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Mathematical Modeling of Landforms: Optimality and Steady-State Solutions

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Abstract. The purpose of this paper is to first show how an Euler-Lagrange theorem can be applied to the steady-state fluvial landscape model, thereby allowing the geomorphic solution surfaces to be understood as those that globally optimize the difference between kinetic and potential energy dissipation (Hamilton's Principle) while conserving mass. It is then shown how this variational formulation of the steady-state problem makes it possible to exploit the well-known Ritz-Galerkin method for finding finite-element numerical solutions to partial differential equations. A "local equation" is derived that shows how elevations at a lattice node are related to the elevations of neighboring nodes. A variety of special solutions to this local equation are given that exhibit many of the at-a-point geometric features that characterize real fluvial landscapes, including peaks, ridges, saddles, hillslopes and forks. It is also shown that pits cannot occur as a solution to the local equation, so that numerical solutions will always be hydrologically sound.

Keywords: Fluvial Landscape Model, Steady-State Solutions, Optimality, Hamilton's Principle, Geomorphology

INTRODUCTION

More than three decades of work by many authors has resulted in a physically-based, mathematical model that can be used to model the time evolution of an idealized fluvial landscape. Despite fairly minor differences in formulation, there is widespread agreement on the basic components of this model, which will be referred to here as the standard model. A key feature of this model is that solution surfaces rapidly evolve from initial conditions to relatively stable dendritic channel networks. That is, after a (geologically) brief adjustment or transition period during which channels are formed and captured, the landscape enters into an "intermediate asymptotic" regime (Barenblatt, 1996; Smith *et al*., 1997, 2000) where the evolution is relatively slow and the branching structure of the river network is preserved as the elevations are gradually lowered. In previous work, Peckham (1995, 1999, 2003) has studied a steady-state version of the standard model in order to elucidate some of the relatively stable geometric forms that characterize this intermediate part of fluvial landscape evolution. In this paper, connections to optimality will be developed and explored in connection with a numerical solution scheme.

THE STANDARD MODEL

Contributions to the study of landscape evolution have been made by numerous authors over several decades, as reviewed by Peckham (2003). There is now considerable agreement on the basic set of equations that should be used to build a fluvial landscape model. The first two equations express conservation of mass for water and sediment

$$\nabla \cdot \underline{q} = R - d_t \tag{1}$$

$$\nabla \cdot \underline{q_s} = U - b_t \tag{2}$$

where $\underline{q}$ and $\underline{q_s}$ are 2D vector fields that give the unit-width discharge of water and sediment as a function of position in the landscape, R is the excess rainrate, U is the rate of tectonic uplift or the rate of soil production, d is the water depth and b is the landscape elevation. The second two equations are

$$\underline{q} = -q\left(\nabla f \,/\, S\right) \tag{3}$$

$$\underline{q_s} = -q_s\left(\nabla f \,/\, S\right) \tag{4}$$

and express the simple fact that the 2D flow direction is in the opposite direction of the water-surface gradient vector. Here $q(x, y)$ is the magnitude of the unit-width discharge vector, $f(x, y)$ is the water surface height (bed height plus water depth) and $S(x, y)$ is the slope, which is given by the magnitude of ∇f. The fifth equation is a generalized Manning-type flow resistance formula, typically written as

$$q = K\, d^{p} S^{1/2} \tag{5}$$

where $d(x, y)$ is the flow depth and $p \geq 5/3$ due to downstream fining of bed material. If the Manning's roughness parameter did not decrease in the downstream direction, as documented by Leopold *et al*. (1964, 1995, p. 244) then we would have $p = 5/3$ and K would simply be the reciprocal of the roughness parameter, n. Equations (3), (4) and (5) stem from conservation of momentum. Finally, the sixth equation is a generalized sediment transport law (which contains many of the well-known laws as special cases), usually written as

$$q_s = E\, q^{m} S^{n} \tag{6}$$

In some recent models, two different sediment transport laws are used, one for fluvial processes and one for hillslope processes. In addition, the formulas for shear stress, τ_0, and unit stream power, P, are sometimes used to express these in terms of different variables. This set of five equations will be referred to here as the standard model. A more complete discussion of them is given by Smith *et al.* (1997a) and Peckham (2003). Taken as a coupled set of equations, they give a closed mathematical system that can, in principle, be solved for all of the variables of interest: $\underline{q}(x, y, t)$, $\underline{q_s}(x, y, t)$, $f(x, y, t)$, $v(x, y, t)$, $d(x, y, t)$ and $\tau_0(x, y, t)$ in terms of a specified $R(x, y, t)$ and $U(x, y, t)$. Here, v is the vertically-averaged flow velocity. Smith *et al.* (1997a) have obtained several important analytical results involving such things as the existence, uniqueness and stability of solutions. Smith *et al.* (2000) have also obtained several closed-form solutions to a time-evolving 1D version of the problem, subject to a wide variety of boundary conditions, and have clarified the concept of a graded stream.

