Elementary Processes in Planetary Accretion

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1. Introduction

In the last two decades, many authors have studied the growth of the planets, i.e., the growth from planetesimals to planets (e.g., Safronov, 1969; Greenberg et al., 1978; Nakagawa et al., 1983; Wetherill and Stewart, 1989). They have been widely different from each other in adopting the physical circumstances under which planetesimals grow to the planets, physical assumptions, initial conditions, as well as the mathematical description of planetary growth. For instance, some authors have considered the planetary accretion in the nebular gas (as to the gaseous planetary accretion, see Hayashi et al. (1985)) and others without the gas. Some have taken into account the effect of fragmentation of colliding planetesimals as well as their coalescence but others have not. The growth of planets has been formulated, in many cases, by means of statistical mass distribution function and, in other cases, pursued as the direct N-body problem. Regardless of the wide differences among the previous works, it should be noted that serious defects exist in common in these works.

One of the most serious problems is that the formation period of the Jovian planets previously estimated is too long. For example, Nakagawa et al. (1983) evaluated the growth times of the cores of Jupiter and Saturn with a mass of 5 $M_E$ ($M_E$ being the terrestrial mass), beyond which the
core can gather a large amount of the nebular gas (Mizuno et al., 1978), to be of about $3 \times 10^7$ yrs and $4 \times 10^8$ yrs, respectively. However, recent observations of T Tauri stars presume that the nebular gas might have dissipated within a period shorter than $1 \times 10^7$ yrs after the formation of the protosun (Adams et al., 1987; 1988; Strom et al., 1989). Since Saturn has a massive H/He envelope, Saturn’s core should have grown to the mass of $5 \, M_E$ within $1 \times 10^7$ yrs, i.e., before the escape of the nebular gas. Even though Stevenson and Lunine (1988) tried to explain the rapid growth of proto-Jupiter as a result of the diffusion of water vapor toward the region of the Jovian orbit, their theory cannot be applicable to the regions far from the protosun. Unfortunately, we have no theory at the present time which can give an explanation for such a rapid growth of the Jovian planets.

Another important problem is “spacing of the planets”. As prescribed typically by Titius-Bode’s law, the semimajor axes of two adjacent planets differ by a factor of 1.3 or so. However, N-body simulation in coplanar system showed that we have ten or more small planets in the region between 0.6 and 1.5 AU unless we introduce the unrealistic assumption that the mean eccentricity of planetesimals is as large as 0.15 in the early stage of planetary growth (Cox and Lewis, 1980). Nobody has succeeded in explaining the number of planets in our solar system.

The serious defects mentioned above certainly come from oversimplification or overlooking of important processes, and/or the introduction of inappropriate assumptions into the theory of planetary formation. Recently, from this point of view, we have re-considered and re-evaluated the elementary processes relevant to the planetary growth, i.e., (a) collision probability between planetesimals, (b) statistical behavior of planetesimals, (c) inelastic collisions between planetesimals, and (d) mathematical (and numerical) description of the planetary growth. In the previous works the first three, (a) to (c), were treated as the sum of the two-body encounter in a free space, but we have re-evaluated them recently in the framework of the three-body problem, in which the effect of the solar gravity is taken into account. Concerning the fourth problem, we have studied the validity of a numerical method to solve the coagulation equation of planetesimals as well as the validity of the coagulation equation itself. In this article, we will describe in brief the above-mentioned elementary processes we have developed recently.
2. Collision rate

The accretion of a protoplanet (exceptionally large planetesimal) is determined by the collision rate between the protoplanet and planetesimals, where “collision” means physical contact. As shown in Section 4, colliding planetesimals almost accrete, and furthermore, tidal capture is not so important (which can be represented as an effective correction factor to protoplanetary radius; Watanabe and Miyama, 1990). Hence we will examine the physical collision rate as the accretion rate.

Collision probability explicitly depends on the relative velocity between the protoplanet and planetesimals. First we consider relative motion between bodies orbiting around the sun. The relative motion is the sum of the Keplerian shear (systematic relative motion due to difference in heliocentric distance) and the random motion of the bodies. The random motion is described by (heliocentric) eccentricity $e$ and inclination $i$, where $e$ represents ellipticity of an orbit (0 for a circular orbit and 1 for a parabolic orbit) and $i$ is the inclined angle of orbital plane, in which both show the deviation from the local circular non-inclined orbit. We will often call eccentricity $e$ and inclination $i$ “random velocity”, since the velocity of such a deviant motion is approximately written as

\[ v \simeq (e^2 + i^2)^{1/2} v_K, \]  

(1)

where $v_K$ is the Keplerian velocity of circular orbit.

