Accumulation of Materials for the Formation of the Giant Planets
– Ring Model under the Flow-out Motion of Disc Gas –

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1. Introduction

The proto solar system was born in the proto cluster of the solar nebula that has the total mass of about 1000$M_\odot$, where $M_\odot$ is the present solar mass (Spitzer, 1963). When the material of 1000$M_\odot$ is concentrated within a distance of 0.1pc, the gas can collapse into the unit region forming the origin of the present sun.

The concentration of large mass of the proto-solar system has been swept out after the initiation of the nuclear fusion in the center of the nebula. The sweeping up process of the material is not well understood in the previous studies of the origin of the solar system. It is also the question how and when the giant planets start to concentrate at the present positions under the condition of the ejecting motion of the material in the nebula.

In this study we propose an origin of the mass concentration process of the giant planets under the condition of the outwards flow of the nebula material after initiation of the proto-sun. The process of the star formation has been reviewed by Spitzer (1963) where he divided the formation periods into four stages; as i) The stage of the formation of an interstellar cloud in a region of the radius about 20pc with mass of $10^4M_\odot$. ii) The stage of the collapse and fragmentations. The mass of 1000$M_\odot$ within a region of 0.2pc can start collapse into the mass density of the present sun in the center
of the nebula. iii) Stage of Helmholtz contraction of proto-sun where the radiation pressure of the infrared resists to simple free falls of the material increasing the gas temperature to 1500°K to 2000°K. iv) Final collapse into the sun. After passing through the period of Helmholtz contraction there appeared nucleus region fragmented into the large clouds that became the proto-sun and other similar stars.

The period of the study of the present work is related to the succeeding period after above mentioned stage iv) where the proto-sun entered into the stage of the bipolar flow stage that has recently been clarified before stage of T-Tauri (Kawabata et al., see this issue chapter 1). Even for T-Tauri star phase, Hayashi made a proposal of the large convective situation at the surface. In this rather chaotic situation of the gas surrounding the proto-sun, Spitzer had already suggested the mass loss processes from the center part after the start of the nuclear fusion of the proto-sun. The observations of bi-polar flow then definitely suggest the mass-loss processes even in the stage earlier than T-Tauri phase.

In the present paper, we have studied the origin of giant planets for the initial phase when the seed material is accumulated in the out flowing gas which making disc with high density. We have here a hypothesis that the nebula gas were accelerated by the rotating magnetic field in the portion of the center of the cloud, that can possibly be formed within the distance of the present Mercury. The gas starts moving in the direction perpendicular to the rotation axis of the main magnetic field generated by the dynamo action of the proto-sun due to the electro-magnetic effects that take place in the rotating magnetized planets (Oya and Aoyama, 1985). There are possibly complicated change of states in the flow-out processes of gas as the plasma is neutralized while flowing out because of the cooling and rather dense situation of gas. Our present studies are concentrated to the phase when the initial accumulation processes of the material were taking place in the disc region of the out-flow gas of the proto-solar system. The study starts from the MHD equation where only changes of arguments takes place in radial direction with the ring symmetry in azimuthal direction for the growth of the density waves. We propose then mechanism of the growth of the fluctuation that enhance the stationary components of the mass concentration through the stages of weak turbulence at positions so as to satisfy Titus-Bode’s law. The growth of the stationary density waves became the
seeds of the giant planets accumulating the material at each point of the present giant planets. To keep the consistency of the constant out flow of the background gas the large viscous like interaction is needed; this viscous interaction is however cannot be achieved by usual viscosity of fluid. While the gas makes flow out with high speed, the neutral particles are partly ionized due to the velocity shear between the stational component gas. The ionization of the gas causes the interaction with the rotating magnetic field even in long distant place from the proto-sun and gives the angular momentum into the gas in the cloud. We considered that this rotation is the origin of the transport of the angular momentum and also the cause of the escape of the material out of the proto solar nebula.

2. Model of the proto-solar nebula

The proto-clusters that was the parent clouds of the stars have been estimated to have $10^3M_\odot$, at least, to achieve the condition of the initial contraction of the gas (Spitzer, 1963). When the cluster continued to collapse, the cloud had possibly been fragmented into the mass size of the present sun when the cluster size was diminished to be less than 0.1pc. In some stellar system (e.g., the case of $\alpha$-Centauri) there are well known three members of stars within 0.3pc. In general, there are also many multiple stellar systems. However, the sun is one of the cases where the mass concentrations has been taken place only into a main star within the proto-cluster with size less than 0.1pc.

