Chapter 1

Finite Strain

Tectonics is the natural processes that deform the shallow solid part of rocky or icy bodies of the Solar System. The deformation proceeds with the movements of rock or ice masses. In this chapter, we will study how to describe the geometric relationship between deformation and displacement—the kinematic aspect of tectonics.

1.1 Definition of strain

Deformation and strain are similar words. However, their distinction is sometimes important. Strain is the change of shape or size, or both; deformation includes both strain and rotation. We discuss the treatment of rotation later, and here we describe how strain is quantified.

Suppose a rock mass with a length $L_0$ was deformed and the length has become $L$, its strain $E$ is defined as

$$E = \frac{L - L_0}{L_0}. \quad (1.1)$$

The ratio $L/L_0$ is called stretch or elongation, and $(L/L_0)^2$ is quadratic elongation. The quantity, defined by the equation

$$E_n \equiv \ln(L/L_0).$$

is a natural or logarithmic strain. If we know length $L'$ every moment in the deformation process, the logarithmic strain of the final shape is

$$E_n = \int_{L_0}^{L} \frac{dL'}{L'} = \ln \frac{L}{L_0}. \quad (1.2)$$

The integral indicates that logarithmic strain is the sum of infinitesimal strains during a deformation process.

$$E_n = \ln \frac{L}{L_0} = \ln \left(1 + \frac{L - L_0}{L_0}\right) = \ln(1 + E).$$

Therefore, $E_n$ is identical with $E$ if each strain is very small.
Geodesy can detect a tiny strain of a rock mass, but strains that are observable as geologic structures are not so small. Most of them are acknowledged by the naked eye. The ammonite in Fig. 1.1 is an example. The original shape of an ammonite must be round, but this one was flattened by a few tens of percents.

It is obvious that we should know the original shape of deformed objects to quantify the strain, although, this is not always possible. We call such objects strain markers. Fossils are often used as markers\(^1\). Zenolith blocks shown in Fig. 1.2 appear to be elongated. However, we do not know their initial shapes. Therefore, they are not useful as strain markers\(^2\).

Fine grained sedimentary layers lie almost horizontally, so that they tell us how the strata were tilted—they are markers of rotation about a horizontal axis. Vertical axis rotations are revealed by paleomagnetism.

Changes in shape are quantified not only by a variation of lengths, but by that of angles between marker lines. Angle changes are an important aspect of the deformation resulting in shear strain\(^3\). If the angle at the corner \(\angle ABC\) that was originally at 90° decreases by \(\phi\) and becomes an acute angle, the angular change is represented by the engineering shear strain (Fig. 1.3)

\[
\gamma = \tan \phi.
\]

If an obtuse angle is the result, the engineering shear strain is negative. It is a matter of course that shear strain depends on the direction of the marker lines in the initial configuration. The dependence is given by Eq. (1.32).

\(^1\)Many kinds of strain markers and procedures for quantification are described in the textbooks [170, 189].

\(^2\)Each of the blocks is not a strain marker. However, if it is allowed to assume that they had random orientations before deformation, we can estimate strain from the present shapes and long-axis orientations (Exercise 1.4).

\(^3\)The words ‘shear strain’ is commonly used for the non-diagonal components of infinitesimal strain tensor (§2.1) which is just the half of engineering shear strain.
1.2. DISPLACEMENT AND DEFORMATION

Geological observations are not so accurate as geodetic ones, so that we should describe both small and large deformations. Although the accuracy is so, geologic data provide information on long-term processes.

Figure 1.2: Elongated zenolith blocks in granite. Takanuki, Northeast Japan.

Figure 1.3: Definition of engineering shear strain $\gamma \equiv \tan \phi$. 
Rock masses deform and their portions move. If entire parts move in the same orientation by the same distance, the rock mass moves as a rigid body. If they move in different directions, deformation occurs. Accordingly we can use the tracks of the portions to describe the deformation. There are two ways to carry out the task. Suppose that we have to describe population movements. One way to describe them is to distinguish individuals by their address at time $t = 0$. We assume in this case a one-to-one correspondence between people and loci for simplicity. Let the vector $x$ indicate the address at arbitrary time $t \neq 0$ for the person who lived at $\xi$ at time $t = 0$. The vector acts as the person’s name. The function $x(\xi, t)$ describes the movement of the person. Taking the vector $\xi$ as a variable, the function indicates the movements of all the people. The other way to describe the population movements is to use the inverse of this function, $\xi(x, t)$. If we input an address $x$ and time $t$, we get the legal domicile $\xi$.

Now suppose the vectors $\xi$ and $x$ represent the original and present position of a rock portion, either of the functions

$$ x = x(\xi, t), \quad (1.3) $$
$$ \xi = \xi(x, t). \quad (1.4) $$

Obviously, the displacement is

$$ u = x - \xi. $$

Equations (1.3) and (1.4) represent the orbit of the portion. The dotted lines in Fig. 1.4 show the orbit of two nearby portions. If we know the orbit of all portions of a rock mass, we know the deformation and translation of the mass. For simplicity, we assume that these functions are continuous at any place and at any time, and we can get derivatives of any order of the functions.

When we describe the variation of a quantity $F$, there are two ways to describe the variation of quantity $F(\xi, t)$ and $F(x, t)$. If the quantity is money, the function $F(\xi, t)$ shows the variation of the fortune of the person who lived at $\xi$ at $t = 0$. The function shows the economic ups and downs of the person. On the other hand, the other function $F(x, t)$ describes how great fortune the person living at $x$ has at time $t$. That is the fixed point observation of the variation. If $T$ is temperature, for example, $T(\xi, t)$ indicates the temperature variation for the rock portion that was at $\xi$. $T(x, t)$ is the temporal variations of temperature at spatially fixed point at $x$. We call $(x_1, x_2, x_3)$, spatial or Euler coordinates and $(\xi_1, \xi_2, \xi_3)$, material or Lagrangian coordinates. $F(\xi, t)$ is the Lagrangian or material description of the variation of $F$. $F(x, t)$ is the spatial or Euler description of the variation. Equations (1.3) and (1.4) are those of movement.

### 1.3 Deformation Gradient

As we have assumed above that Eqs. (1.3) and (1.4) are continuous and smooth, the transformation between two coordinates $(\xi_1, \xi_2, \xi_3)$ and $(x_1, x_2, x_3)$ is in one-to-one correspondence and we obtain
1.3. DEFORMATION GRADIENT

Figure 1.4: Deformation of a rock mass can be calculated from the relative position of two nearby portions in the mass before and after the deformation. Their initial positions are represented by the vectors \( \xi \) and \( \xi + d\xi \), and the final positions by \( x \) and \( x + dx \). Their tracks during deformation is indicated by dotted lines.