When the distance between the bodies is so large as to neglect their mutual gravitational force, they orbit individually around the sun (Keplerian regime). On the other hand, when two bodies approach closely to each other, their motion is determined only by the mutual gravitational force (two-body regime). In the intermediate cases, their orbits are influenced by both, mutual and solar gravitational forces (three-body regime). If $e$ and $i$ are sufficiently large, Keplerian shear can be neglected, compared to the random motion, and the Keplerian regime would overlap the two-body regime; the disappearance of the three-body regime validates the so-called two-body approximation (particle-in-a-box approximation). In the two-body approximation, kinetic quantities for planetesimals are calculated
using the method for the kinetic theory of gases: Solar gravitational force is neglected in the gravitational encounters and incident (isotropic) velocity is given by a random velocity (1). Originally, particle-in-a-box calculations for planetesimals were developed by Safronov (1969); he obtained collision and scattering rates using this method. Almost all subsequent studies of planetary accretion also adopted the two-body approximation.

Obviously, in encounters with small $e$ and $i$, the three-body regime is important and the Keplerian shear dominates relative motion: the two-body approximation breaks down. We must examine the validity limit of the two-body approximation and clarify gravitational encounters in the three-body problem.

Collision rates must be given as functions of eccentricity and inclination, assuming a local, uniform surface number density of planetesimals. Thus, to investigate the accretion rate of the protoplanet, we must determine the collision rate as a function of $e$ and $i$ and clarify the velocity-distribution ($i.e., e$- and $i$-distribution) of a planetesimal swarm. In this section we present the former, and the latter is described in subsequent sections.

Collision rates with a three-body calculation were examined by some authors. Since there is no general analytic solutions in the three-body problem, examining the three-body collision rate needs a large number of numerical orbital integrations. Nishida (1983) examined two simple cases where a protoplanet and planetesimals revolve on the same plane and the orbit of the protoplanet is circular. Wetherill and Cox (1985) studied the cases where $e = 2i$ for the planetesimals and the orbit of the protoplanet is circular.

A series of our papers (Nakazawa et al., 1989a; 1989b; Ida and Nakazawa, 1989) extensively investigated the collision rate with three-body calculations, using Hill's framework. The masses of the protoplanet and planetesimals are much smaller than the solar mass, and their orbits do not appreciably deviate from a circular non-inclined orbit ($e, i \ll 1$). In this case, equations of motion in the three-body problem are reduced to simple form by Hill's equations; these reductions are called Hill's approximations. Hill's framework provides the following useful properties (Hénon and Petit, 1986; Petit and Hénon, 1986; Nakazawa and Ida. 1988; Nakazawa et al., 1989a): First, relative motion can be separated from barycenter motion, and barycenter motion is purely Keplerian, which can be integrated ana-
lytically. Second, mass of protoplanet (and planetesimals) can be scaled out. Consequently, results can be applied to the cases with arbitrary protoplanet mass. In particular, the condition for the occurrence of collision is independent of the protoplanet mass since scaled protoplanet radius does not depend on the mass.

Ida and Nakazawa (1989) obtained the collision rate with wide ranges of $e$ and $i$. The collision rate along $e = 2i$ (gravitational scattering favors a state where $e = 2i$; see Ida, 1990) is shown in Fig. 1, where we find that (i) for $v/v_e > 0.1$ (where $v$ is defined by Eq.(1) and $v_e$ is the escape velocity from the protoplanet surface), the two-body approximation is valid, (ii) for $0.1 > v/v_e > 0.01$, collision rate is enhanced over that in the two-body approximation by a factor $\sim 3$, and (iii) for $v/v_e < 0.01$, collision rate takes constant value, which is much smaller than that in the two-body approximation. Ida and Nakazawa (1989) presented collision rates with the velocity-distribution where all planetesimals have the same (relative) $e$ and $i$. However, Nakazawa et al. (1989a) gave a formula in an integral form deriving collision rates with arbitrary velocity-distribution; for example, the collision rate with Gaussian velocity-distribution was given by Ida and Nakazawa (1988).

Fig. 1. Collision rate obtained by the three-body calculation, $P$, and that in the two-body approximation $P_{2B}$ as functions of $v/v_e$ for the case of $e = 2i$, where $v$ is given by $(e^2 + i^2)^{1/2}v_K$. 
Greenzweig and Lissauer (1990) also adopted Hill’s framework and examined the collision rate with homogeneous velocity distribution satisfying $e = 2i$. They defined the relative velocity in terms of $e$ and $i$, slightly different from Eq.(1), including the effect of the Keplerian shear.