Here we have made a hypothetical model for the proto-solar nebula with respect to the stage of bi-polar flow where the proto-sun had started the mass loss process. In the nucleus region of the mass condensation in the proto-cluster, there might be generated a star with the mass of $4M_\odot$ and the luminosity of $1000L_\odot$ (Cameron, 1963), where $L_\odot$ is the luminosity of the present sun. The nucleus star might be dressed by the large volume of cloud with average density of $10^{-3}g/cm^3$ that was extended to the range of the present Mercury where the number density is $2.5\times10^{20}/cm^3$. The dense atmosphere of the proto-sun is here called the proto-solarsphere. This stage of the dense and turbulent cloud of the dressed gas with the total mass of the nearly $4M_\odot$ are heated by the nucleus star and the surface temperature at the position of Mercury might be approached to 3000°K.
Though it has not yet been established, we have the following scenario for the plasma out flow processes: Inside of the proto-solarsphere, there is a threshold region where the temperature becomes higher than $10^5$ °K and all of the hydrogen and helium clouds are completely ionized. There is the possible magnetic field emanating from the nucleus region that had been rotating with high speed with rotation of the proto-sun. Owing to this set up of the plasma situation the high density plasma is accelerated outward being enforced by $\mathbf{I} \times \mathbf{B}$ force where $\mathbf{I}$ and $\mathbf{B}$ the current density and magnetic field. The acceleration process has been applied for the case of the Jovian planetary wind (Oya and Aoyama, 1985). The outflowing plasma forms disc gas after recombination to neutral gas in a cool region in the equatorial plane apart from the proto-sun. We assume that a high speed and dense gas flow from the proto solar sphere. Because the surface of the proto solar sphere is smoothly connected to the gas media of the proto solar nebula, high rate of the mass flow can be assumed as given in Table I.

We separate the proto solar nebula into two regions here; \textit{i.e.}, i) the inner nebula that can be defined within the radius of the present asteroid belts centered at the sun and ii) the outer nebula in the region outside of the position of the present asteroid belts. As we can see in the distribution of the planets of the solar system given in Fig. 1, there is a break point in the distribution of the planets at a point corresponding to the asteroid belt; \textit{i.e.}, the ratio of the distance $r_{m+1}/r_m$ defined between the planets at the order of $m$-th and $(m+1)$-th from the sun for an arbitrary integer $m$, is 1.79 for the giant planets while the value is 1.50 for the terrestrial type planets. If the rate $r_{m+1}/r_m$ is related to the mechanism of the mass concentration, it

<table>
<thead>
<tr>
<th>Number Density at 4 AU</th>
<th>Flow Out Velocity (km/sec)</th>
<th>Duration Time for Loosing $3M_\odot$ (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{13}$/cc</td>
<td>10</td>
<td>$6.7 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$6.7 \times 10^4$</td>
</tr>
<tr>
<td>$10^{12}$/cc</td>
<td>10</td>
<td>$6.7 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$6.7 \times 10^5$</td>
</tr>
<tr>
<td>$10^{11}$/cc</td>
<td>10</td>
<td>$6.7 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>$6.7 \times 10^6$</td>
</tr>
</tbody>
</table>

Table I. Density, flow out velocity, and duration time of models.
Fig. 1. Logarithmic plots of the distance $r_m$ of the $m$-th planets from the sun. For a constant $c$, ($c = r_{m+1}/r_m$) the plot shows a straight line; the break point of two straight lines fitted to the giant planets and to the terrestrial planets show apparent change of the $r_{m+1}/r_m$ value between these two groups suggesting the difference of the formation processes of plante.

is reasonable to consider the different situations of the formation processes between the terrestrial type planets and giant planets. The studies are in this paper therefore mainly focused on the mass concentration processes of the giant planets in the outer solar nebula.

In the middle part of the nebula with distance less than 100AU from the central body, the gas might be making the high speed rotation where the centrifugal force is well balanced with the gravity and finally overcomes the gravity to make the gas escape out from the solar system. To obtain the rotation of the nebula, we could not avoid the consideration of the interaction effects between the plasma inferred to be produced in the flowing gas by interaction with turbulent states with the intense magnetic field emanating from the proto-sun. But this plasma effect was carried out only in partially ionized stage where almost majority of gas indicate the neutral stage. For this situation, the velocity within the range of several km/s to several 10km/s at the position of proto Jupiter can escape from the region
of a large concentration of mass of $4M_\odot$.

In the present studies, we concentrate the stage of the homogeneous circulation motion in the azimuthal direction in the gas cloud; and the studies are focused on the growth and evolution of the density fluctuation in the radial direction. Therefore the density fluctuation is taking place in the disc making form of rings. In Fig. 2, an illustration of the frame of the present studies are given; for this frame the fluid dynamics in the disc with convection motion in the radial direction is solved as will be given in Section 3. We have theoretically treated the processes of trapping of the outward flow of the gas due to the generated density waves in the disc. The density waves grown in the disc generates stationary waves by coupling of the density waves with the velocity fluctuation through quasi-linear processes.

![Diagram](image)

Fig. 2. Model of the disc under the condition of the flow-out gas.

The deposited material of 1% to the background density of the nebula gas is still sufficient as the source material of the giant planets even only another percent out of the deposited material is concentrated to final formation of the proto-planets.