AN EMPIRICAL SLOPE-DISCHARGE FORMULA

The hydraulic geometry equations (Leopold *et al.*, 1964, 1995) express the width, depth, velocity and slope of channels as powers of the mean-annual river discharge, Q, and are given by $w = w_1 Q^b$, $d = d_1 Q^f$, $v = v_1 Q^m$, and $S = S_1 Q^z$, respectively. The subscript of 1 on the coefficients in these equations refers to the value obtained when Q equals one cubic meter per second and SI units are used. By defininition, $Q = vwd$, which implies that $w_1 d_1 v_1 = 1$ and $(b + f + m) = 1$. Typically observed values for the exponents are $b = 0.5$, $f = 0.4$, $m = 0.1$ and $z = -0.5$. Since the unit-width discharge is given by $q = Q/w$, the width-discharge and slope-discharge formulas can be combined to get

$$q = q_1 S^{\gamma} \tag{7}$$

as an alternate version of the slope-discharge formula. This implies that $\gamma = (1 - b)/z$, and $q_1 = 1/(w_1 S_1^{\gamma})$, with $\gamma = -1$ corresponding to the typical values of $b = 0.5$ and $z = -0.5$. The data given by Leopold *et al.* (1964, p. 244–246) suggests approximate values of $w_1 = 9.0$ meters, $S_1 = 0.0636$ (unitless), and $q_1 = 0.007$ square meters per second. Note that $\gamma = -1$ corresponds to spatially constant unit stream power, since $P = \rho g q S$. Similarly, the depth-discharge formula can be rewritten as a slope-depth formula

$$d = d_1 \left(\frac{S}{S_1} \right)^{f/z} \tag{8}$$

Comparing these to Eq. (5), we have

$$p = \frac{z}{f} \left(\gamma - \frac{1}{2} \right) \tag{9}$$

which for the typical values of z, f and γ gives $p = 15/8 = 1.875$. This is greater than $5/3 = 1.667$ due to the tendency for the Manning roughness parameter to decrease downstream due to fining of bed material. Note that the empirical equations of hydraulic geometry were obtained from data for mature drainage networks and they may not hold for the early, transient forms in the evolution of a fluvial landscape.

A STEADY-STATE FLUVIAL LANDSCAPE MODEL

In an effort to better understand how the geometric attributes of channels and channel networks depend on the physical parameters in the standard model equations, Peckham (1995, 1999) has studied a simplified steady-state version of the standard model. This model has three simple components: (1) steady-state mass conservation for water, $\nabla \cdot \underline{q} = R$, (2) $\underline{q}$ parallel to $-\nabla f$, and (3) a slope-discharge formula as given by Eq. (7). A known drawback of the slope-discharge formula is that it does not allow for a change in process dominance near hilltops, and may therefore give rise to cusp-like drainage divides, such as the ones that are often observed in badlands. Combining these three components leads to the following partial differential equation

$$\nabla \cdot \left(S^{\gamma - 1} \nabla f \right) = -R' \tag{10}$$

where $S = S(x, y) = |\nabla f|$ is the water-surface slope, and $R' = R/q_1$ is the rescaled "geomorphically-effective" rainrate with units of inverse length. As mentioned earlier, the exponent γ should be fairly close to –1. However, it would also be reasonable to use this equation to model convex hillslope forms, since hillslope processes are often modeled by a slope-discharge formula with $\gamma > 0$. Longitudinal profiles are concave down for $\gamma > 0$ (e.g. parabolic for $\gamma = 1$) and concave up for $\gamma < 0$ (e.g. logarithmic for $\gamma = -1$); see Peckham (1995, 1999, 2003). The type of this nonlinear, second-order equation can be shown to be hyperbolic for $\gamma < 0$, parabolic for $\gamma = 0$ and elliptic for $\gamma > 0$. The elliptic case of this equation is known as the p–Laplace equation ($p = \gamma - 1$), since it reduces to Laplace's equation for $\gamma = 1$ and $R' = 0$. The hyperbolic case does not appear to have received much prior attention in the literature. Its nonlinearity for $\gamma \neq 1$ presents a mathematical challenge.

