Astronomical Periods in the Solar System

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1. Introduction

From the central Sun the position of the planets can be observed changing regularly relative to the fixed stars. From the Earth they are observed under entirely different stars, so that the regularities of the sidereal revolutions of the planets cannot directly be found by a geocentric observer.

Essentially different are the apparent motions of the planets relative to the Sun. Here the observer is defined in the system by taking the Sun as a reference, whereas in the former, the apparent sidereal motions, the observer is involved with his entire motion during the solar year. It follows that only in the synodic revolutions are the regularities of the planetary motions immediately observable from the Earth. It is to the geocentric observation of the synodic periodicities of all celestial bodies together with the yearly revolution of the apparent Sun that the fundamental principles of astronomy are ascribed.

Due to the eccentricities and perturbations of the planetary orbits the synodic periods are not constant but vary greatly. Although this trivial fact is frequently mentioned, the question has, so far as I know, never been quantitatively analysed.

The present paper is concerned with derivation of some fundamental formulae for the variable synodic periods of the planetary motions.

Symbols

- A mean anomalistic period (^{d})
- α mean anomaly (°)
- c daily heliocentric motion of the planets (°/d), e.g., 0.52407116 (Mars, tropical, 1900)
- c_n coefficients of the longitudes, e.g. $c_1 = 365.25c$, (°/d)
- D mean period relative to the ascending node (^{d})
- d solar day
- E heliocentric elongation of the planets from the Earth (°)
- e mean eccentricity
- i mean inclination of the orbit to the ecliptic (°)
- L mean longitude (°)

l true longitude in the plane of the ecliptic (°)
λ true longitude in the orbit (°)
M mean anomalistic displacement relating to the number of synodic revolutions (°)
N, N_S mean longitude of the ascending node (°)
n_n coefficient of the motion of the ascending node
P mean longitude of perihelion, or of perigee (Moon) (°)
Q mean angular displacement relative to the ascending node (°)
S mean synodic period (^{d}) = $360 \cdot 365.25/(c_1' - c_1) = 360/(c' - c)$
s number of synodic revolutions
$\overline{\Delta S}$ mean deviation of one synodic revolutions from the mean length $S(^d)$, used as $s\overline{\Delta S}$
$s\overline{\Delta S}$ deviation of s synodic revolutions from the mean length $sS(^d)$
$s\overline{\Delta U}$ deviation of s synodic revolutions from the mean angular displacement $sU(^d)$
t time in general
U mean angular displacement after one synodic revolution, sidereal or tropical, $=cS$(°)
$\overline{\Delta U}$ mean deviation of one synodic revolution from the mean angular displacement U(°), used as $s\overline{\Delta U}$.

Contrary to the above symbols for the outer planets (fugitives), those for the inner planets (pursuers) as seen from the apparent central body are marked by ($'$), thus e.g. c' ($> c$).

The fundamental orbital elements are given as functions of time as

$$L_t = L_0 + c_1 t + c_2 t^2 + \cdots$$

$$P_t = P_0 + p_1 t + p_2 t^2 + \cdots$$

$$N_t = N_0 + n_1 t + n_2 t^2 + \cdots$$

$$e_t = e_0 + e_1 t + e_2 t^2 + \cdots$$

$$i_t = i_0 + i_1 t + i_2 t^2 + \cdots \quad (1)$$

t = time interval from the reference epoch, 1900 (Expl. suppl., Connaiss. des temps), in Julian years, where in general

$$|c_1 t| >> |c_2 t^2| >> \cdots,$$

$$|p_1 t| >> |p_2 t^2| >> \cdots,$$
$$\vdots \qquad \vdots \qquad \vdots \quad .$$

All above quantities except c_1 are small and a great part of the quantity for p_1 is covered by the precession of the equinoxes.