3. Basic equations

The governing equations of the proto solar nebula under the conditions of the gas out flow can be expressed by the equation of the continuity, and
the equation of the dynamics of the fluid with the effects of the gravity field which is caused not only by the mass concentration of the proto-sun but also by the local mass distribution in the nebula. That is,

\[
\frac{\partial N}{\partial t} + \nabla (N \mathbf{V}) = 0
\]  
(1)

and

\[
Nm_0 \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -Nm_0 \nabla \phi - \nabla (N \kappa T)
\]  
(2)

where \( \kappa \) is the Boltzmann constant, \( m_0 \) and \( N \) are the average mass and the number density of the nebular particles, respectively, and others are the usual meanings. The gravitational potential \( \phi \) is governed by the Poisson equation,

\[
\nabla^2 \phi = 4\pi G Nm_0.
\]  
(3)

The above equations (1) to (3) are basically nonlinear equations. We apply, then, a linearized approach extending to the quasi-linear regime for the continuity equation. That is, by taking perturbations (quantities with suffix-1) around the quasi-equilibrium quantities (with suffix-0) as

\[
N = N_0 + N_1,
\]  
(4)

\[
\phi = \phi_0 + \phi_1,
\]  
(5)

and

\[
\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1.
\]  
(6)

We can separate the basic equation into the following set of linear and quasi-linear combination of the equations. The equilibrium equations are expressed for the dynamic equation as
\[-m_0 N_0 (\mathbf{V}_0 \times \text{rot} \mathbf{V}_0) + \frac{1}{2} m_0 N_0 \nabla V_0^2 + m_0 N_0 \nabla \phi_0 + \nabla (N_0 \kappa T) = 0. \] (7)

The background density of the nebula was modulated by the generated density wave in the solar nebula. To provide suitable expressions for this procedure we employ here a quasi-linear treatment for the continuity equation as

\[
\frac{\partial N_0}{\partial t} + \nabla (N_0 \mathbf{V}_0) = -\nabla (N_1 \mathbf{V}_1). \] (8)

The linearized equations for the first order perturbation is then expressed as

\[
\frac{\partial N_1}{\partial t} + \nabla (N_0 \mathbf{V}_1) + \nabla (N_1 \mathbf{V}_0) = 0, \] (9)

\[
N_0 m_0 \frac{\partial \mathbf{V}_1}{\partial t} - m_0 \{ N_0 (\mathbf{V}_0 \times \text{rot} \mathbf{V}_1) + N_0 (\mathbf{V}_1 \times \text{rot} \mathbf{V}_0) \} + N_0 m_0 \nabla (\mathbf{V}_0 \cdot \mathbf{V}_1) \] (10)

\[+ \frac{1}{2} m_0 N_1 \nabla V_0^2 + \kappa T \nabla N_1 + m_0 N_0 \nabla \phi_1 + m_0 N_1 \nabla \phi_0 = 0,
\]

and

\[-\nabla^2 \phi_1 = 4\pi G m_0 N_1. \] (11)

4. Background density

We treat here a disc of the constant thickness that is initiated at the edge of 0.38AU from the center of the proto-sun, i.e., at the point very close to the position of the present Mercury, the outward flow with the high density of $10^{13} \sim 10^{14}/\text{cm}^3$ starts to flow out from this edge in radial direction. Here we employ a cylindrical coordinate system $(r, \theta, z)$ whose $z$-axis coincides with the rotation axis of the proto-sun; the origin of the coordinate
is taken at the proto-sun. The equilibrium density \( N_0(r, t) \) is assumed only to change in radial direction so as to satisfy the basic equation Eq.(8), and gradually changes in time due to the quasi-linear effects that has also been given in Eq.(8). As will be discussed in Section 5, the underlying waves has unique nature that the replacement of the signs of \( \omega \) and \( k \) changes the dispersion relation itself. Therefore we cannot have the wave with the phase of \((\omega_a t + \int k_a dr)\) together with the wave with the phase \((\omega_a t - \int k_a dr)\), \((-\omega_a t + \int k_a dr)\) or \((-\omega_a t - \int k_a dr)\). Considering this point, we define here, all kinds of waves with phase \( \varphi \) treated here, as

\[
A = \text{Re}\{e^{i\varphi}\},
\]

instead of

\[
A = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi}).
\]

In this context, we express the density \( N_0 \) as

\[
N_0 = N_{00}\text{Re}\{1 + a_+^*(t)e^{i\Theta} + a_-^*(t)e^{-i\Theta}\} \frac{r_0}{r},
\]

where \( N_{00} \) is a constant and \( a_+^*(t) \), \( a_-^*(t) \) and \( \Theta(r) \) are the functions of the time \( t \) and the position \( r \).

The selection of Eq.(14) and \( a_+^*(t) \), \( a_-^*(t) \) and \( \Theta(r) \) is not necessarily unique but they should satisfy the relation

\[
\frac{\partial N_0}{\partial t} + V_0 r \frac{\partial N_0}{\partial r} + N_0 \frac{1}{r} \frac{\partial}{\partial r} (rV_0 r) + \frac{1}{r} \frac{\partial}{\partial r} (rN_1 V_1 r) = 0,
\]

which is reduced from Eq.(8), for the radial component of the velocity. We therefore have selected a solution given in Eq.(14) with

\[
\Theta = \int_{r_0}^{r} k_B dr
\]
with \( k_B = k_0 (r_0 / r) \), for constants \( k_0 \) and \( r_0 \). To satisfy Eq.(15), the quantities \( N_0 \) in Eq.(14) with Eq.(16) give constraint as

\[
\frac{\partial a_+^*}{\partial t} + i k_B V_0 r a_+^* + V_0 r \frac{\partial a_+^*}{\partial r} = \left( \frac{-1}{N_0 r_0} \right) \frac{\partial}{\partial r} (r [N_1 V_{1r}]_+) \tag{17}
\]

and

\[
\frac{\partial a_-^*}{\partial t} - i k_B V_0 r a_-^* + V_0 r \frac{\partial a_-^*}{\partial r} = \left( \frac{-1}{N_0 r_0} \right) \frac{\partial}{\partial r} (r [N_1 V_{1r}]_-). \tag{18}
\]