AN EULER-LAGRANGE FORMULATION OF THE STEADY-STATE EQUATION

The Euler-Lagrange Theorem from the calculus of variations (Logan, 1987) establishes a correspondence between partial differential equations (PDEs) such as Eq. (10) and extremized functionals. One version of this theorem states that if $J(f)$ is a functional over the domain Ω of the form

$$J(f) = \iint_{\Omega} L\left(x, y, f, f_x, f_y\right) dxdy \tag{11}$$

where L depends on a function $f(x, y)$ and its first derivatives f_x and f_y, then the "extremal surfaces" (or stationary functions) that correspond to $\delta J = 0$ must satisfy the following partial differential equation (PDE):

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial f_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial f_y}\right) = 0 \tag{12}$$

The best-known example of this theorem is the case where $L = \sqrt{1 + f_x^2 + f_y^2}$ is the surface area functional and the surfaces which minimize the surface area for a given set of boundary conditions must satisfy the famous minimal surface equation:

$$\left(1 + f_y^2\right) f_{xx} - 2\, f_x\, f_y\, f_{xy} + \left(1 + f_x^2\right) f_{yy} = 0 \tag{13}$$

As shown by Peckham (1995), a simple application of the Euler-Lagrange Theorem to the steady-state equation (10) implies that the modeled landscapes either minimize or maximize the following integral subject to boundary conditions on the boundary of the domain, Ω:

$$J(f) = \iint_\Omega \left[\frac{S^{\gamma+1}}{\gamma+1} - f R'\right] dxdy \quad (\gamma \neq -1) \tag{14}$$

$$J(f) = \iint_\Omega \left[\log(S) - f R'\right] dxdy \quad (\gamma = -1) \tag{15}$$

This result, and somewhat similar ones by Sinclair and Ball (1996) and Smith *et al.* (1997a, b), establish a firm connection between physically-based models and the concept of optimality. For $\gamma \neq -1$, the first term in the integrand that is being optimized is proportional to the unit stream power, since $q = q_1 S^\gamma$ and $P = \rho g q S$. Note that the first term is essentially the rate at which kinetic energy is dissipated by the channel network and the second term is the rate at which the system gains potential energy through the introduction of rainfall inputs at various elevations throughout the watershed. As a globally optimized difference between kinetic and potential energy dissipation, this may be interpreted as a continuous-system case of a famous theorem of classical mechanics known as Hamilton's Principle, as applied to fluvial landscapes. See Goldstein (1980, ch. 12) for a discussion of Hamilton's Principle in the context of continuous systems and fields.

Many previous authors have postulated that fluvial landscapes may evolve in such a way that local and global energy dissipation or stream power is minimized, and this is the basis of the optimal channel network (OCN) simulation models that have been studied by Howard (1990), Rinaldo *et al.* (1992, 1993), Rigon *et al.* (1993), Rodriguez-Iturbe *et al.* (1992a, b, c) and Sun *et al.* (1994a, b). However, it is much more satisfying to start from first principles like mass conservation and then deduce optimality principles using an Euler-Lagrange theorem than to attempt to guess what quantity may be optimized and then validate this guess with the realism of simulations. To the author's knowledge, the potential energy generation component of Eq. (15) has not appeared in the previous OCN studies. Also, the logarithmic dependence on slope for $\gamma = -1$ (spatially constant unit stream power) is something that would be difficult to anticipate.

A FINITE-ELEMENT APPROACH TO THE STEADY-STATE EQUATION

For PDEs that can be written in a variational form, the well-known Ritz-Galerkin method (Dahlquist and Björck, 1974, p. 395) provides a numerical solution strategy. Solutions $f(x, y)$ to Eq. (10) must extremize the functional $J(f)$ that is given in Eq. (15). On a triangular lattice with N facets, $J(f)$ can be written as:

$$J(f) = \sum_{j=1}^{N} \iint_{\Delta j} \left[\frac{S^{\gamma+1}}{\gamma+1} - fR' \right] dxdy \quad (\gamma \neq -1) \tag{16}$$

$$J(f) = \sum_{j=1}^{N} \iint_{\Delta j} \left[\log(S) - fR' \right] dxdy \quad (\gamma = -1) \tag{17}$$

