathbf{x} \cdot \mathbf{n}) = Sd / V$. Note that $S$ is the area of the fault surface. Let the geometric seismic moment of $S$ and $d$ of this fault be

$$M_g \equiv Sd. \quad (2.51)$$
Figure 2.12: (a) Paleomagnetic declinations [73, 79, 84, 85, 162, 165, 166, 242, 271, 273] and major faults that were probably activated when the Japan Sea opened in the Early Miocene [271]. Note that the forearc (Pacific side) of Northeast Japan ceased paleomagnetic rotation by 16–17 Ma, earlier than the termination of the rotation in the backarc (Japan Sea side) [84]. Major faults in Northeast Japan (FF, Futaba Fault; HI, Hachirogata–Ichinoseki Tectonic Line; HM, Honjo–Matsushima Tectonic Line, NT, Nishikawa–Takarazawa Tectonic Line; NM, Nihonkoku–Miomote Tectonic Line) after [100]. OF, Oisawa Fault; TSZ, Tanakura Shear Zone; KTL, Kanto Tectonic Line. (b) Paleo-positions of Northeast Japan and Hokkaido. (c) A simple model to account for the rotation and transtensional deformation in the drifting NE Japan arc.

Using this moment, Eq. (2.50) is rewritten as

$$ F = I + \frac{M_g M_a}{V}. $$

The second term in the right-hand side is composed of variables that can be determined from the fault parameters and the size of the rock mass.

The faults are numbered by their ages. That is, the number 1 is the oldest and the number $N$ is the youngest. The total deformation of the mass by the fault activity is indicated by the product, $F = F^{(N)} \cdot F^{(N-1)} \ldots F^{(1)}$, where $F^{(i)}$ stands for the deformation gradient tensor for the $i$th fault.
Figure 2.13: Deformation of a rock body by the movement of a fault. Gray, thick, dotted line shows the initial shape of the body. The resultant shape is shown by light gray rectangles. The latter shape is approximated by a parallelopiped indicated by dotted line. The unit vectors \( n \) and \( b \) is the direction of fault plane and slip, respectively. Displacement \( d \) is assumed to be very small compared to the dimension of the rock body.

Substituting Eq. (2.52) into this equation, we obtain

\[
F = F^{(N)} \cdots F^{(1)} = \left[ I + \frac{M^{(N)}_g M^{(N)}_a}{V} \right] \cdots \left[ I + \frac{M^{(1)}_g M^{(1)}_a}{V} \right] \approx I + \frac{1}{V} \sum_k M^{(k)}_g M^{(k)}_a. \tag{2.53}
\]

Based on Eq. (2.53), we have the strain and rotation of the rock mass

\[
E = \frac{1}{2} \left( \delta F + \delta F^\top \right) = \frac{1}{V} \sum_k M^{(k)}_g \left\{ M^{(k)}_a + \left[ M^{(k)}_a \right]^\top \right\}, \tag{2.54}
\]

\[
\Omega = \frac{1}{V} \sum_k M^{(k)}_g \left\{ M^{(k)}_a - \left[ M^{(k)}_a \right]^\top \right\}. \tag{2.55}
\]

Since the deformation for each fault is very small, second-order terms are neglected in the last approximation. The sum of tensors does not depend on their order. As a result, we do not need to know the order of activity of the faults to estimate the total deformation of the rock mass. This tolerance is important for structural geology, because it is difficult to determine the order\(^7\).

If the tectonic plates were literally rigid, tectonic activity would have occurred only at plate boundaries. However, there are extensive deformation zones in the world, especially continental collision zones are usually wide. Applying the above theory, the rate of deformation in a zone are estimated from the seismic moment of great earthquakes that are released within the zone during a certain period of time [89].

\(^7\)Cross-cutting relationship among faults tells the relative chronology of the faults. However, we can only date a part of fault assemblage.
CHAPTER 2. INFINITESIMAL STRAINS AND THEIR ACCUMULATIONS

The theory is applied to geological faults [133]. However, the accurate estimation of deformation by this method is probably difficult. The reason is that exposed faults are only a part of the faults in the rock mass. Large faults are often hidden by young sediments, though the faults contribute significantly to the deformation of the rock mass. In addition, the area of the fault surface, \( S \), is difficult to estimate. The fractal structure of the size and frequency of faults is sometimes employed to fill this gap, but it may not have enough precision.

2.8 Exercises

2.1 Derive the stretching and spin tensors of simple shearing with a stationary velocity field.

2.2 Derive Eq. (2.28).

2.3 The rate of strain of a pure shear rift is related to the \( \beta \) factor by Eq. (2.39). Show that \( \beta \) increases exponentially with time if the rate of strain is constant. Determine the width of the rift zone as the function of time for the two cases: a constant strain rate and exponentially decaying rate.

2.4 Show that vorticity vanishes for pure shear.

2.5 Finite rotation is observable as paleomagnetic deflection. Suppose a vector \( \mathbf{a} \) that is fixed in a sedimentary body and indicates due north when it deposited. A rotation with a constant angular velocity vector \( \mathbf{w} \) and fixed rotation axis has carried the vector \( \mathbf{a} \) to \( \mathbf{b} \) during a time interval \( t \). Let \( \tilde{\mathbf{R}} \) be the orthogonal tensor corresponding to the rotation (\( \mathbf{v} = \tilde{\mathbf{R}} \cdot \mathbf{a} \)). Derive a relationship between \( \tilde{\mathbf{R}} \) and \( \mathbf{W} \), the axial vector of which is \( \mathbf{w} \). Note that both \( \tilde{\mathbf{R}} \) of this problem and \( \mathbf{R} \) in Section 1.5 are orthogonal tensors, they represent different phenomena. The present one indicates spinning of a rock mass, but that in Section 1.5 is the rotation of passive strain markers.

2.6 Many fault scarps have been found on Mercury. They are called lobate scarps because of their meandering trace on the surface, suggesting that they are the surface expression of low-angle, thrust faults [232]. Some impact craters are horizontally shortened (Fig. 2.14), supporting the thrust hypothesis. If so, the fault activity must have decreased the surface area of the planet, though other types of faults may exit. The reduction of the area is attributed to ancient global cooling that resulted in a decrease of the radius and surface of Mercury. If the fault parameters used in the method of geometric seismic moment were known for the mercurial thrust faults, we would be able to estimate the deformation of the planet associated with the fault activity. Instead, a simple and very rough estimate is possible.
Let $L_i$, $h_i$ and $d_i$ be the surfacial length, vertical displacement and dip of the $i$th fault. Show that the decrease of radius $\Delta R$ can be roughly estimated by the equation

$$\frac{\Delta R}{R} = \frac{1}{8\pi R^2} \sum_{i=1}^{N} \frac{L_i h_i}{\tan d_i}, \quad (2.56)$$

where $R$ is the initial radius, and $N$ is the number of fault.

2.7 The strata shown in Fig. 1.18(a) have been deformed by minor normal faults. The strain ellipse in (e) was approximately drawn from visual estimation of (c) and (d). Quantify the deformation using the asymmetric moment tensor of the faults. Assume that all the fault planes are perpendicular to the surface of the outcrop and have the same area.