Terms \([N_1 V_{1r}]_+\) and \([N_1 V_{1r}]_-\) are defined for the term \( \partial (r N_1 V_{1r}) / r \partial r \) in Eq.(15) as

\[
\frac{1}{r} \frac{\partial}{\partial r} (r N_1 V_{1r}) = \frac{1}{r} \frac{\partial}{\partial r} [N_1 V_{1r}]_+ e^{i \Theta} + \frac{1}{r} \frac{\partial}{\partial r} [N_1 V_{1r}]_- e^{-i \Theta}. \tag{19}
\]

To solve Eqs.(17) and (18), the Laplace and Fourier transformation methods are applied; i.e., for \( a_+^* \)

\[
\tilde{a}_+^* = \int_0^\infty dt \int_{-\infty}^\infty dr a_+^*(t, r) e^{-(st - i kr)}, \tag{20}
\]

with the inverse transformation of

\[
a_+^* = \frac{1}{(2\pi)^2 i} \int_{\sigma - i \infty}^{\sigma + i \infty} ds \int_{-\infty}^\infty dk \tilde{a}_+^*(s, k) e^{(st - i kr)}. \tag{21}
\]

For \( a_-^* \) we can also use Eqs.(20) and (21); applying Eqs.(20) and (21) to Eqs.(17) and (18), it follows that

\[
a_+^* = \frac{-A_+}{(2\pi)^2 i} \int_{-\infty}^\infty dk \int_{\sigma - i \infty}^{\sigma + i \infty} ds \left. e^{(st - i kr)} \right|_{k \{s + i(kB V_{0r} + kV_{0r})\} \{s - \gamma_+\}}, \tag{22}
\]
and

\[ a_+^* = \frac{-A_-}{(2\pi)^2 i} \int_{-\infty}^{\infty} dk \int_{\sigma i \infty}^{\sigma + i \infty} ds \frac{e^{(st - i kr)}}{k(s + i(-k_B V_{0r} + k V_{0r}))} \{s - \gamma_+\} \]  \hspace{1cm} (23) 

In Eqs. (22) and (23), \( A_+ \), \( A_- \), \( \gamma_+ \) and \( \gamma_- \) are defined in the following expressions; \( i.e., \)

\[ [N_1 V_{1r}]_+ = [N_{10} V_{1r0}]_+ e^{\gamma_+ t}, \]

\[ [N_1 V_{1r}]_- = [N_{10} V_{1r0}]_- e^{\gamma_- t}, \]

\[ A_+ = [N_{10} V_{1r0}]_+ e^{i\varphi_+} / N_{00} r_0, \]  \hspace{1cm} (24) 

and

\[ A_- = [N_{10} V_{1r0}]_- e^{-i\varphi_+} / N_{00} r_0, \]

where \( \varphi_+ \) and \( \varphi_- \) are phases corresponding to \( e^{i\Theta} \) and \( e^{-i\Theta} \) terms. Detailed expressions for \( \gamma_+ \) and \( \gamma_- \) will be given in the next section. After mathematical manipulation, Eqs. (22) and (23) with Eq. (24) give solution

\[ a_+^* = \frac{A_+}{2} \cdot \frac{k_B V_{0r} + i \gamma_+}{(k_B V_{0r})^2 + \gamma_+^2} \{e^{-ik_B V_{0r} t} - e^{\gamma_+ t}\} \]  \hspace{1cm} (25) 

and

\[ a_-^* = \frac{A_-}{2} \cdot \frac{(-k_B V_{0r}) + i \gamma_-}{(k_B V_{0r})^2 + \gamma_-^2} \{e^{ik_B V_{0r} t} - e^{\gamma_- t}\}. \]  \hspace{1cm} (26) 

For consideration of the quasi-equilibrium density given by Eq. (14), time averaged value for \( N_0 \) during an interval \( T \) becomes suitable expression; then \( T \) is set to be \( T > 1/k_B V_{0r} \) to have the following equation:
\[ a_+^*(t)e^{i\Theta} + a_-^*(t)e^{-i\Theta} = \frac{1}{T} \int_0^T \{ a_+^*(t)e^{i\Theta} + a_-^*(t)e^{-i\Theta} \} dt. \] (27)

From Eqs. (25) and (26), therefore, \( N_0 \) is expressed by

\[ N_0 = N_{00} \left[ 1 + \left\{ b_1 e^{\gamma_1 t} \cos(\Theta + \varphi_+) + b_2 e^{\gamma_1 t} \cos(\Theta - \varphi_-) \right\} \right] \frac{r_0}{r}. \] (28)

where

\[
\begin{align*}
    b_1 &= -\frac{A_+}{2} \frac{1}{\sqrt{(k_B V_0)^2 + \gamma_+^2}}, \\
    b_2 &= \frac{A_-}{2} \frac{1}{\sqrt{(k_B V_0)^2 + \gamma_+^2}}, \\
    \varphi_+ &= \tan^{-1}(\gamma_+/k_B V_0), \\
    \varphi_- &= \tan^{-1}(\gamma_-/k_B V_0).
\end{align*}
\] (29)

and

\[ \varphi_b = \tan^{-1}\left( \frac{\cos \varphi_- + (b_1/b_2)e^{(\gamma_+ - \gamma_1)t} \cdot \cos \varphi_+}{\sin \varphi_- - (b_1/b_2)e^{(\gamma_+ - \gamma_1)t} \cdot \sin \varphi_+} \right). \] (32)