The following equation from Eq. (1.3):

\[
dx_i = \frac{\partial x_i}{\partial \xi_1} d\xi_1 + \frac{\partial x_i}{\partial \xi_2} d\xi_2 + \frac{\partial x_i}{\partial \xi_3} d\xi_3,
\]

where \( i = 1, 2, \) or \( 3 \). The small vectors \( dx \) represents the distance between the rock portions that occupied at nearby points \( P_1 \) and \( P_2 \) at \( t = 0 \). The vector \( d\xi \) stands for the distance between the same portions at time \( t \) (Fig. 1.4). The matrix

\[
F = \frac{\partial x}{\partial \xi} = \begin{pmatrix}
\frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\
\frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\
\frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3}
\end{pmatrix}
\]

is useful to describe deformation. Then we obtain the following formula from Eq. (1.5):

\[
dx = F \cdot d\xi,
\]

showing that the transformation between \( dx \) and \( d\xi \) is linear, although the transformation between \( x \) and \( \xi \) is not necessarily linear. \( F \) is called a deformation gradient tensor, and is a measure of deformation. Such a linear transformation maps a sphere to a ellipsoid and a rectangular solid to a parallelepiped.

1.3.1 Strain ellipsoid

A state of strain is graphically represented in structural geology by a strain ellipsoid that is the result of the strain from an unit sphere (Fig. 1.5). The principal radii of the ellipsoid and their directions are called principal strains and principal axes, respectively. The principal strains are labelled as \( X, \)
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Figure 1.5: Strain ellipsoid and its principal axes.

Y, and Z, in descending order, and the corresponding principal axes are called X-, Y-, Z-axes. The axes cross each other at right angles. Pairs of axes define three principal planes of strain. A strain ellipsoid is symmetric with respect to those three planes, therefore, a state of strain has orthorhombic symmetry. Strain markers are often observed on the surface of rocks; the ammonite in Fig. 1.1 was found in a bedding plane. In two dimensional problems, we use strain ellipse to show the strain. The ellipsoid that becomes a unit sphere by the same deformation is called a reciprocal strain ellipsoid.

According to linear algebra, \(|F| \neq 0\) is the necessary and sufficient condition for the transformation being linear. This is also the condition for the existence of its inverse matrix. The determinant

\[ J = |F| \quad (1.7) \]

is called the Jacobian of the deformation. To show that this quantity represents the volume change during deformation, suppose three vectors \(d_\xi^{(1)} = (d_\xi^1, 0, 0)^T\), \(d_\xi^{(2)} = (0, d_\xi^2, 0)^T\), \(d_\xi^{(3)} = (0, 0, d_\xi^3)^T\) were perpendicular to each other when \(t = 0\). After deformation, the vectors have become \(dx_1\), \(dx_2\), \(dx_3\). The initial vectors span an initial volume equal to \(V_0 = d_\xi^1 d_\xi^2 d_\xi^3\). As the initial vectors are infinitesimally small, the final ones are also small. Hence, the volume of the parallelepiped that is spanned by the final vectors is equal to their triple scalar product

\[ V = dx^{(1)} \cdot (dx^{(2)} \times dx^{(3)}) = \| dx^{(1)} \cdot dx^{(2)} \cdot dx^{(3)} \|. \]

The triple scalar product is identical to the determinant of the matrix that is composed by the components of the vectors (see Appendix A), so that

\[ V = \| dx^{(1)} \cdot dx^{(2)} \cdot dx^{(3)} \| = \| F \cdot d_\xi^{(1)} \cdot F \cdot d_\xi^{(2)} \cdot F \cdot d_\xi^{(3)} \| \]

\[ = |F| \| d_\xi^{(1)} \cdot d_\xi^{(2)} \cdot d_\xi^{(3)} \| = JV_0. \quad (1.8) \]

Therefore, the Jacobian \(J = |F|\) is equal to the volume ratio of the initial and final configurations.

In the case of \(|F| = 0\), a rock mass vanishes. As negative volume is meaningless in tectonics, we can always assume the inequality

\[ J = |F| > 0. \quad (1.9) \]

\(^4\)This property will be used to consider the strain of a rock mass by many faults therein.
If $|\mathbf{F}| = 1$, then the volume of any portion of the mass is retained during deformation. This type of deformation is called *incompressible*.

The linearity represented by Eq. (1.6) is useful for structural geology. The Triassic ammonite shown in Fig. 1.1 originally should have been of round shape, but was flattened by Cretaceous orogeny that affected the eastern margin of Eurasia. The strata that contained the fossil were folded. Folding itself is a nonlinear transformation between $\mathbf{x}$ and $\xi$. However, the ammonite suffered a simple flattening: the round shape became oval, because the ammonite is very small and any vectors between portions within the fossil acted as the small vector $d\mathbf{x}$. In general, $\mathbf{F}$ is a function of
position and the transformation is non-linear: rocks may be distorted. However, \( F \) is assumed to be a continuous and smooth function of position. Therefore, the linear transformation

\[
x = F \cdot \xi
\]  

(1.10)

is a good approximation of the deformation field for a small region. The whole space is approximated by tessellation of the regions. If this equation is satisfied, the deformation is called \textit{homogeneous} for a specific region.

1.3.2 Special types of deformation

\textbf{Zero and infinitesimal strains and rigid body rotation}

If the deformation gradient tensor is equal to the identical tensor, then \( \text{d}x = F \cdot \text{d}\xi = I \cdot \text{d}\xi = \text{d}\xi \).

This means that the initial and final vectors between nearby portions in a rock mass do not change at all. This further indicates \textit{zero deformation}, even if the mass was moved. It should be noticed that zero tensor \( O \) does not stand for zero deformation but for disappearance. If \( F^T = F^{-1} \), the length of the vectors does not change but their directions vary, indicating a \textit{rigid body rotation} (Fig. 1.6).

Let \( \delta F \) be the difference between \( F \) and \( I \):

\[
\delta F = F - I.
\]  

(1.11)

In the case of \textit{infinitesimal deformation}, \( F \approx I \) and \( \delta F \approx O \) where \(|\delta F| \ll 1\). This is called a \textit{finite strain} if \( \delta F \) is far from the zero tensor.