2. Variations of the Synodic Periods in Time

By means of the above elements the true motion of a planet (fugitive) in its unperturbed orbit about the apparent central body may be defined and the true longitude in the orbit and in the plane of the ecliptic is given by

$$\lambda(°) = L + \frac{180}{\pi}\left(2e\sin\alpha + \frac{5}{4}e^2\sin 2\alpha + \cdots\right),$$

$$l(°) = \lambda - \frac{180}{\pi}\left\{\tan^2\left(\frac{i}{2}\right)\sin 2(\lambda - N) - \cdots\right\}, \tag{2}$$

where $\alpha = L - P$.

Denoting the corresponding quantities for the pursuer by ($'$), we may further define the mean synodic period S_t of the planet as

$$S_t = \frac{360 \cdot 365.25}{\frac{\mathrm{d}}{\mathrm{d}t}(L' - L)_t} = \frac{360 \cdot 365.25}{c_1' - c_1 + 2(c_2' - c_2)t + \cdots}. \tag{3}$$

Due to the extremely small quantity of $(c_2' - c_2)t$ comparing to the term $(c_1' - c_1)$, we may further put for our present purpose approximately as

$$S_t \approx S = \frac{360 \cdot 365.25}{c_1' - c_1}, \ (^d). \tag{4}$$

The above mean synodic period, $S_t(\approx S)$, is the mean time interval defined by two successive configurations of the same kind of the three bodies. The synodic period of the three bodies which we define by E, the heliocentric elongation of the planet from the reference point, can be represented at the time t_0 as

$$l_{t_0} = l'_{t_0} + E, \tag{5}$$

and at the time t, distant by a whole number of the synodic revolutions s, as

$$l_t = l'_t + E - 360s. \tag{6}$$

Hence we obtain

$$\begin{aligned}
&\left[l-(l'+E)\right]_t - \left[l-(l'+E)\right]_{t_0} \\
&\quad = 360s \\
&\quad = c_1\left(t-t_0\right) + c_2\left(t^2-t_0^2\right) + \cdots + \frac{180}{\pi}\left\{2e\left(\sin\alpha_t - \sin\alpha_{t_0}\right) + \cdots \right. \\
&\qquad\qquad \left. - \tan^2\left(\frac{i}{2}\right)\left[\sin 2(\lambda-N)_t - \sin 2(\lambda-N)_{t_0}\cdots\right]\right\} \\
&\quad - \left\{c'_1\left(t-t_0\right) + c'_2\left(t^2-t_0^2\right) + \cdots + \frac{180}{\pi}\left[2e'\left(\sin\alpha'_t - \sin\alpha'_{t_0}\right) + \cdots\right]\right\}.
\end{aligned} \tag{7}$$

Putting for t in (3) the mean epoch of the time interval $(t - t_0)$ we may deduce the deviation of the actual synodic time interval from the mean period in days:

$$365.25\left(t-t_0\right) - sS_t = s\overline{\Delta S} = \frac{\dfrac{365.25\cdot 180}{\pi}\left[2e\left(\sin\alpha_t - \sin\alpha_{t_0}\right) + \cdots\right]}{\left(c'_1 - c_1\right) + \left(c'_2 - c_2\right)\left(t+t_0\right) + \cdots}, \ \left({}^{d}\right). \tag{8}$$

In order to eliminate some factors in the above equations we shall express the planetary positions in the orbit at t by those at t_0. From (1) we obtain

$$\alpha_t = (L-P)_t = L_0 - P_0 + \left(c_1 - p_1\right)t + \left(c_2 - p_2\right)t^2 + \cdots,$$

hence

$$\begin{aligned}
\alpha_t &= \alpha_{t_0} + \left(t-t_0\right)\left[\left(c_1-p_1\right) + \left(c_2-p_2\right)\left(t+t_0\right) + \cdots\right] \\
&= \alpha_{t_0} + \left(t-t_0\right)\frac{\mathrm{d}}{\mathrm{d}t}(L-P),
\end{aligned}$$

where $\mathrm{d}(L - P)/\mathrm{d}t$ approximately corresponds to the anomalistic velocity of the planet at the epoch, $(t + t_0)/2$. Likewise

$$(L-N)_t = (L-N)_{t_0} + \left(t-t_0\right)\frac{\mathrm{d}}{\mathrm{d}t}(L-N), \tag{9}$$

where

$$(\lambda - N)_t = (L - N)_t + \frac{\pi}{180}\left[2e\sin(L - P)_t + \cdots\right].$$