This equation is further rewritten as

\[ N_0 = N_{00} \left[ 1 + b^*(t) \sin(\Theta + \varphi_b) \right] \frac{r_0}{r}. \] (30)

with

\[ b^*(t) = b_1^2 e^{2\gamma_1 t} + b_2^2 e^{2\gamma_1 t} + 2b_1 b_2 e^{(\gamma_++\gamma_1)t} \cdot \cos(\varphi_+ + \varphi_-), \] (31)
The phase $\varphi_b$ becomes constant within an early phase of the evolution time because the exponential variation with function $e^{(\gamma_+ - \gamma_-)t}$, $\varphi_b$ approaches as $\varphi_b = \pi/2$ for $\gamma_+ - \gamma_- < 0$ and as $\varphi_b = \frac{3}{2} \pi$ for $\gamma_+ - \gamma_- > 0$.

The expression given by Eq.(30) shows that the accumulation component $b^*(t)$ makes exponential growth with time constant $\gamma_+$ and $\gamma_-$. When the underlying density waves with $N_1$ and $V_{1r}$ make growth, as that will be discussed in the next section.

5. Linearized formulae and the solutions

In the local point at a given distance $r_s$ from the center the linearized formulae with the perturbations for Eqs.(9), (10) and (11), are expressed in the same cylindrical coordinate system as follows; the component equations are:

\[
\begin{align*}
\frac{\partial V_{1r}}{\partial t} + V_{0r} \frac{\partial V_{1r}}{\partial r} + \frac{\partial \phi_1}{\partial r} + \left( \frac{\kappa T}{N_0 m_0} \right) \frac{\partial N_1}{\partial r} &+ \left( \frac{1}{N_0} \frac{\partial \phi_0}{\partial r} \right) N_1 = 0, \\
N_0 \frac{\partial V_{1r}}{\partial r} + \left( \frac{N_0}{r} + \frac{\partial N_0}{\partial r} \right) V_{1r} + \frac{\partial N_1}{\partial t} &+ V_{0r} \frac{\partial N_1}{\partial r} + \left( \frac{V_{0r}}{r} + \frac{\partial V_{0r}}{\partial r} \right) N_1 = 0, \\
\frac{\partial^2 \phi_1}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_1}{\partial r} - 4\pi G m_0 N_1 &= 0.
\end{align*}
\]

By taking Fourier transformations for $V_{1r}$, $\phi_1$, and $N_1$ with angular frequency $\omega$ and wave number $k$, it follows that

\[
\begin{align*}
\left( -i\omega + i k V_{0r} \right) \tilde{V}_{1r} + i k \tilde{\phi}_1 + \left( \frac{V_{0r}^2}{2N_0} \right) i k \tilde{N}_1 &- \left( \frac{V_{0r}^2}{2N_0^2} \frac{\partial N_0}{\partial r} \right) \cdot \tilde{N}_1 = 0, \\
\left( -i\omega + i k V_{0r} + \left( \frac{V_{0r}}{r} \right) \right) \tilde{N}_1 &+ N_0 \left( i k + \frac{1}{r} + \frac{1}{N_0} \frac{\partial N_0}{\partial r} \right) \tilde{V}_{1r} = 0, \\
\left( -k^2 + \frac{i k}{r} \right) \tilde{\phi}_1 - 4\pi G m_0 \tilde{N}_1 &= 0.
\end{align*}
\]
where $\tilde{V}_{1r}$, $\tilde{\phi}_1$ and $\tilde{N}_1$ are Fourier transformed quantities corresponding to $V_{1r}$, $\phi_1$, and $N_1$; $V_{th}^2$ is given by $V_{th}^2 = \kappa T/m_0$.

Then by eliminating $\tilde{\phi}_1$ in Eq.(33) it follows that

\[
\left\{ 4\pi Gm_0 i k + \frac{V_{th}^2}{2N_0} \left( ik - \frac{1}{N_0} \frac{\partial N_0}{\partial r} \right) \left( -k^2 + \frac{ik}{r} \right) \right\} \tilde{N}_1 \\
+ \left( -k^2 + \frac{ik}{r} \right) (-i\omega + ikV_{0r}) \tilde{V}_{1r} = 0.
\]

(35)

To obtain non-trivial solution, then, we have to solve the matrix equation as

\[
\begin{vmatrix}
- \left( k^2 + \frac{ik}{r} \right) & (-i\omega + ikV_{0r}) & 4\pi Gm_0 ik + \left( -k^2 + \frac{ik}{r} \right) \\
N_0 \left( ik + \frac{1}{r} + \frac{1}{N_0} \frac{\partial N_0}{\partial r} \right) & 2N_0 \left( ik - \frac{1}{N_0} \frac{\partial N_0}{\partial r} \right) & -i\omega + ikV_{0r} + \frac{V_{0r}}{r}
\end{vmatrix} = 0.
\]