There are two kinds of growth feature of planets: One is the runaway growth, where a planet is formed by runaway planetesimals accreting other small planetesimals, and the other is the orderly growth where a planet is formed by many similar-sized planetesimals. Ohtsuki and Ida (1990) investigated in detail the conditions for the occurrence of the runaway growth, using the collision rate obtained by Ida and Nakazawa (1989). They found that the three-body effect on the collision rate facilitates runaway growth in the low velocity case ($v/v_e \sim 0.1$) while it strongly suppresses runaway growth in the very low velocity case ($v/v_e < 0.01$). They also found that the mass dependence of $e$ and $i$ largely influences the occurrence of runaway growth: Smaller random velocity for massive planetesimal favors runaway growth. As shown in Fig. 1, smaller values of $e$ and $i$ give large values of collision rate and, hence, runaway growth would lead to rapid growth of planets.

Thus, the remaining problem is to determine the magnitude and mass-dependence for the random velocity. Random velocity is determined by mutual gravitational scattering, gas drag, and inelastic collision. In the next two sections, we will introduce the studies of gravitational scattering and inelastic collision.

3. Gravitational scattering

The random velocities of the protoplanet and planetesimals change through mutual gravitational scattering in two different ways: One is viscous stirring which converts solar gravitational energy into the random energy. The other is dynamical friction which transfers random energy from the larger to the smaller planetesimals. So far, in most of the previous studies of planetary accretion, the viscous stirring time scale was simply assumed to be equal to the two-body relaxation time in stellar dynamics, studied by Chandrasekhar (1942), and the dynamical friction was neglected.

The effects of dynamical friction were taken into account by Stewart and Wetherill (1988) by means of the Boltzmann equation following Hor-
nung et al. (1985). But their treatment for gravitational encounters was still based on the two-body approximation. It seems likely that the two-body approximation is inadequate to calculate the dynamical friction and viscous stirring rates in the low random velocity as in the collision rate. Wetherill and Cox (1984) made a preliminary examination for the validity of the two-body approximation, which suggests that the two-body approximation breaks down for the case of \( v/v_e < 0.07 \).

To obtain the dynamical friction and viscous stirring rates by the three-body calculations, we must integrate the orbits numerically. However, in distant encounters where the minimum distance during the encounter is sufficiently large, changes in the orbital elements are very small and can be calculated by a perturbation method. Goldreich and Tremaine (1982) derived change in eccentricity for the encounter between bodies both of which are on coplanar and circular \((e \text{ and } i = 0)\) orbits. Generalizing approximately these results to encounters with \( e, i \neq 0 \), Weidenschilling (1989) calculated the viscous stirring rate for distant encounters. Hasegawa and Nakazawa (1990) exactly derived changes in the eccentricity and inclination for general \((e, i \neq 0)\) distant encounters (these exact results almost coincide with Weidenschilling’s results). They also showed that changes in eccentricity and inclination due to horse shoe orbits are very small.

Ida (1990) formulated and calculated viscous stirring and dynamical friction rates for general three-body cases without restriction to distant encounters, using Hill’s framework. He showed that the contribution of viscous stirring and dynamical friction rates from the distant encounters are much smaller than those from the close encounters. He considered the bimodal population of planetesimals with masses \( m_1 \) and \( m_2 \), and examined the change in the mean random velocity of population 1 caused by gravitational perturbation of planetesimals of population 2. The results are as follows:

\[
\frac{d\langle v_1^2 \rangle}{dt} \approx \frac{m_2}{(m_1 + m_2)^2} \left\langle \frac{m_2 v^2}{T_{VS}} + \frac{(m_2 v_2^2 - m_1 v_1^2)}{T_{DF}} \right\rangle,
\]

(2)

where \( v_1 \) and \( v_2 \) are random velocities of planetesimals in populations 1 and 2 (related to eccentricities and inclinations as in Eq.(1)), \( v \) is the random
velocity for relative motion \( (v^2) = \langle v_1^2 \rangle + \langle v_2^2 \rangle \), and \( T_{VS} \) and \( T_{DF} \) are time scales for viscous stirring and dynamical friction. It is shown that a state with \( e = 2i \) will be favored for \( v/v_e > 0.1 \) (for \( v/v_e < 0.1 \), the time scales are independent of \( e, i \) ratio). Thus, \( T_{VS} \) and \( T_{DF} \) in the case of \( e = 2i \) are shown in Fig. 2 together with Chandrasekhar’s two-body relaxation time \( T_{2B} \). The factor \( (m_2 v_2^2 - m_1 v_1^2) \) in r.h.s. of Eq.(2) shows that dynamical friction works toward “energy equipartition”. Ida (1990) concluded from Eq.(2) and Fig. 2 that; (i) for \( v/v_e > 0.1 \) the random velocity of larger planetesimals suffers dynamical friction due to the smaller planetesimals, while the random velocity of the smaller planetesimals is increased entirely by viscous stirring, and (ii) for \( v/v_e < 0.1 \) the dynamical friction is less effective and the random velocity increases by viscous stirring for both the smaller and larger planetesimals. In a state of “energy equipartition”, a larger planetesimal has a smaller random velocity, which leads to the runaway planetary growth as shown by Ohtsuki and Ida (1990), and consequently, the growth time of the planets is substantially shortened as shown by Wetherill and Stewart (1989).