\textbf{Plane strain}

If the deformation gradient tensor has the following shape by choosing an orientation of orthogonal coordinates, it is called \textit{plane strain}:

\[
F = \begin{pmatrix}
\bullet & \bullet & 0 \\
\bullet & \bullet & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where closed circles represent any value. In this case, the movement of every portion of a rock mass is in translation along the third coordinate axis. We can deal with the deformation by a two-dimensional problem on the \( O-12 \) coordinate plane.

There are many tectonic problems that can be treated as plane strain. For example, orogenic belts, island arcs, and mid oceanic ridges are linear or elongated tectonic features, so that plane strain on their cross-section is a good approximation for the movements in the zones.
1.4. GEOLOGICAL OBSERVATION OF DEFORMATION

Figure 1.7: Coaxial deformation and principal strain axes. If the fan OAB is deformed to OA’B’ with OA and OB being parallel to the principal axes, the line segments OA’ and OB’ are not rotated, but OC is rotated to OC’. Arrows indicate displacements.

Pure and simple shear

Pure shear and simple shear often appear in the literature on tectonics. They are special cases of plane plain strain, and are written as

\[
F = \begin{pmatrix}
 p & 0 \\
 0 & 1/p \\
\end{pmatrix}, \quad (1.12)
\]

pure shear

\[
F = \begin{pmatrix}
 1 & 2q \\
 0 & 1 \\
\end{pmatrix}, \quad (1.13)
\]

simple shear

where \( p \) and \( q \) are arbitrary numbers. The factor of 2 is attached in the upper-right component in the last matrix only for the convenience of later calculations. Simple shear is often assumed for fault zones, where positive \( q \)s represent dextral shear (Fig. 1.6(b)). Note that \( |F| = 1 \) for both pure and simple shear, indicating that the volume of any part of a rock mass is kept constant during deformation. Structural geologists uses the term coaxial deformation in the case of

\[
F = \begin{pmatrix}
 p & 0 & 0 \\
 0 & q & 0 \\
0 & 0 & r \\
\end{pmatrix}. \quad (1.14)
\]

Linear markers parallel to the coordinate axes do not rotate by this deformation. Hence, this is also called irrotational (Fig. 1.7).

1.4 Geological observation of deformation

Structural geologists have a grasp of the deformation of a mappable-scale rock mass by, for example, air- and space-borne remote-sensing techniques in arid areas, and seismic sounding. For areas where those remote-sensing techniques are not useful, they measure \( F \) at outcrops that are indicated by strain markers, and plot the data on a map to understand the large-scale deformation.
The movement of a rock mass is characterized by the movement of the center of the mass, rigid body rotation, and deformation. However, deformation gradient tensor, $F$, lacks the information on the first factor, translation. Long-distance translation of a rock mass is determined by, for example, the change in paleomagnetic declination and paleoclimate indicated by fossils. They indicate latitude changes. East-west movements are difficult to identify. Offset of old geologic structures across a fault zone indicates relative horizontal displacement, but does not indicate their absolute movements. To constrain them, we need some hypothesis, such as the fixity of hotspots.

On the other hand, horizontal-axis rotations are found by the tilting of sedimentary strata. Vertical-axis rotations are observed by paleomagnetism and twisting of old geologic structures that penetrate rotating and surrounding blocks. Disorder in the trend of dike swarms are used to infer vertical-axis rotations in the western United States [199].

**Primary structures**

Quantification of strain is done by comparing the initial and final shape of a rock mass. So, we need to know the shape before tectonic deformation, that is called a *primary structure*.

The offset of key beds shows the displacement of faults. If there are talus or alluvial fan deposits accompanied by a fault scarp, we are able to know when the faulting occurred by the age of the deposits.

Sediments lie almost horizontally when they were deposited. Stratification in sediments is often used to know the tilting. However, there are following exceptions, so that we are careful to deal with strata as strain markers. Talus breccias at the foot of scarps can lie at a significant angle from the horizontal. Volcanic rocks accumulates on the slope of volcanoes. In addition, folding sometimes turns strata upside down.

Lithologic stratification is important not only as a strain marker but also as an agent of tectonic movements. Mechanical property of rocks is dependent on lithology, therefore, once a stratified rock mass is loaded, lithologic boundary act as a rheological discontinuity. Bedding planes are often used as fault planes (§8.4.2). In addition, sediments themselves with a thickness of hundreds of meters indicate the presence of a basin or subsidence of the basement when they were deposited. Sedimentary sequence is an indicator of vertical tectonic movements (§3.10).

Sediments are useful indicators of ancient tectonics. This is true not only on the Earth but also on other planets and satellites. For example, lunar maria are sedimentary basins that are filled with horizontally lying, very low viscosity, flood basalts and breccias derived from nearby and distant impact craters, and are covered by a veneer of regolith (Fig. 1.8).

### 1.5 Tensorial representation of strain ellipsoid

Strain ellipses are represented by tensors. As long as we consider tectonics, we can assume that $|F| > 0$ (Eq. (1.9)). Accordingly, using the polar decomposition theorem, $F$ is decomposed into the
1.5. TENSORIAL REPRESENTATION OF STRAIN ELLIPSOID

Figure 1.8: Sedimentary layers (L) cropping out at the margin of the Euler crater in the Mare Imbrium on the Near Side of the Moon. The crater is about 30 km in diameter with a central peak (P). The marginal slope had collapsed to form terraces (T) the bases of which sunk in the impact melt with a level surface (IM). Apollo 15 frame P10274. Courtesy of NASA.

product of an orthogonal tensor \( R \) and symmetric tensors \( U \) and \( V \):

\[
F = R \cdot U = V \cdot R.
\]  

(1.15)

For given \( F \), the tensors are uniquely determined. Substituting this equation into \( x = F \cdot \xi \), we have \( x = R \cdot (U \cdot \xi) \). Therefore, the deformation indicated by \( F \) is equal to the two-stage process: pure shear represented by \( U \) followed by the rigid body rotation \( R \) (Fig. 1.9). \( U \) and \( V \) are called right and left stretch tensors. As the names suggest, the tensors are the extension stretch defined in p. 3 to three-dimensional strain. Their principal axes (eigenvectors) are not parallel except for coaxial deformation (\( R = I \)).

Tensors \( U \) and \( V \) are real symmetric tensors, so that they are represented by diagonal matrices like Eq. (1.14) by taking proper Cartesian coordinates. Therefore, the tensors stand for coaxial deformations.