(36)

This gives the equation for $\omega$ as

\[
a\omega^2 + b\omega + c = 0
\]

(37)

with

\[
a = \frac{k^*}{\omega_s^2} - i \left( \frac{1}{\omega_s^2 r^*} \right),
\]

(38)

\[
b = \left\{ \frac{1}{\omega_s} (-2k^* + \frac{1}{r^*}) \right\} + i \left( \frac{3k^*}{\omega_s r^*} \right).
\]

(39)
and
\[
c = \left\{1 - \frac{\eta^2}{2}(1 + \alpha^2)\right\} k^*^3 + \left(\frac{\alpha \eta^2}{2r^*}\right) k^*^2 + \frac{1}{r^*} \left(r^*^2 \zeta^2 + \frac{\eta^2}{2} - 1\right) k^*
\]
\[+ i \left[ \frac{1}{r^*} \left\{(2 + \alpha^2) \frac{\eta^2}{2} - 1\right\} k^*^2 - \frac{\alpha k^*}{r^*^2} \left\{\zeta^2 r^*^2 + \frac{\eta^2}{2}\right\}\right]. \quad (40)
\]

In the complex constants \(a\), \(b\), and \(c\), new symbols denote the following quantities:
\[
\begin{align*}
k^* &= kr_0, \\
r^* &= r/r_0, \\
\omega_s &= \frac{V_{0r}}{r_0}, \\
\eta &= \frac{V_{th}}{V_{0r}}, \\
\zeta &= 4\pi G m_0 N_0 r_0^2 / V_{0r}, \\
\alpha &= \frac{b^*(t) \cos(\Theta + \varphi_b)}{1 + b^*(t) \sin(\Theta + \varphi_b)}.
\end{align*}
\quad (41)
\]

From the dispersion relation, Eqs. (36) and (37), we can find the possible \(\omega-k\) relation of the existing underlying waves; the real part of angular frequency is related to the wave number \(k^*\) that is included in Eqs. (38) to (40) giving
\[
\omega = \frac{1}{2a} \left\{-b \pm \sqrt{b^2 - 4ac}\right\} \quad (42)
\]
for \(a\), \(b\), and \(c\) given in Eqs. (38), (39) and (40), respectively.

The dispersion curves corresponding to Eq. (42) are given in Figs. 3(a) to (d) for the conditions \(N_M = 10^{13}/\text{cm}^3\) at the position of 0.38AU \((N_J = 7.69 \times 10^{12}/\text{cm}^3\) at the point of the present Jupiter), and for the solar wind velocity of 10km/s. The results in Figs. 3(a) to (d) is calculated with 1% of the fluctuation to the back ground density, \(i.e.,\) for \(a^* = 1/100.\)
The imaginary part of the obtained mode indicates a positive value for \( \exp(-i(\omega t - kr)) \); this means the continuous growth of the waves. The growth rate for the out flowing velocity of 10km/s is fairly fast giving the time constant of around a few tens of year for the growth of the stationary density waves, as has been given in Table II.

![Diagram](image)

Fig. 3(a). Dispersion curves, \( \omega_T-k \) with the growth rate \( \gamma \). Normalized wave number is given for \( kr_0 \) selecting \( r_0 \) to be 4AU for temperature \( T = 1000^\circ \text{K} \), the flow out velocity \( V_0 = 5 \text{km/s} \), with the background density \( N = 7.3 \times 10^{11} / \text{cm}^3 \) corresponding to the position of Jupiter. Thick curve, curve with two dots chain and dotted curves respectively indicate positive real, negative real and imaginary parts \( \gamma \) of \( \omega \). Positive \( \gamma \) shows the growth of the waves.

6. Steady growth of the density waves

The proto-solar nebula gas that moves out forming the disc wind car-
ries out the material outwards. The accumulation of material inside solar system disc is therefore essentially related to the stationary state of the perturbations with respect to the proto-sun. Formation of the stationary waves in the solar system in the linear regime is impossible as we can see $\partial \omega / \partial k$ values from the dispersion relation given in Figs. 3(a) to (d); i.e., we cannot find any point where $\partial \omega / \partial k = 0$. Around the point $k_{0R}$ where $\omega = 0$, however, we can find a special state where non-linear coupling of two waves satisfies the condition of the stationary perturbations that make the steady growth.

As has been given by Eqs. (14) and (20), the coupling term $\partial(rN_1V_{1r})/\partial r$ is the source of the accumulation of the material to make the steady growth.
of the density waves. The possible solution of $N_1$ and $V_{1r}$ that can give the effective coupling conditions are then expressed by

$$
N_1 = N_{1a} \exp\{\gamma_a t + i(-\omega t + \int k_a dr)\} \\
+ N_{1b} \exp\{\gamma_b t + i(\omega t + \int k_b dr)\} \\
+ N_{1c} \exp\{\gamma_c t + i(-\omega t - \int k_a dr)\} \\
+ N_{1d} \exp\{\gamma_d t + i(\omega t - \int k_b dr)\}
$$

(43)
Fig. 3(d). Same format with Fig. 3(a) for \( N = 1.26 \times 10^{11}/\text{cm}^3 \) corresponding to the position of Neptune.