For the unknown time interval $(t - t_0)$ containing a whole number of the synodic revolutions s we adopt the mean synodic period sS_t as the first approximation:

$$t - t_0 \approx \frac{sS_t}{365.25}, \ \left({}^{y}\right). \tag{10}$$

Since, e.g.,

$$\frac{\mathrm{d}(L - P)_t}{\mathrm{d}t} = \frac{360 \cdot 365.25}{A_t}, \ \left({}^{\circ}/y\right), \tag{11}$$

we shall introduce the following terms into (8) in order to simplify our procedure:

$$\left(t - t_0\right)\frac{\mathrm{d}(L - P)}{\mathrm{d}t} \approx 360s\left(\frac{c_1 - p_1}{c_1' - c_1}\right) \approx 360s\left(\frac{S}{A}\right)_t \equiv M \,\mathrm{mod.}\, 360(^{\circ}),$$

$$\left(t - t_0\right)\frac{\mathrm{d}(L' - P')}{\mathrm{d}t} \approx 360s\left(\frac{c_1' - p_1'}{c_1' - c_1}\right) \approx 360s\left(\frac{S}{A'}\right)_t \equiv M' \,\mathrm{mod.}\, 360(^{\circ}),$$

$$\left(t - t_0\right)\frac{\mathrm{d}(L - N)}{\mathrm{d}t} \approx 360s\left(\frac{c_1 - n_1}{c_1' - c_1}\right) \approx 360s\left(\frac{S}{D}\right)_t \equiv Q \,\mathrm{mod.}\, 360(^{\circ}),$$

hence simply

$$\alpha_t = \alpha_{t_0} + M,$$

$$\alpha_t' = \alpha_{t_0}' + M',$$

$$(L - N)_t = (L - N)_{t_0} + Q. \tag{12}$$

The above three quantities M, M' and Q are the average displacements of the planetary positions in the orbits relative to the reference points, the perihelions and the ascending node, in the time interval of s synodic revolutions .

By means of (9)–(12) we may simplify (8) and obtain the deviation of the actual synodic time interval of the two planets from the mean synodic periods to the first approximation as

$$s\overline{\Delta S_1} = \frac{180 \cdot 365.25}{\pi(c_1' - c_1)} \Big\{ 4e\cos\left(\alpha_{t_0} + \frac{M}{2}\right)\sin\frac{M}{2} + \frac{5}{2}e^2\cos\left(2\alpha_{t_0} + M\right)\sin M \cdots$$

$$- 4e'\cos\left(\alpha'_{t_0} + \frac{M'}{2}\right)\sin\frac{M'}{2} - \frac{5}{2}e'^2\cos\left(2\alpha'_{t_0} + M'\right)\sin M' - \cdots$$

$$-2\tan^2\frac{i}{2}\cos\left[2(L-N)_{t_0} + Q + \cdots\right]\sin Q + \cdots \Big\}$$

$$= f\left[e, e', i, \alpha_{t_0}, \alpha'_{t_0}, (L-N)_{t_0}, s\right]$$

$$= f(M, M', Q), \ \left({}^d\right). \tag{13}$$

We may express α' and $(L - N)$ as functions of α or conversely, hence, the time variation of s synodic revolutions of a planet can be represented as a function of the mean anomaly or longitude of the planets:

$$s\overline{\Delta S_1} = f(\alpha) = f(L),$$

$$= f(\alpha') = f(L'). \tag{14}$$

The first approximation (13) is now to be employed as the correction to the mean synodic period in (10), (12), (13) followed by the procedure

$$M_{n+1} = 360s\left(\frac{S + \overline{\Delta S}_n}{A}\right), \ M'_{n+1} = 360s\left(\frac{S + \overline{\Delta S}_n}{A'}\right),$$

$$Q_{n+1} = 360s\left(\frac{S + \overline{\Delta S}_n}{D}\right),$$

$$s\overline{\Delta S}_{n+1} = f\left(M_{n+1}, M'_{n+1}, Q_{n+1}\right),$$

$$n = 1, 2, 3, \ldots, \text{ according to (13).} \tag{15}$$

For any given value for α (α') the deviation of the actual s synodic revolutions from the mean period, $s\overline{\Delta S}$ $(=s\overline{\Delta S}_{n=\infty})$, can be accurately obtained by repeating the above process, (13), (15). Knowing this we shall express the whole computing procedure into a single equation.