Since slope is constant over each triangle, the integrals can be approximated as

$$J(f) \approx \sum_{j=1}^{N} \left[\frac{S_j^{\gamma+1}}{\gamma+1} - \left(f_{j,1} + f_{j,2} + f_{j,3} \right) R_j' / 3 \right] A_j \quad (\gamma \neq -1) \tag{18}$$

$$J(f) \approx \sum_{j=1}^{N} \left[\log\left(S_j\right) - \left(f_{j,1} + f_{j,2} + f_{j,3} \right) R_j' / 3 \right] A_j \quad (\gamma = -1) \tag{19}$$

The set of node heights that extremize $J(f)$ must satisfy the equation:

$$\frac{\partial J}{\partial f_i} = 0, \quad \text{for every node index, } i \tag{20}$$

J is a sum over all triangles in the lattice, but a given node height, f_i, is only shared by 6 facets. Hence, there are only 6 nonzero terms in the expression for $(\partial J/\partial f_i)$. Let k be a second index that runs clockwise through the 6 neighbor nodes or facets

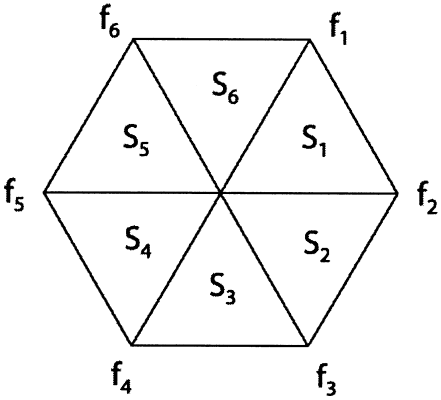

Fig. 1. Neighbor node elevations and facet slopes for a node in a lattice of equilateral triangles.

associated with node index *i*. We then have:

$$\frac{\partial}{\partial f_i}\left\{\sum_{k=1}^{6}\left[\frac{S_k^{\gamma+1}}{\gamma+1} - R_k'\left(f_i + f_k + f_{k+1}\right)/3\right]A_k\right\} = 0 \quad (\gamma \neq -1) \tag{21}$$

$$\frac{\partial}{\partial f_i}\left\{\sum_{k=1}^{6}\left[\log(S_k) - R_k'\left(f_i + f_k + f_{k+1}\right)/3\right]A_k\right\} = 0 \quad (\gamma = -1) \tag{22}$$

Note that the $(k + 1)$ index is cyclic (modulo 6). Both of the previous equations simplify to:

$$\sum_{k=1}^{6} A_k S_k^{\gamma}\left(\frac{\partial S_k}{\partial f_i}\right) = \frac{1}{3}\sum_{k=1}^{6} R_k A_k \tag{23}$$

In the general case, A_k can be computed as the determinant

$$A_k = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_i & x_k & x_{k+1} \\ y_i & y_k & y_{k+1} \end{vmatrix} = \left[y_i\left(x_{k+1} - x_k\right) + y_k\left(x_i - x_{k+1}\right) + y_{k+1}\left(x_k - x_i\right)\right]/2 \tag{24}$$

where (x_i, y_i), (x_k, y_k) and (x_{k+1}, y_{k+1}) are the planar coordinates of the vertices of the *k*th triangle. However, for the remainder of this discussion we will specialize to the case of equilateral triangles and spatially uniform rain, so that $A_k = A$ and

$R'_k = R'$. This yields a “local equation” that must hold at each node (see Fig. 1):

$$\sum_{k=1}^{6} S_k^{\gamma} \left(\frac{\partial S_k}{\partial f_i} \right) = 2R' \tag{25}$$

For the 6 facets that touch node i, the slope can be expressed in terms of local node heights and the lattice scale, ε, as:

$$S_k = \frac{2}{\sqrt{3}\varepsilon} \left[f_i^2 + f_k^2 + f_{k+1}^2 - f_i \left(f_k + f_{k+1} \right) - f_k f_{k+1} \right]^{1/2} \tag{26}$$

This implies that

$$\frac{\partial S_k}{\partial f_i} = \frac{2}{S_k} \left[\frac{2 f_i - \left(f_k + f_{k+1} \right)}{3\varepsilon^2} \right] \tag{27}$$

For the linear case of $\gamma = 1$, the local equation (25) is:

$$\sum_{k=1}^{6} S_k \left(\frac{\partial S_k}{\partial f_i} \right) = 2R' \tag{28}$$

Inserting the previous expressions for S_k and $(\partial S_k/\partial f_i)$ and simplifying, we get

$$f_i = \left(\frac{1}{6} \sum_{k=1}^{6} f_k \right) + \frac{R' \varepsilon^2}{4} \tag{29}$$

For $R' = 0$, the height of the center node is equal to the average height of the six equidistant neighbor nodes, a well-known result for Laplace’s equation.