Let us consider the relationship between the strain ellipse and these tensors. If \( |\xi| = 1 \), the terminal point of the vector \( \xi \) is on the surface of the unit sphere. The vector is transformed to \( x = F \cdot \xi = V \cdot (R \cdot \xi) \) of which the terminal point is on the strain ellipsoid. The transition from unit sphere to ellipsoid is due to \( V \). Therefore, \( V \) represents the strain ellipsoid of which the principal axes and principal strains are equivalent to the eigenvectors and eigenvalues of \( V \). Note that \( V \) also represents a coaxial deformation.

Let us derive the tensorial representation of strain ellipsoids. Remember that a unit sphere is transformed into a strain ellipsoid by a deformation represented by \( F = V \cdot R \). The right-hand side of this equation indicates what the sphere has undergone, firstly by rotation \( R \) and secondly by
Figure 1.9: Polar decomposition of the deformation gradient tensor $F$. The upper and lower deformation processes have the initial (circle) and final (ellipse) shapes in common. The cross in the ellipse indicates the principal strain axes. White line segments are strain markers.

coaxial deformation $V$. The rotation does not change the shape or size of the sphere, so that the final shape is controlled solely by the tensor $V$. In this case, the deformation is written as the equation $x = F \cdot \xi = V \cdot \xi$, where the vector $\xi$ is the radius of the unit sphere so that $|\xi| = 1$. The strain ellipsoid of this deformation is represented by the end point of $x$. As the deformation due to $V$ is coaxial, the line segments parallel to the principal directions of the ellipsoid do not rotate during the deformation. Therefore, let $\lambda$ be the elongation of the segments and $\xi$ be one of the directions, then the two quantities satisfy the equation $V \cdot \xi = \lambda \xi$. This indicates that we obtain the shape and attitude of the strain ellipsoid by solving the eigen-equation $V \cdot \xi = \lambda \xi$. Since we have assumed $|F| > 0$, the eigenvalues are all positive. The principal strains $X$, $Y$ and $Z$ are calculated as the maximum, intermediate, and minimum eigenvalues of $V$.

Based on these observations, we shall derive the equation of strain ellipsoid. The unit sphere before deformation is described by the equation $|\xi|^2 = 1$. Combining the equation $\xi = F^{-1} \cdot x$, we have $(F^{-1} \cdot x) \cdot (F^{-1} \cdot x) = 1$. This is the equation of strain ellipsoid (Fig. 1.10). The left-hand side is rewritten by the components as

$$\sum_{i,j,k} (F^{-1}_{ij} x_j) (F^{-1}_{ik} x_k) = \sum_{i,j,k} x_j (F^{-1}_{ji}) x_k = \sum_{i,j,k} x_j [(F^{-1}_{ij})^T F^{-1}_{ik}] x_k.$$

$$\therefore \ x \cdot (F^{-1})^T \cdot F^{-1} \cdot x = 1.$$  \hspace{1cm} (1.16)

We define the left Cauchy-Green tensor, $B$, and the right Cauchy-Green tensor, $C$, by the following equations:

$$B \equiv F \cdot F^T$$  \hspace{1cm} (1.17)

$$C \equiv F^T \cdot F.$$  \hspace{1cm} (1.18)
1.5. TENSORIAL REPRESENTATION OF STRAIN ELLIPSOID

Figure 1.10: Strain ellipse and tensors that represent the identical finite strain associated with the displacement field indicated by short arrows. The dilatation of this deformation is 25% (\(|F| = 1.25\)).

Since \(|F| > 0\) and \(|F^T| > 0\), \(C\) and \(B\) are symmetric tensors. Combining Eqs. (1.17) and (1.18), we get the relationship between these tensors:

\[
C = F^{-1} \cdot B \cdot F, \quad B = F \cdot C \cdot F^{-1}.
\]

From Eq. (1.18), we have

\[
C = F^T \cdot F = (R \cdot U)^T \cdot (R \cdot U) = U^T \cdot R^T \cdot R \cdot U = U^T \cdot U,
\]

where Eqs. (C.19) and then (C.15) are used. Likewise, we get \(B = V \cdot V^T\). Since \(U\) and \(V\) are symmetric,

\[
C = U^2, \quad B = V^2. \quad (1.19)
\]

These equations indicate that the left and right Cauchy-Green tensors are the expansion of quadratic elongation (§1.1). According to the discussion in Section C.5 and Eq. (1.19), the symmetric tensors \(C, U\), and \(U^{-1}\) have eigenvectors in common. Three symmetric tensors \(B, V\), and \(V^{-1}\) have common eigenvectors (Fig. 1.10).

Using Eqs. (C.21) and (C.22), we have

\[
B^{-1} = (F \cdot F^T)^{-1} = (F^T)^{-1} \cdot F^{-1} = (F^{-1})^T \cdot F^{-1}. \quad (1.20)
\]

Thus, the equation of strain ellipsoid (Eq. (1.16)) becomes

\[
x \cdot B^{-1} \cdot x = 1.
\]
$B^{-1}$ is a real symmetric tensor, so that its eigenvalues are all positive in sign (see §C.7). Accordingly, let $B_1$, $B_2$, $B_3$ be the eigenvalues of $B$ in a descending order of magnitude, then they are related to the principal strains by the equations:

$$X = \sqrt{B_1}, \quad Y = \sqrt{B_2}, \quad Z = \sqrt{B_3}.$$  

Since $V = B^{1/2}$, the principal strains are identical with the eigenvalues of $V$. On the other hand, $U^{-1}$ represents the reciprocal strain ellipsoid (Fig. 1.10) for the given deformation gradient $F$.

1.6 Green’s and Almansi’s finite strain tensors

**Definitions**  We have seen how $F$ is related to elongation in the previous section. Now, we consider the relationship between $F$ and strain. Suppose that there were two near-by points $P_1$ and $P_2$ whose positions are indicated by the vectors $\xi$ and $\xi + d\xi$, respectively. The materials that existed at the points $P_1$ and $P_2$ moved to the points $Q_1$ and $Q_2$ that are indicated by the vectors $x + dx$ (Fig. 1.4). Let $u$ be the displacement vector of the rock portion that moved from $P_1$ to $Q_1$. The distance between the points changed from $d\xi^2 = \sqrt{d\xi \cdot d\xi}$ to $ds^2 = \sqrt{dx \cdot dx}$. The square of the former is

$$ds^2 = dx \cdot dx = (F \cdot d\xi)^T \cdot (F \cdot d\xi) = d\xi \cdot (F^T \cdot F) \cdot d\xi = d\xi \cdot C \cdot d\xi.$$  

therefore we have

$$ds^2 - ds_0^2 = d\xi \cdot C \cdot d\xi - d\xi \cdot I \cdot d\xi = d\xi \cdot (C - I) \cdot d\xi.$$  

Accordingly, we introduce the tensor

$$G \equiv \frac{1}{2} \left( F^T \cdot F - I \right) = \frac{1}{2} \left( C - I \right).$$  

(1.22)

where $C$ is the right Cauchy-Green tensor and is symmetric ($C^T = C$), hence, $G$ is a symmetric tensor (§C.7): ($G = G^T$). The length change is then rewritten as

$$ds^2 - ds_0^2 = 2d\xi \cdot G \cdot d\xi.$$  

(1.23)

If the movement of a rock mass is a translation or rigid-body rotation, the distance between any points in the mass do not change. Only if the mass changes its shape, there are pairs of points with $ds^2 - ds_0^2 \neq 0$. Accordingly, $G$ is a measure of strain. Zero strain is indicated by $G = 0$. $G$ is called Green’s finite strain tensor.