The process of substituting newly obtained values for old ones in (13), (15) can be described as

$$s\overline{\Delta S}_2 \approx s\overline{\Delta S}_1 + s\overline{\Delta S}_1 \cdot \Delta_1 = s\overline{\Delta S}_1\left(1+\Delta_1\right)$$

$$s\overline{\Delta S}_3 \approx s\overline{\Delta S}_1 + s\overline{\Delta S}_2 \cdot \Delta_2 = s\overline{\Delta S}_1\left(1+\Delta_2+\Delta_1\Delta_2\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$s\overline{\Delta S}_{n+1} \approx s\overline{\Delta S}_1 + s\overline{\Delta S}_n \cdot \Delta_n = s\overline{\Delta S}_1\left(1+\Delta_n+\Delta_{n-1}\Delta_n+\cdots\right), \tag{16}$$

where Δ_n is given by

$$\Delta_n = \frac{180\cdot 365.25\cdot 4}{\left(c_1'-c_1\right)}\left\{\frac{e}{A}\left[\left(1-\frac{m_n^2}{3!}+\frac{m_n^4}{5!}-\cdots+(-1)^i\frac{m_n^{2i}}{(2i+1)!}\right)\cos(\alpha+M)\right.\right.$$
$$\left.+\left(-\frac{m_n}{2!}+\frac{m_n^3}{4!}-\cdots+(-1)^i\frac{m_n^{2i-1}}{(2i)!}\right)\sin(\alpha+M)\right]$$
$$-\frac{e'}{A'}\left[\left(1-\frac{m_n'^2}{3!}+\frac{m_n'^4}{5!}-\cdots\right)\cos(\alpha'+M')\right.$$
$$\left.\left.+\left(-\frac{m_n'}{2!}+\frac{m_n'^3}{4!}-\cdots\right)\sin(\alpha'+M')\right]\right\}$$
$$= f\left(m_n, m_n'\right);\; n = 1,2,3,\ldots, \tag{17}$$

where the corrections, m_n and m_n', to be successively applied to M_n and M_n' in (15) are expressed in radians as

$$m_1 = 2\pi s\left(\frac{\overline{\Delta S}_1}{A}\right),$$

$$m_2 = 2\pi s\left(\frac{\overline{\Delta S}_2}{A}\right) = m_1\left(1+\Delta_1\right),$$

$$m_n = 2\pi s\left(\frac{\overline{\Delta S}_n}{A}\right) = m_1\left(1+\Delta_{n-1}+\Delta_{n-2}\Delta_{n-1}+\cdots+\Delta_1\Delta_2\cdots\Delta_{n-1}\right). \tag{18}$$

Any order of accuracy is, therefore, obtainable as functions of the first approximation, $s\overline{\Delta S}_1$, by following the procedure

$$
\begin{array}{l}
\overline{\Delta S}_1 \rightarrow m_1,\ m_1' \rightarrow \Delta_1 \rightarrow s\overline{\Delta S}_2 = f\left(s\overline{\Delta S}_1\right) \\
\qquad\quad \downarrow \qquad \swarrow \\
\qquad\quad m_2,\ m_2' \rightarrow \Delta_2 \rightarrow s\overline{\Delta S}_3 = f\left(s\overline{\Delta S}_1\right) \\
\qquad\quad \downarrow \qquad \swarrow \\
\qquad\quad m_3,\ m_3' \rightarrow \Delta_3 \rightarrow s\overline{\Delta S}_4 = f\left(s\overline{\Delta S}_1\right) \\
\qquad\quad \downarrow \qquad \swarrow \\
\qquad\quad \vdots \qquad\quad \vdots \quad \vdots \ .
\end{array} \tag{19}
$$