For the nonlinear case of $\gamma = -1$, the local equation (25) is:

$$\sum_{k=1}^{6} S_k^{-1} \left(\frac{\partial S_k}{\partial f_i} \right) = 2R' \tag{30}$$

Inserting the previous expressions for S_k and $(\partial S_k/\partial f_i)$ and simplifying, we get

$$\sum_{k=1}^{6} \left[\frac{f_k + f_{k+1} - 2 f_i}{f_k^2 + f_{k+1}^2 + f_i^2 - f_i \left(f_k + f_{k+1} \right) - f_k f_{k+1}} \right] = -4R' \tag{31}$$

If we now make the change of variables $h_k = (f_k - f_i)$, we have

$$\sum_{k=1}^{6}\left[\frac{h_k + h_{k+1}}{h_k^2 + h_{k+1}^2 - h_k h_{k+1}}\right] = -4R' \tag{32}$$

This can be written in a number of different ways, such as

$$\sum_{k=1}^{6}\left[\frac{\left(h_k + h_{k+1}\right)^2}{h_k^3 + h_{k+1}^3}\right] = -4R' \tag{33}$$

and

$$\sum_{k=1}^{6}\left[\frac{h_k + h_{k+1}}{h_k^2 + h_{k+1}^2 + \left(h_k - h_{k+1}\right)^2}\right] = -2R' \tag{34}$$

This latter form of the local equation makes it easy to see that finite-element solutions for $\gamma = -1$ will be hydrologically sound or free of pits. If there were a pit at node i, then all 6 outer node heights would be higher than the center height, which would imply that $h_k > 0$ for all k. However, this would make every term on the left side positive, while the term on the right is negative for $R' > 0$. So pits cannot occur. The same argument applies to any value of γ. The analogous local equation for any value of γ is

$$\sum_{k=1}^{6}\left[\frac{h_k + h_{k+1}}{\left[h_k^2 + h_{k+1}^2 + \left(h_k - h_{k+1}\right)^2\right]^{p(\gamma)}}\right] = -R' 2^{p(\gamma)}\left(\sqrt{3}\varepsilon\right)^{1+\gamma} \tag{35}$$

where $p(\gamma) = (1 - \gamma)/2$. Note that $\gamma = -1$ is the only case where the equation is independent of the lattice scale parameter, ε. This represents a form of scale invariance.

If heights are known at the center node of a hexagon and 5 of the 6 outer nodes, then only two of the terms in the sum will contain the unknown node height and the other 4 terms can be evaluated. For example, if the unknown node has index $k = 2$, we can write Eq. (32) as

$$\left[\frac{h_1 + h_2}{h_1^2 + h_2^2 - h_1 h_2}\right] + \left[\frac{h_2 + h_3}{h_2^2 + h_3^2 - h_2 h_3}\right] = -4R' - K = G \tag{36}$$