Now, substituting $d\xi = F^{-1} \cdot dx$ into the right-hand side of the equation $d\xi = F^{-1} \cdot dx$, we have

$$ds^2 - ds_0^2 = dx \cdot dx - (F^{-1} \cdot dx) \cdot (F^{-1} \cdot dx) = dx \cdot \left[ I - (F^{-1})^T \cdot (F^{-1}) \right] \cdot dx$$
$$= dx \cdot \left[ I - (F \cdot F^T)^{-1} \right] \cdot dx = 2dx \cdot A \cdot dx.$$  

(1.24)
1.6. GREEN’S AND ALMANSI’S FINITE STRAIN TENSORS

where the tensor $A$ is called Almansi’s finite strain tensor and is defined by the equation

$$A = \frac{1}{2} \left[ I - (F \cdot F^T)^{-1} \right] = \frac{1}{2} \left( I - B^{-1} \right). \quad (1.25)$$

where $B$ is the left Cauchy-Green tensor. If $A = 0$, strain is nil. This tensor is symmetric, also ($A = A^T$), and is related with Green’s finite strain tensor by the equation

$$G = F^T \cdot A \cdot F. \quad (1.26)$$

The length change was written as the product of the material vector $\xi$ and the strain tensor $G$ (Eq. (1.22)). On the other hand, the same quantity is written as the product of the spatial vector $x$ and the strain tensor $A$ (Eq. (1.24)). It follows that $G$, $U$, and $C$ indicate the material (Laglangian) expressions of strain, whereas $A$, $V$, and $B$ represent the spatial (Euler) expressions of the strain.

**Strain ellipsoid** We have seen in Section 1.5 that the parameters of a strain ellipsoid are calculated as the eigenvalues and eigenvectors of $B^{-1}$. Now, let us consider the relationship between the parameters and the finite strain tensors, $G$ and $A$. From Eqs. (1.19) and (1.22), we have

$$2G + I = F^T \cdot F = C = U^2. \quad (1.27)$$

and, similarly,

$$2A = I - B^{-1} = I - (V^2)^{-1}. \quad (1.28)$$

Let $A_1$, $A_2$, $A_3$ be the eigenvalues of $A$ in descending order of magnitude, then

$$A_1 = \frac{1}{2} \left( 1 - \frac{1}{B_3} \right), \quad A_2 = \frac{1}{2} \left( 1 - \frac{1}{B_2} \right), \quad A_3 = \frac{1}{2} \left( 1 - \frac{1}{B_1} \right).$$

Thus, the principal strains $X$, $Y$, and $Z$ are related to the eigenvalues as

$$X = U_1 = V_1 = \sqrt{B_1} = \sqrt{2G_1 + 1} = \frac{1}{\sqrt{1 - 2A_3}}. \quad (1.29)$$

$$Y = U_2 = V_2 = \sqrt{B_2} = \sqrt{2G_2 + 1} = \frac{1}{\sqrt{1 - 2A_2}}. \quad (1.30)$$

$$Z = U_3 = V_3 = \sqrt{B_3} = \sqrt{2G_3 + 1} = \frac{1}{\sqrt{1 - 2A_1}}. \quad (1.31)$$

The principal strain axes ($X$, $Y$, and $Z$ in Fig. 1.5) are parallel to the eigenvectors of the strain tensors $V$, $B$, and $A$. Those of the tensors $U$, $C$, and $G$ are identical with those of the former three tensors if the deformation is coaxial.
CHAPTER 1. FINITE STRAIN

Shear strain  The Cauchy-Green tensors, \( B \) and \( C \), not only indicate length changes, but also shear strains in relation to the initial direction of marker lines. Suppose that the angle between two unit vectors \( u^{(1)} \equiv \xi^{(1)} / |\xi^{(1)}| \) and \( u^{(2)} \equiv \xi^{(2)} / |\xi^{(2)}| \) was \( \Theta \). After deformation, it becomes \( \theta \), and the vectors \( \xi^{(1)} \) and \( \xi^{(2)} \) become \( dx^{(1)} \) and \( dx^{(1)} \), respectively. Shear strain \( \phi \) is defined as \( \phi \equiv \Theta - \theta \).

Using Eq. (1.21), we obtain

\[
\cos \theta = \cos(\Theta - \phi) = \frac{dx^{(1)} \cdot dx^{(2)}}{|dx^{(1)}| |dx^{(2)}|} = \frac{d\xi^{(1)} \cdot C \cdot d\xi^{(2)}}{|d\xi^{(1)}| |d\xi^{(2)}|} = \frac{u^{(1)} \cdot C \cdot u^{(2)}}{|u^{(1)} \cdot C \cdot u^{(2)}|}.
\]

(1.32)

Given the initial angle \( \Theta = u^{(1)} \cdot u^{(2)} \), we can calculate the shear strain for a couple of marker lines in any direction. For the case of \( \Theta = 90 \), Eq. (1.32) is equal to \( \sin \phi \), and the engineering shear strain \( \gamma = \tan \phi \) is positive and negative if acute and obtuse angles are the results, respectively. Equation (1.32) is the Lagrangian description of the shear strain, whereas that of the Eulerian description is expressed with \( B^{-1} \) instead of \( C \). Determination of a strain ellipse from angle changes in fossils and other geological strain markers is explained in great detail in [189].