Since $e, e' << 1$, we may estimate a rough value for $s|\overline{\Delta S}_1|_{\max}$ by considering only the first power of e and e' in (13) as

$$
s\left|\overline{\Delta S}_1\right|_{\max} = \frac{180 \cdot 365.25 \cdot 4}{\pi\left(c_1' - c_1\right)}(e + e'),\ \left({}^{d}\right). \tag{20}
$$

The above two expansions in series in (17), $\sin m/m (=1 - m^2/3! + \cdots)$ and $(\cos m - 1)/m (= -m/2! + \cdots)$, converge rapidly [$|m|$, $|m'| \le 1.2$ (Mercury)], with the successive ratios respectively

$$
\frac{|X_n|}{|X_{n-1}|} = \frac{m^2}{2i(2i+1)},\ \frac{m^2}{(2i-1)2i}, \quad i = 1,2,3,\ldots \tag{21}
$$

It is, therefore, necessary to take only a few m- and m'-terms into account. Considering this, we may approximately describe

$$
\Delta_1 = f(m_1, m_1'), \qquad (17)
$$

$$
\Delta_2 = \Delta_1\left[1 - \frac{180 \cdot 365.25 \cdot 4e}{(c_1' - c_1)A} \cdot \frac{m_1}{2}\sin(\alpha + M)\right] = f(\Delta_1, m_1)
$$

$$
\Delta_3 = \Delta_1\left[1 - \frac{180 \cdot 365.25 \cdot 4e}{(c_1' - c_1)A} \cdot \frac{m_1}{2}\sin(\alpha + M)(1 + \Delta_1)\right] = f(\Delta_1, m_1)
$$

$$
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \quad . \tag{22}
$$

Since $|\Delta|_{\max} < 1$, we put in (16)

$$1 + \Delta_n + \Delta_{n-1}\Delta_n + \cdots = \frac{1}{1 - \Delta_n}, \tag{23}$$

and we obtain a simple formula for deducing the deviation searched for in terms of the first approximation $s\overline{\Delta S}_1$ and Δ_n

$$s\overline{\Delta S} = \frac{s\overline{\Delta S}_1}{1 - \Delta_n} = f\left(s\overline{\Delta S}_1\right). \tag{24}$$

Putting $\Delta_n(n = 1, 2, 3,...)$ into (24) we may compute the variation of the synodic periods to any desired order of accuracy.

With, e.g., Δ_3 above (22) we obtain the variation of the synodic period eventually as a function of a initial mean anomaly α_{t_0}:

$$\begin{aligned} s\overline{\Delta S} &\approx \frac{s\overline{\Delta S}_1}{1 - \Delta_3} = \frac{s\overline{\Delta S}_1}{1 - \Delta_1\left[1 - \dfrac{180 \cdot 365.25 \cdot 4e}{(c_1' - c_1)A} \cdot \dfrac{m_1}{2}\sin(\alpha + M)(1 + \Delta_1)\right]} \\ &= f\left(s\overline{\Delta S}_1, \Delta_1, m_1\right) = f\left(s\overline{\Delta S}_1\right) = f(e, e') \\ &= f\left(\alpha_{t_0}\right). \end{aligned} \tag{25}$$

This is the formula to be recommended for a planet with great eccentricity and short anomalistic period.