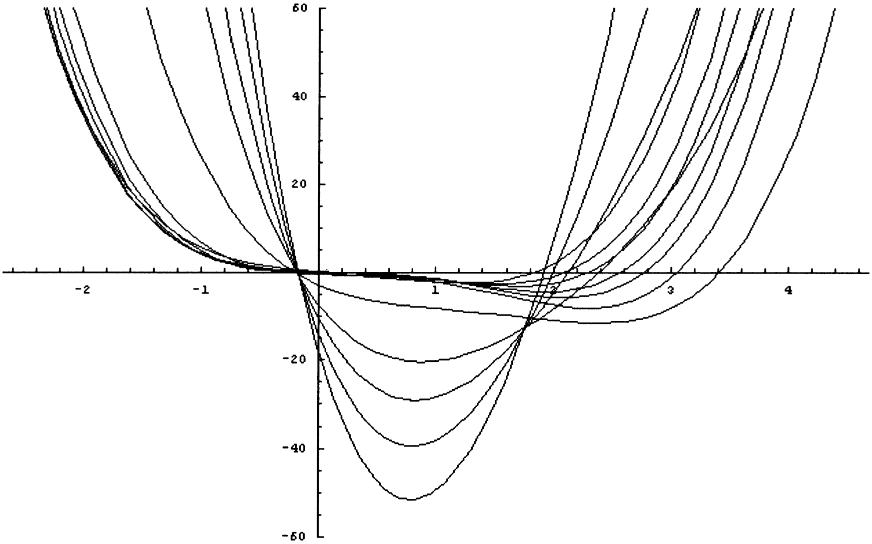

Fig. 2. Plot of the quartic for different combinations of u and w, showing the tendency for there to be two real roots.

where K is the sum of the four terms that do not contain h_2. When the value of G is nonzero, we can make the further change of variables $u = h_1G$, $v = h_2G$ and $w = h_3G$ to get

$$\left[\frac{u+v}{u^2+v^2-uv}\right]+\left[\frac{v+w}{v^2+w^2-vw}\right]=1 \tag{37}$$

Putting the two terms on the left over a common denominator and simplifying leads to a fourth-order polynomial for v, namely

$$P(v)=v^4+a_1v^3+a_2v^2+a_3v+a_4=0 \tag{38}$$

where

$$a_1=-(2+u+w)$$

$$a_2=\left(u^2+w^2+uw\right)$$

$$a_3=2uw-u^2(1+w)-w^2(1+u)$$

Table 1. Special solutions to the local equation for $R' = 1$, translated so $f_0 = 0$. Any of these can be rescaled to give a solution for $R' > 0$ by dividing all six node heights by R'.

Solution	f_1	f_2	f_3	f_4	f_5	f_6	**Note**
Isolated peak	-3	-3	-3	-3	-3	-3	or $f_3 = -\frac{3}{2}$
Symm. ridge	-2	-2	0	-2	-2	0	
Linear ridge	1	a	-1	-1	a	1	$a = -0.682328$
Desc. ridge	-2	-2	-1	-2	-2	2	or $f_3 = -2$
Simple saddle	-1	-1	1	-1	-1	1	
Monkey saddle	a	b	a	b	a	b	$\frac{-(a+b)}{a^2+b^2-a\,b} = \frac{2}{3}$
Monkey saddle	a	$-\frac{1}{2}$	a	$-\frac{1}{2}$	a	$-\frac{1}{2}$	$a = -1 \pm \sqrt{\frac{3}{2}}$
Hillslope	a	a	0	b	b	0	$a = \frac{-b}{1+b}$
Channel	b	0	a	0	b	b	$a = \frac{-b}{2\,b+3}$
Channel 1	1	0	$-\frac{1}{5}$	0	1	1	
Fork	1	0	$-\frac{1}{4}$	0	1	2	
Amphitheater	1	1	a	a	1	1	$a = -0.175971$

$$a_4 = uw(uw - u - w)$$

For the special case of $u = w$, the quartic reduces to a quadratic and we have

$$v_1 = (2 + u + B)/2 \quad (39)$$

$$v_2 = (2 + u - B)/2 \quad (40)$$

$$B = \sqrt{4 + 12u - 3u^2} \quad (41)$$

Preliminary results suggest that when $u \neq w$, 2 of the roots are always complex and 2 are real. This is supported by plotting the quartic as a graph for a wide variety of (u, w) pairs; see Fig. 2. With multiple real roots, one is faced with the issue of which one to use. Realizing that solutions make the functional $J(f)$ an extremum, it appears as though we should decide at the outset whether to minimize or maximize $J(f)$ and then select the roots that achieve this goal. It also seems likely based on results for optimal channel networks (OCNs) that roots should be chosen so as to minimize $J(f)$, both locally and globally over the problem domain

SPECIAL SOLUTIONS TO THE LOCAL EQUATION

The author has found several special solutions to the local equation for $\gamma = -1$ that demonstrate many of the local geometries that one wants to be able to reproduce and understand with a fluvial landscape model. These are summarized in Table 1. For each of these, the height of the center node, f_i, is zero and $h_k = f_k$;