![Diagram showing time scales for \( T_{VS}, T_{DF}, \) and \( T_{2B} \) as functions of \( v/v_e \) in the case of \( e = 2i \).](image)

Fig. 2. \( T_{VS}, T_{DF}, \) and \( T_{2B} \) as functions of \( v/v_e \) in the case of \( e = 2i \).

However, a state of “energy equipartition” cannot be obtained if the time required is longer than the accretion time of a protoplanet: the problem was examined by Hasegawa et al. (1991). He developed the Fokker-Planck equation with the help of the scattering matrices obtained by the three-body calculations and examined the evolution of eccentricity and inclination of a
protoplanet. For the typical cases he found that the time within which a protoplanet approaches to the state of energy equipartition is shorter than that of planetary accretion. Thus, for relatively large velocity cases where \( v/v_e > 0.1 \), it seems likely that a state of “energy equipartition” will be achieved.

4. Inelastic collisions and accretion

Not only gravitational scattering, inelastic collisions and accretion may also contribute to the evolution of random velocity at the early stages of accumulation where the collision frequency is large. In previous works on planetary accumulation, the Keplerian motion of planetesimals in the solar gravitational field was not considered and the collisions were treated as those in a free space (e.g., Safronov, 1969; Greenberg et al., 1978; Stewart and Wetherill, 1988; Wetherill and Stewart, 1989).

Recently, Ohtsuki (1990) examined the random velocity evolution of planetesimals due to direct collisions in the solar gravitational field, as follows. First, he examined the orbital change caused by collisions for a given restitution coefficient through an orbital calculation of the three-body problem. The colliding planetesimals accrete or rebound depending on the impact velocity and restitution coefficient. In Fig. 3, we plot the gravitational sticking probability, i.e., the fraction of the number of collision orbits which ultimately lead to the accretion. The random velocity of planetesimals is at most comparable to the escape velocity and mostly less than it, because the drag force due to the solar nebular gas effectively damps a higher velocity (Ohtsuki, 1990). The restitution coefficient of planetesimal is expected to be much less than 0.5 through the high velocity impact experiments of rocks (Hartmann, 1978). In such ranges of random velocity and restitution coefficient, we may expect that most of the colliding planetesimals will ultimately accrete, as shown in Fig. 3. This situation is quite different from the collisions between planetary ring particles, where colliding particles rebound almost without accretion (Goldreich and Tremaine, 1978). It also conflicts with the assumption made in previous works on planetary accretion that the equilibrium state in random velocity of planetesimal assemblage for certain masses is achieved through the balance between the inelastic dissipation and the enhancement by gravitational scattering (Safronov, 1969; Greenberg et
Fig. 3. Gravitational sticking probability of planetesimals as functions of $v/v_e$ with $i = 0$ for given restitution coefficient $\varepsilon$.

al., 1978; Nakano, 1987; Ohtsuki et al., 1988); in these works, change of mass-distribution caused by accretion was not included explicitly.

Ohtsuki (1990) derived the equation for the random velocity evolution due to accretion, which is coupled with the coagulation equation. He found that the inelastic dissipation of random velocity has the tendency to lead to the equipartition of random kinetic energy of the planetesimals, as in the case of velocity evolution due to gravitational scattering described in the last section.

The random velocity at early stages, where inelastic collision contributes to the velocity evolution, is as large as the escape velocity, as mentioned above. The collision rate in such a high velocity region has $d\ln P/d\ln v \simeq 0$ as shown in Fig. 1. The runaway growth of a protoplanet is possible only in the moderate velocity region, where the collision rate $P$ has strong negative power dependence on $v$ (Ohtsuki and Ida, 1990). Thus, we cannot expect runaway growth at these stages even if energy equipartition is achieved.