\[
V_{1r} = V_{1ra} \exp\{\gamma_{a} t + i(\omega t + \int k_{a} dr + \varphi_{a})\} \\
+ V_{1rb} \exp\{\gamma_{b} t + i(\omega t + \int k_{b} dr + \varphi_{b})\} \\
+ V_{1rc} \exp\{\gamma_{c} t + i(-\omega t - \int k_{a} dr + \varphi_{c})\} \\
+ V_{1rd} \exp\{\gamma_{d} t + i(\omega t - \int k_{b} dr + \varphi_{d})\}. 
\]

(44)

When we select only the quasi-stationary or stationary solutions, for the
time and the space by substituting Eqs. (43) and (44) into Eqs. (17) and (18) with Eq. (19), it follows that

\[
N_1 V_{1r} = (N_{1a} V_{1rb} e^{i\varphi_b} + N_{1b} V_{1ra} e^{i\varphi_a}) \exp\{(\gamma_a + \gamma_b)t + i\Phi\} \\
+ (N_{1c} V_{1rd} e^{i\varphi_d} + N_{1d} V_{1re} e^{i\varphi_c}) \exp\{(\gamma_c + \gamma_d)t - i\Phi\} \\
+ (N_{1a} V_{1rd} e^{i\varphi_d} + N_{1d} V_{1ra} e^{i\varphi_a}) \exp\{(\gamma_a + \gamma_d)t + i\Theta\} \\
+ (N_{1b} V_{1rc} e^{i\varphi_c} + N_{1c} V_{1rb} e^{i\varphi_b}) \exp\{(\gamma_b + \gamma_c)t - i\Theta\},
\]

(45)

where

\[
\Phi = \int (k_a + k_b)dr
\]

(46)

and

\[
\Theta = \int (k_a - k_b)dr.
\]

(47)

When we select shorter wavelength case as given in Eq. (47) \([N_{10} V_{1r0}]+, [N_{10} V_{1r0}]-, \varphi_+\) and \(\varphi_-\) in Eqs. (24); and (29) can be expressed as

\[
[N_{10} V_{1r0}]_+ = \sqrt{(N_{1a} V_{1rd})^2 + (N_{1d} V_{1ra})^2 + 2N_{1a} N_{1d} V_{1rd} V_{1ra} \cos(\varphi_a - \varphi_d)},
\]

(48)

\[
[N_{10} V_{1r0}]_- = \sqrt{(N_{1b} V_{1rc})^2 + (N_{1c} V_{1rb})^2 + 2N_{1b} N_{1c} V_{1rc} V_{1rb} \cos(\varphi_b - \varphi_c)},
\]

(49)

\[
\varphi_+ = \tan^{-1}\left(\frac{N_{1a} V_{1rd} \sin \varphi_d + N_{1d} V_{1ra} \sin \varphi_a}{N_{1a} V_{1rd} \cos \varphi_d + N_{1d} V_{1ra} \cos \varphi_a}\right),
\]

(50)

and

\[
\varphi_- = \tan^{-1}\left(\frac{N_{1b} V_{1rc} \sin \varphi_c + N_{1c} V_{1rb} \sin \varphi_b}{N_{1b} V_{1rc} \cos \varphi_c + N_{1c} V_{1rb} \cos \varphi_b}\right).
\]

(51)
The solution shows a procedure of the accumulation of the material due to the coupling of the density waves $N_1$ and the velocity waves $V_{lr}$ around extremely low characteristic frequency. The growth rate of $\gamma_+$ and $\gamma_-$ for the accumulation of the density is given in Table II for the positions at the present giant planets. The value is almost around 30yrs. In Fig. 4, the density $N_0$ where the level of $b^*(t)$ component makes growth to 40%, as stationary components, with respect to the back ground flowing density is indicated. After this period, the situation of the density distribution will enter the strong nonlinear stage.

<table>
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<tr>
<th>Position</th>
<th>Characteristic time (year)</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>0.235</td>
</tr>
<tr>
<td>S</td>
<td>0.644</td>
</tr>
<tr>
<td>U</td>
<td>0.861</td>
</tr>
<tr>
<td>N</td>
<td>1.152</td>
</tr>
</tbody>
</table>

7. Importance of the local magneto-plasma effects

The present studies are restricted in the homogeneous ring system. For the constant out flow of the gas through the proto solar system, conditions of the constant azimuthal velocity should be assumed in the ring system. That is, for the equilibrium condition in Eq.(7) it is written that

$$N_{0a}m_0 \left\{ \frac{1}{2}(V_{0r}^2 + V_{0\theta}^2) \right\} + \phi_0 + \kappa TN_{0a} = 0. \quad (52)$$