1.7  Shear zone

1.7.1  Strain tensor for simple shear

In order to study finite deformations in shear zones, consider a simple shear deformation (Fig. 1.11). In this case, the deformation gradient is

\[
F = \begin{pmatrix}
1 & 2q & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(1.33)

so that the right and left Cauchy-Green tensors are

\[
B = V^2 = F \cdot F^T = \begin{pmatrix}
1 + 4q^2 & 2q & 0 \\
2q & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad C = U^2 = F^T \cdot F = \begin{pmatrix}
1 & 2q & 0 \\
2q & 1 + 4q^2 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

In order to calculate the eigenvalues of \( B \), we take the upper-left \( 2 \times 2 \) submatrix, and we have its characteristic equation, \( \lambda^2 - 2(2q^2 + 1) + 1 = 0 \). The solutions of this equation are \( 1 + 2q^2 \pm 2q\sqrt{1 + q^2} \), therefore, we have the maximum, intermediate, and minimum eigenvalues are, respectively,

\[
B_1 = 1 + 2q^2 + 2q\sqrt{1 + q^2}, \quad B_2 = 1, \quad B_3 = 1 + 2q^2 - 2q\sqrt{1 + q^2}.
\]

The principal strains are equivalent to the eigenvalues of \( V = B^{1/2} \). Thus,

\[
X = \sqrt{1 + 2q^2 + 2q\sqrt{1 + q^2}}, \quad Y = 1, \quad Z = \sqrt{1 + 2q^2 - 2q\sqrt{1 + q^2}}.
\]

(1.34)
The principal strain axes are parallel to the eigenvectors of \( B \), hence, the \( X \), \( Y \), and \( Z \) axes lie in the directions
\[
\left( q + \sqrt{q^2 + 1}, 1, 0 \right)^T, \quad \left( 0, 0, 1 \right)^T, \quad \left( q - \sqrt{q^2 + 1}, 1, 0 \right)^T
\]
On the other hand, the reciprocal strain ellipsoid has the principal radii, \( 1/X \), \( 1/Y \), \( 1/Z \), and the corresponding principal axes are
\[
\left( -q - \sqrt{1 + q^2}, 1, 0 \right)^T, \quad \left( 0, 0, 1 \right)^T, \quad \left( -q + \sqrt{1 + q^2}, 1, 0 \right)^T
\]
respectively. Note that the maximum principal radius is not \( 1/X \) but \( 1/Z \). The strain and reciprocal strain ellipsoids are symmetric with respect to the coordinate planes (Fig. 1.11) that are defined so as to write simple shear deformations in the form of Eq. (1.33).

Now let us calculate \( U \) and \( R \) for this deformation. We shall determine the components of \( U \) and \( R \) from the polar decomposition, \( F = R \cdot U \). To this end, let the orthogonal tensor be
\[
R = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
where \( \varphi \) stands for the angle of rigid-body rotation. Substituting this into the equation \( F = R \cdot U \),
we have
\[
U = R^T \cdot F = \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2q & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\cos \varphi & 2q \cos \varphi + \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi - 2q \sin \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (1.35)

However, the angle \(\varphi\) must satisfy the equation \(2q \cos \varphi + \sin \varphi = -\sin \varphi\) because of the symmetry \(U = U^T\). Thus,
\[-q = \tan \varphi.\] (1.36)

Therefore, \(-q\) is equal to the engineering shear strain (Fig. 1.11), and we obtain
\[
R = \begin{pmatrix}
\cos(\arctan q) & \sin(\arctan q) & 0 \\
-\sin(\arctan q) & \cos(\arctan q) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (1.37)

From Eqs. (1.35) and (1.36), we have
\[
U = \begin{pmatrix}
\cos \varphi & -2 \tan \varphi \cos \varphi + \sin \varphi & 0 \\
-\sin \varphi & \cos \varphi + 2 \tan \varphi \sin \varphi & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
-\sin \varphi & \cos \varphi + 2 \tan \varphi \sin \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\] (1.38)

The eigenvalues are calculated from the matrix in the right-hand side of this equation. They are \(\sec \varphi \pm \tan \varphi\) and 1, and are equal to the eigenvalues of \(V\). These are the principal strains with a different form from Eq. (1.34). For \(q > 0\), \(\varphi < 0\). Thus, \((\sec \varphi - \tan \varphi)\) is the greatest eigenvalue, and the principal strains are
\[
X = \sec \varphi - \tan \varphi, \quad Y = 1, \quad Z = \sec \varphi + \tan \varphi.
\] (1.39)

Consider infinitesimal and infinite simple shear deformations. They are represented by the limits \(q \to +0\) and \(q \to \infty\). According to Eq. (1.36), \(\varphi\) approaches zero from minus for the case of the infinitesimal simple shear. Therefore, \(R\) becomes an infinitesimal, clockwise rotation in Fig. 1.11. For the case of infinite simple shear (\(q \to \infty\)), \(\varphi \to -\pi/2\), indicating that \(R\) approaches the clockwise rotation by 90°. What are the components of \(U\) at the limits? Using the approximations, \(\cos \varphi \approx 1\), \(\sin \varphi \approx \varphi\), and \(\tan \varphi \approx \varphi\) for an infinitesimal \(\varphi\), Eq. (1.38) becomes
\[
\lim_{\varphi \to -0} U = \begin{pmatrix}
1 & -\varphi & 0 \\
-\varphi & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

This tensor has three eigenvalues, \((1 + \varphi) \lesssim 1 \lesssim (1 - \varphi)\). The eigenvectors corresponding to the greatest and smallest eigenvalues have the components \((1, 1, 0)^T\), and \((1, -1, 0)^T\), respectively. Therefore, the principal strain axes intersect the displacement vectors at 45° for an infinitesimal simple shear.
1.7. SHEAR ZONE

Figure 1.12: Deformation patterns within a shear zone. Note the different sense of en echelon arrays accompanied by horizontal shortening and extension for the same sense of shear. (a) The patterns within a right-lateral shear zone. (b) En echelon arrays of folds and reverse faults within a right-lateral shear zone. (c) En echelon arrays of normal faults within a right- and a left-lateral shear zones.

As for the infinite simple shear, the angle goes to $\varphi \to -\pi/2$. Thus, the $U_{22}$ goes to infinity:

$$\lim_{\varphi \to -\pi/2} U = \lim_{q \to \infty} U = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \infty & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The eigenvector paired with the maximum eigenvalue rotates to the direction $(1, x, 0)^T$ with $x \to \infty$.

1.7.2 Deformation accompanied by wrench faulting

Using the equations derived above, deformation patterns within a shear zone are investigated here. Consider an east-west trending dextral shear zone accompanied by wrench faulting\(^5\), and assume that the shear zone is subject to a simple shear for which the amount of shear is evaluated by $q$ (Fig.