With Δ_1 we have

$$\begin{aligned} s\overline{\Delta S} &\approx \frac{s\overline{\Delta S}_1}{1 - \Delta_1} \\ &\approx \frac{s\overline{\Delta S}_1}{1 - 2S\left\{\dfrac{e}{A}\left[\cos(\alpha + M) - \dfrac{m_1}{2}\sin(\alpha + M)\right] - \dfrac{e'}{A'}\left[\cos(\alpha' + M') - \dfrac{m_1'}{2}\sin(\alpha' + M')\right]\right\}} \\ &= f\left(s\overline{\Delta S}_1\right) = f\left(\alpha_{t_0}\right), \left({}^{d}\right). \end{aligned} \tag{26}$$

In most cases one may further neglect some terms above, thus e.g. a simple formula already applied to the Moon (pursuer):

$$s\overline{\Delta S} \approx \frac{s\overline{\Delta S}_1}{1 + \dfrac{2Se'}{A'}\cos(\alpha' + M')}, \left({}^{d}\right). \tag{27}$$

3. Variations of the Synodic Periods in Position

The fact that synodic periods vary greatly in time indicates immediately that they also vary in position.

From (13), (25)–(27), it is evident that the synodic time intervals vary as a function of the mean anomalies, which can further be expressed by means of the longitude. This means that we can deduce the relation between the variation of the synodic periods in time, $s\overline{\Delta S}$, and that in position, $s\overline{\Delta U}$, both in observable quantities. In the following we shall therefore demonstrate the variable synodic periods as a function of the true longitude, such that the celestial observations derivable from our general astronomical tables can be directly reproduced by our formulae. For this aim we shall, for simplification, choose two most usual cases, conjunctions and oppositions of a planet with the Earth, namely where the heliocentric elongation $E = 0$ and 180° [(5), (6)]. It should be noted that for any value of E the problem does not become substantially more complex.

The true longitudes of a fugitive—or a pursuer with (′)—at t_0 and t are given by

$$l_{t_0} = L_{t_0} + 2e\sin\alpha_{t_0}\cdots,$$

$$l_t = L_{t_0} + \frac{c_1}{365.25}s\left(S+\overline{\Delta S}\right) + 2e\sin\left[\alpha_{t_0} + M + 360s\left(\frac{\overline{\Delta S}}{A}\right)\right]\cdots. \qquad (28)$$

Since further, neglecting the i-term,

$$l'_t - l'_{t_0} = l_t - l_{t_0},$$

we have the angular deviation of the actual synodic periods from the mean periods as a function of the time deviation:

$$s\overline{\Delta U} = \left(l_t - l_{t_0}\right) - \frac{c_1 sS}{365.25} = \left(l'_t - l'_{t_0}\right) - \frac{c'_1 sS}{365.25}$$
$$= \frac{c_1}{365.25}s\overline{\Delta S}$$
$$+\frac{180}{\pi}\left\{4e\cos\left[\alpha_{t_0} + \frac{M}{2} + 180s\left(\frac{\overline{\Delta S}}{A}\right)\right]\sin\left[\frac{M}{2} + 180s\left(\frac{\overline{\Delta S}}{A}\right)\right]\right.$$
$$\left.+\frac{5}{2}e^2\cos\left[2\alpha_{t_0} + M + 360s\left(\frac{\overline{\Delta S}}{A}\right)\right]\sin\left[M + 360s\left(\frac{\overline{\Delta S}}{A}\right)\right]\cdots\right\},\ (^\circ),$$

$$= \frac{c_1'}{365.25} s\overline{\Delta S} + \frac{180}{\pi}\{4e' \ldots\}, \ (^\circ),$$
$$= f\left(s\overline{\Delta S}\right). \tag{29}$$

This is the fundamental relationship between position $s\overline{\Delta U}$ and time $s\overline{\Delta S}$ in synodic periods.

From (13), (25)–(27) and (29) it follows

$$s\overline{\Delta S}, s\overline{\Delta U} = 0, \quad \text{when} \quad M, M', Q \equiv 0 \quad \text{mod.} 360^\circ,$$

$$s\left|\overline{\Delta S}\right|, s\left|\overline{\Delta U}\right| = \text{max. (approx.)}, \quad \text{when } M, M' \equiv 180^\circ \text{mod.} 360^\circ. \tag{30}$$

Consequently the time interval and the angular displacement between two synodic phenomena at the same orbital positions always correspond to the mean values. Dealing with two opposite positions, initial and final, the variation becomes extreme.

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