As the accumulation proceeds, gravitational scattering and gas drag dominate the velocity evolution. In this case, dynamical friction caused by gravitational scattering as well as gas drag force are a much more effective mechanism for velocity damping of large planetesimals. We think that the inelastic collisions and the accretion hardly contribute in the random velocity evolution except for the earliest stages. It may hardly contribute even after the dispersal of the solar nebula, where gravitational scattering should
control the velocity evolution.

5. Numerical simulation of planetary growth

Using the studies of elementary physical processes described in Sections 2 to 4, we can simulate the planetary accretion in detail by solving the evolution of planetesimal mass distribution. In this section, we first discuss the approach to simulate the planetesimal accumulation.

In general, we cannot obtain the analytic expressions describing the evolution of mass distribution. Safronov (1969) assumed simple functional forms for the collision rate and velocity of planetesimals, which enabled him to estimate analytically the growth time of planets. Without such simplifications, Greenberg et al. (1978) numerically solved the coagulation equation given by

$$\frac{\partial n(m, t)}{\partial t} = -n(m, t) \int_0^\infty \alpha(m, m') n(m', t) dm'$$
$$+ \frac{1}{2} \int_0^m \alpha(m - m', m') n(m - m', t) n(m', t) dm', \quad (3)$$

where $n(m, t) dm$ is the number density of planetesimals with a mass between $m$ and $m + dm$, and $\alpha(m, m')$ is the collision rate. Greenberg et al. solved the evolutions of mass and velocity distribution simultaneously, and found that the runaway growth of a protoplanet occurred in the early stage of accumulation.

However, Ohtsuki et al. (1990) showed that numerical simulation using too coarse mass-coordinate divisions results in artificial acceleration during the accumulation. The analytic solutions to Eq.(3) were found, for example $\alpha(m, m') = \text{const.}$ (Smoluchowski, 1916) and $\alpha(m, m') \propto m + m'$ (Safronov, 1963). Ohtsuki et al. used these two analytic solutions to show that the numerical calculations of planetesimal accumulation lead to a substantial overestimation of the rate of planetary growth when a mass ratio between neighboring mass coordinates exceed $\sqrt{2}$. Previously, $\sqrt{2}$ (Nakagawa et al., 1983; Ohtsuki et al., 1988), 2 (Hayakawa et al., 1989), and 8 (Greenberg et al., 1978) were used for this mass ratio. In Fig. 4, we show an example of calculation of planetary growth using these three mass ratios. It strongly
suggests that the runaway growth in the early stages found by Greenberg et al., who used 8-ratio divisions, came from a numerical error.

In contrast to the fixed mass-coordinate algorithms mentioned above, Wetherill and Stewart (1989) represented the mass distribution with a number of moving "batches", each containing a large number of planetesimals of the same mass. Wetherill (1990) presented test calculations using this method which showed a better agreement with the analytic solutions than those obtained by Ohtsuki et al. (1990) using the $\sqrt{2}$-ratio.

Wetherill (1990) also tested the more difficult case for runaway growth ($\alpha(m,m') \propto mm'$), the analytic solutions in this case being found by Trubnikov (1971). Applying the Trubnikov's solution in this case, only to the continuous mass distribution of small planetesimals but not to the runaway protoplanet, Wetherill concluded that his algorithm can also reproduce the runaway case.

It is generally conjectured, however, that the ordinary coagulation equation (3) itself breaks down in the case of the runaway growth. In order to make clear when and how the coagulation equation should be replaced by the direct N-body calculation, Tanaka and Nakazawa (1991) derived an exact stochastic coagulation equation for the planetary growth. Although it is very difficult to solve, even numerically, these equations, and hard to see the general properties of the solutions, they obtained an important result. That is, these stochastic equations reduce to the familiar coagulation
equation (3) when the following two conditions are satisfied: (a) the correlation \( \langle n(m, t)n(m', t) \rangle \) can be described well by a product of individuals, \( \langle n(m, t) \rangle \langle n(m', t) \rangle \) (brackets mean the ensemble average) and (b) the number of planetesimals is very large (or more exactly, \( \langle n(m, t)n(m', t) \rangle \) is much greater than \( \langle n(m, t) \rangle \) for all combinations of \( m \) and \( m' \)). Especially, in the case of the runaway growth, both of the above conditions are hardly satisfied. If we use Eq.(3) even in this case, the growth of a protoplanet is artificially accelerated by the collisions with its “goast”. Thus, we should give up the idea that, by a suitable manner, we can simulate the runaway growth in the framework of the coagulation equation. We must develop a new method combining the ordinary statistical description for a large number of small planetesimals and the N-body calculation for a small number of massive protoplanets.

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