\(^5\)A wrench fault is a strike-slip fault with a vertical fault plane or shear zone. We assume a wrench fault here, only because sections perpendicular to the fault zone are horizontal.
1.12(a)). If $q$ is small, the principal strain axes intersect the fault zone at $\sim 45^\circ$. The $X$ and $Y$ axes are oriented NE–SW and NW–SE, respectively. Due to the strain, sedimentary layers blanketing the shear zone may be deformed to form a right-step en echelon array of folds (Fig. 1.12(b)). Anderson's theory (§6.3) predicts that a right-step array of reverse faults and a left-step normal faults can be formed there. Namely, folds and reverse faults are the features of horizontal shortening, so that they are arranged in a right-step en echelon manner. Normal faults are accompanied by horizontal extension, so that they are arranged in a left-step extension.

En echelon cracks filled with formation water or magma are sometimes formed at depth. Figure 1.13 shows an example; they were filled with quartz that precipitated in the cracks, and were twisted by incremental shearing. The S-shaped veins are the older and twisted ones.

There are many surface manifestations of geologic structures indicating a horizontal shortening or extension on the Moon. Lunar mare basins are topographic depressions formed by large impact cratering about 4 billion years ago. The basins are blanketed by layers mainly of basaltic flows and breccias ejected by impact cratering. Figure 1.14 shows the surface features in southwestern Mare Selenitatis. The mare deposits are divided into geologic units I, II and III in this region. Unit I is older and unit III is the youngest.

The layers lie horizontally, but are folded in places to form topographic ridges, called wrinkle ridges. Large wrinkle ridges are called arches. The ridges evidence that the deposits were affected by horizontal shortening. Note that NNW–SSE trending ridges make left-step en echelon arrays.

Long linear rilles are believed to be grabens. There is no rille in unit III, suggesting that the graben formation had ceased before this unit was deposited in this region. However, ridges were formed in all the units.

### 1.8 Deformation history

We understand strain by comparing the present and initial shapes of strain markers. However, even if we know both shapes, there are many deformation paths from the initial to the final configurations. Therefore, the configurations are not enough to understand the deformation history. Other lines of evidence are needed to constrain the path.$^6$

We have studied the means to quantify strain by small strain markers such as the deformed ammonite (Fig. 1.1). However, if such strain markers show little deformation, a rock mass that encompasses the markers can show a map-scale deformation. For example, Neogene formations in the western side of Northeast Japan were folded and faulted since the late Miocene, resulting in the horizontal shortening of the crust by $\sim 10\%$ [201]. However, fossils yielded from the strata exhibit little deformation.

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$^6$It is a further problem as to how small tectonic movements represented by, e.g., earthquakes are related to geologic features that are formed by the integral of those movements. See the next chapter for detail.
1.8. DEFORMATION HISTORY

Figure 1.13: (a) Schematic pictures showing mineral veins filling en echelon cracks accompanied by dextral shear. Black arrows indicate the maximum compressive orientation, which is parallel to each of the cracks. (b) Incremental shear rotates old veins and forms new ones, which cut the old veins. If the new shear zone is narrower than the older, the ends of old veins may be left unrotated to form an array of Z-shaped veins. (c) Example of the array in the Sambagawa metamorphic belt, Wakayama, Southwest Japan. Metamorphic foliation is folded showing the dextral sense of shear along the en echelon veins.

1.8.1 Stratigraphy

Sedimentary rocks are useful to study tectonics, because they have preserved a variety of information since they were deposited, and their depositional ages can be easily determined. A sedimentary sequence can be compared to a magnetic tape. Therefore, a sedimentary basin provides a wide
variety of evidence to constrain its tectonic history. The law of superposition and cross-cutting relationships are the clues. In addition, sedimentary layers are often assumed to be horizontal when they were deposited. Particular species of fossils are the indicators of paleo-environments. Some benthic fossils, e.g., molluscs and benthic foraminifera, are used to infer a paleo water depth. If a sedimentary pile is inferred to have accumulated upon a shallow continental shelf and is a few kilometers thick, the strata indicates syn-sedimentary subsidence of the basement, because eustatic sea-level changes have amplitudes smaller than the thickness by an order of magnitude.

Figure 1.15 shows a seismic profile taken from the southwestern margin of the Japan Sea. The margin was subject to a folding event at the end of the Miocene. To the south of the MITI Tottori-Oki Well, a north-dipping reverse fault affects the Koura, Josoji, and the lower part of the Furue Formations, suggesting that the faulting was terminated in the middle of the Furue stage. Folded
1.8. DEFORMATION HISTORY

Figure 1.15: Seismic profile through the southwestern margin of the Japan Sea [237]. The Furue, Josoji and Koura Formations deposited in the Miocene, and were folded at the terminal Miocene time. Note that Pleistocene sediments are not affected by the folding.

Structural geology has a weak point that geologic structures do not tell the ages of their formation. Therefore, some articles implicitly assume that deformation followed immediately after deposition, and that deformation and depositional ages are virtually the same. Coexisting structures are sometimes considered to have been formed at the same time. However, evidence other than the structures is needed to establish the age of deformation. For the case of the off-shore region shown in Fig. 1.15, the timing of folding was revealed by the fossil and radiometric ages of the strata that were cored at the well.

1.8.2 Balanced cross-section

The procedure of cross-section balancing [48] has become popular in recent years as a means of helping to analyze and improve cross-sections through folded or faulted sedimentary layers. Geometrically consistent cross-sections tell not only the present configurations but also the history of the
Figure 1.16: Balanced cross-section (upper panel) through western Pakistan where the convergence between the Eurasian and Indo-Australian plates is accommodated by folding and faulting [204]. The lower panel shows the configuration of the strata prior to the deformation. Rock masses bounded by faults are restored to their initial position relative to the right side of the section. Comparison of the panels indicates a horizontal shortening of this area by 2.6 km.