In this model, the equilibrium quantities are given as

$$\begin{align*}
N_{0a} &= N_{00}(r_0/r), \\
V_{0r} &= \text{const.}, \\
\phi_0 &= -\phi_{00}(\frac{r_0}{r}), \\
\kappa T &= \text{const.}
\end{align*} \quad (53)$$
Fig. 4. Example function of \( r_0 \{1 + b^* \sin(\Theta + \varphi)\}/r \) for \( b^* = 0.4. \ k r_0 = 10.75, \) and \( \varphi = 0. \) The curves correspond to a stage of the growth of stationary component \( b^*(t) \) as the results of quasi-linear processes between \( N_1 \) and \( V_1r. \) Each maximum portion coincides with the point satisfying Titus-Bode's law with \( r_{m+1}/r_m = 1.79. \)
This gives constraint to $V_{\theta 0}$ as

$$V_{\theta} = \sqrt{\frac{r_0}{r}} V_{0\theta}. \quad (54)$$

There is an apparent discrepancy between Eq.(53) and the $r$ dependence that resulted from the Euler equation; i.e., from Eq.(7) it follows that

$$\rho \frac{1}{r} \frac{\partial}{\partial r} (r V_{\theta}) = 0. \quad (55)$$

This gives the condition

$$V_{\theta} = (\frac{r_0}{r}) V_{0\theta} \quad (56)$$

For the frictional media with viscosity $\eta$ we can rewrite the effects as

$$\frac{1}{r} \frac{\partial}{\partial r} (r V_{\theta}) = \frac{1}{R} \left\{ \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} (r V_{\theta}) \right) - \left( \frac{r V_{\theta}}{r} \right) \right\} + \frac{\partial^2 (r V_{\theta})}{\partial z^2}, \quad (57)$$

where $R$ is the Raylods number that is defined by

$$R = r_e V_r/(\eta/\rho) \simeq r V_r/(\eta/\rho), \quad (58)$$

for the characteristic length $r_e$. Though the precise $R$ number cannot be obtained, we can estimate, by extending the gas dynamic of the regular state gas, high $R$ number ranging from $10^9$ to $10^{12}$ for $r_e$ with order of 1AU with flow velocity of the order of 10km/s. Therefore, the expression of Eq.(57) is almost the same expression with Eq.(55). To avoid this difficulty, it is required to have large equivalent $\eta$ value, or alternate special situations. One of the most possible situation is related to the effect of the
electromagnetism underlying in the flow out processes of gas being caused by the partial ionization of plasma. The ionization may be possibly due to in the processes of interactions of the flowing components of gas with the stationary components. When the moving gas encounter the stational component of gas the speed of the gas which exceeds 60km/s provides the velocity for the ionization of the hydrogen. Though the radial velocity of 10km is lower than this critical velocity, the superposition of the rotation of the gas in the solar nebula may result in the ionization. In this partial ionized gas region, a balance may be raised as

$$ \rho_0 \left( \frac{r_0}{r} \right) V_r \left( \frac{V_\theta}{r} \right) = I_r B_z, \quad (59) $$

where $I_r$ and $B_z$ are the locally generated radial current and the penetrating magnetic field, respectively. The penetrated magnetic field is possibly extended in the disc region with the function given by

$$ B_z = \left( \frac{r_0}{r} \right)^2 B_{0z}. \quad (60) $$

The generation of the local current $I_r$ depends on the local plasma density that is also proportional to the gas density. If we assume the relation that

$$ I_r = I_{0r} \sqrt{r_0/r}, \quad (61) $$

the balance given by Eq.(59) becomes possible origin of the transport of the angular momentum providing the condition, given in Eq.(54), $i.e.$, the $r$ dependence of the azimuthal velocity is slower than the angular momentum conservation. For studying the fluctuation and turbulence in azimuthal component, to trace the evolution of the produced ring to grow into segments that becomes seeds to form the giant planets, investigations of instabilities in the azimuthal direction are needed for future studies.
8. Conclusion

The accumulation of the source material for formation of the giant planets are studied in the present paper. In the studies, we have presented a hypothesis that there was the condition of the high density in the disc of the proto-solar nebula that is formed in the period when the proto-sun entered into the stage of bi-polar flow before the T-Tauri star stage. Due to the high speed rotation of the magnetic field associated with the high temperature plasma inside area of the proto-sun, which is here called the proto-solarsphere, the gas makes out flow forming the disc. The fluid dynamics of the disc gas have been studied theoretically for the quasi-linear stage under the effects of the self-gravity, in the high density disc where the flow out of gas is dominated. The results show that in a fairly dense condition of the cloud with the number density of $10^{11} \sim 10^{12}/\text{cm}^3$ with the flow velocity around 10km/s the disc gas makes the instability of the density waves whose wave lengths are expanded as logarithmic function of the distance with time constant of about $10^8$ s ($\simeq 30$ yrs). When the density becomes tenuous the growth rate becomes slow. The underlying density waves in quasi-linear stage makes coupling in their flux $N_1V_1r$, which produces density perturbation generating density-gravity waves whose wave number depends logarithmically on the distance to satisfy Titus-Bode's law. After the growth of the waves in the initial phase of the process which can be described with the present quasi-linear theory, the evolution may will enter the stage of strong nonlinear process which can only be traced with computer simulation method. The analyses on the phase of the strong nonlinear processes are then deferred for future studies.

For constant flow out of gas, transfer of the angular momentum is needed. For this processes the effects of the magnetic field in the disc gas is important with the consideration of the process of the ionization of the gas for the interaction with the rotating magnetic field. The turbulent state of the plasma may be the possible origin of the ionization. The rotating magnetic field in contribute to transport effects of the angular momentum from the inside region that is called solarsphere in this paper. The theoretical studies including the rotation of the cloud should also be followed to clarify the actual evolution of the formation processes of the giant planets.
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References