The key steps involved in the procedure is the restoration of the beds depicted in the cross-section to the relative positions that they had prior to deformation. The original states of the beds are found with the assumption that sedimentary layers lie horizontally when they were deposited. This is not valid if their basement did not have an horizontal surface or if sedimentation was coeval with the deformation. Sedimentological studies reveal the architecture of a sedimentary basin. For example, it is possible to infer whether the strata in question accumulated upon a slope or vast plain. Syn-sedimentary tectonics often results in horizontal variations of layer thickness and lithofacies that were affected by growing paleo basins and swells. Therefore, the validity can be tested. Consequently, we can assume the initial attitude of the beds for some cases. In addition, the conservation

\[\text{construction}^{7}\] and the amount of deformation (Fig. 1.16).

\[\text{The key steps involved in the procedure is the restoration of the beds depicted in the cross-section to the relative positions that they had prior to deformation. The original states of the beds are found with the assumption that sedimentary layers lie horizontally when they were deposited. This is not valid if their basement did not have an horizontal surface or if sedimentation was coeval with the deformation. Sedimentological studies reveal the architecture of a sedimentary basin. For example, it is possible to infer whether the strata in question accumulated upon a slope or vast plain. Syn-sedimentary tectonics often results in horizontal variations of layer thickness and lithofacies that were affected by growing paleo basins and swells. Therefore, the validity can be tested. Consequently, we can assume the initial attitude of the beds for some cases. In addition, the conservation}\]

\[^{7}\text{See [248, §6.4] for the explanation about the relative timing of the stacking of fault blocks for the case of “hinterland dipping duplex”. The geometry of the fault blocks and their relative positions constrain the history.}\]
1.8. DEFORMATION HISTORY

Figure 1.17: Restoration of faulted soft sediment in the Upper Pleistocene Shimosa Group, central Japan. The left photograph was taken at an outcrop on which a lens cap about 5 cm across was put as a scale. Faulting was due to a landslide [11]. The photograph was separated along the faults into pieces, which were restored to the original positions relative to each other (right panel).

of bed length or bed area on the section is used as a constraint for the restoration. When all the pieces are restored, key beds should be contiguous across faults on the section. The cross-sections are constructed by thoughtful analysis of fault shapes and of the conservation and satisfy the initial condition from the pieces of information on the present attitude and position of strata and faults observed at outcrops or boreholes. The section that satisfies all these requirements are called a balanced cross-section\(^8\). And, the geologic structures depicted on the section is said to be retrodeformable.

Such sections show the amount of deformation. The section in Fig. 1.16 indicates a 13% shortening of a shallow level of the crust. This was estimated by comparing the horizontal displacement of the upper part of the Maliri Pin and the original distance of the Maliri and Indus Pins.

The retrodeformability is useful to estimate not only shortening but also extensional deformations, and apply the concept to meso-scale deformations. Meso-scale normal faults that cut Pleistocene sandy soft sediment are shown in Fig. 1.17(a). The photograph was broken into pieces along the faults and sedimentary layers were restored to their original positions (Fig. 1.17(b)). There are gaps and overlaps in the latter configuration, possibly due to plastic deformation of the fault blocks and more probably to the component of displacements of the blocks perpendicular to the section.

Although each fault represents discontinuous movements, coarse graining of the gross deformation of the sediment allows us to ensure that the deformation is uniform (Fig. 1.18). The strain ellipse that represent the gross deformation has its major axis subparallel to the lamination. The horizontal

---

\(^8\)See [260] for further reading on cross-section balancing.
Figure 1.18: Strain estimated from a balanced cross-section. (a) The external form of the restored section (1.17(b)) and a dark gray circle with a radius of 10 cm drawn on the figure. (b) The circle is broken into arcs by normal faulting. Thin lines indicate the faults and frame of the photograph (1.17(a)). A light gray ellipse is drawn to fit the arcs, and approximate the strain ellipse representing the strain of the sediment by fault movements. (c) Parameters of the strain ellipse. The semi-major and minor are 12 and 8.3 cm long, respectively.

and vertical strains are estimated to be $E_1 \approx (12 - 10)/10 = 20\%$ and $E_2 \approx (8.3 - 10)/10 = -17\%$, respectively.

### 1.9 Exercises

1.1 The increase of $p$ and $q$ in Eqs. (1.13) and (1.13) indicates progressive pure shear and simple shear deformations. Determine how engineering shear strain should vary for the two cases.

1.2 Show that the principal radii of reciprocal strain ellipsoid is determined as the eigenvalues of the tensor $U^{-1}$.

1.3 A two-dimensional homogeneous deformation reshapes an ellipse to a different ellipse. Assume that the equation $\xi \cdot A \cdot \xi = 1$ indicates the initial ellipse. Show the equation of the final shape by the deformation $F$.

1.4 It is possible to quantify two-dimensional strain from the assemblage of objects such as ooids and rounded gravels whose section were elliptical before strain (Fig. 1.19), assuming (1) a homogeneous deformation for the assemblage, (2) random orientations of their pre-strain major-axis\(^9\) and (3) a variation in grain shapes [188, 120]. That is, the optimal strain ellipse for the assemblage is determined from their present eccentricity and long-axis orientations (Table 1.1). The eccentricity

\(^9\)Actually, sediments have anisotropic grain fabric to some extent: major-axes of grains tend to have one or more dominant orientations. A mathematical inverse method for determining the optimal strain ellipse from those grains has been made recently [267]. In addition, the method evaluates the confidence interval of the strain ellipse.
Figure 1.19: (a) Photomicrograph showing deformed calcareous ooids [94]. (b) Ellipses fitted to 18 grains in the photomicrograph. Long-axis orientations, $\phi_f$, are measured from the reference orientation, where the subscript ‘f’ stands for the quantities after strain (final stage).

is defined by the ratio of long and short axes, $R$, called the aspect ratio. Formulate a mathematical inversion to determine the aspect ratio $R_s$ and long-axis orientation $\phi_s$ of the optimal strain ellipse from the aspect ratios and orientations of the deformed grains [120].

Table 1.1: Aspect ratios, $R_f$, and long-axis orientations, $\phi_f$, of the 18 grains in Fig. 1.19.

<table>
<thead>
<tr>
<th>$R_f$</th>
<th>1.8</th>
<th>2.3</th>
<th>2.1</th>
<th>2.0</th>
<th>1.6</th>
<th>1.6</th>
<th>1.9</th>
<th>1.8</th>
<th>1.7</th>
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<tbody>
<tr>
<td>$\phi_f$</td>
<td>18°</td>
<td>16°</td>
<td>24°</td>
<td>2°</td>
<td>12°</td>
<td>28°</td>
<td>11°</td>
<td>11°</td>
<td>18°</td>
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<table>
<thead>
<tr>
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<th>1.6</th>
<th>1.8</th>
<th>1.5</th>
<th>1.5</th>
<th>1.8</th>
<th>1.4</th>
<th>1.6</th>
<th>1.7</th>
<th>1.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_f$</td>
<td>17°</td>
<td>12°</td>
<td>18°</td>
<td>10°</td>
<td>8°</td>
<td>12°</td>
<td>16°</td>
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<td>15°</td>
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