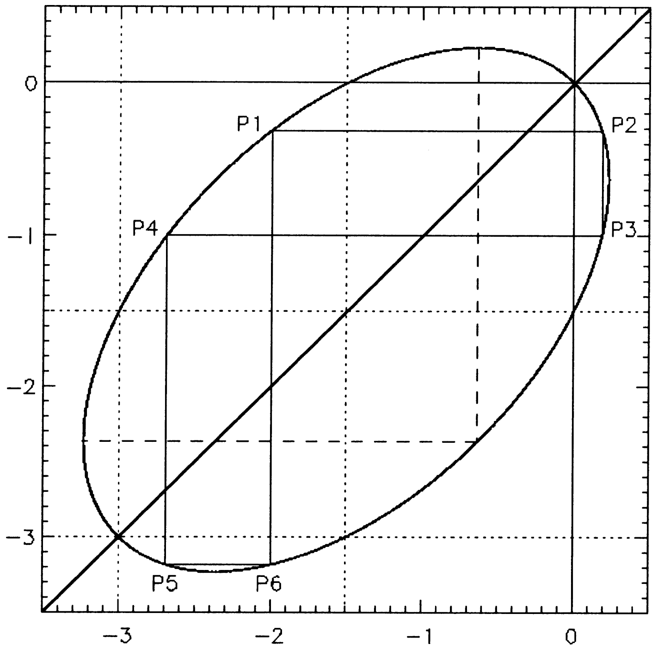

Fig. 3. Ellipse diagram showing a particular path through six points on the ellipse that produces a solution to the local equation.

see Fig. 1. The solutions given here all have a high amount of symmetry, but it should be kept in mind that Eq. (32) has an infinite number of solutions. The hillslope solution is always concave up. The linear ridge and hillslope solutions occur on the surfaces of Fig. 4. Notice also that both divergent and convergent flow geometries are represented. The rescaled excess rainrate, R', serves only to rescale solutions along the z-axis, as can also be seen from analyzing the scaling symmetries of the original PDE, Eq. (10). For the case $R' = 0$, it can be shown that if all of the node heights are chosen so as to lie on an arbitrary plane, they will also satisfy Eq. (10). This is true for any value of γ, and is to be expected since $f(x, y) = ax + by + c$ is a closed-form solution to the PDE for any value of γ when $R' = 0$. The local equation also has the translation property of the original PDE, in that if a set of f_k values provides a solution, then $f_k' = f_k + c$ is also a solution, where c is any constant

An infinite class of special solutions can also be obtained by the following algorithm. Let $\gamma = -1$ and

$$T_k = \frac{h_k + h_{k+1}}{h_k^2 + h_{k+1}^2 - h_k h_{k+1}} \tag{42}$$

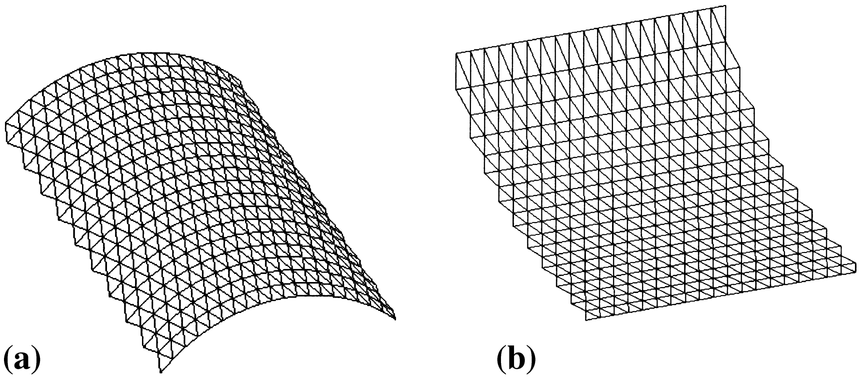

Fig. 4. (a) The numerical solution corresponding to a known analytic solution of the form $f(x, y) = A(x) + cy$, when $\gamma = -1$. Note that the ridge is concave down even though there is no term to model diffusive hillslope processes in the steady-state equation. This solution occurs as a periodic sequence of such shapes, as often seen in road cuts. (b) The numerical solution corresponding to the known analytic solution of a "logarithmic sheet".

so that the local equation is just

$$\sum_{k=1}^{6} T_k = -4R' \tag{43}$$

Now suppose that all of the T_k values are equal to $-2R'/3$. If we take $x = h_k$ and $y = h_{k+1}$, then the (x, y) pairs that satisfy $T_k = -2R'/3$ plot as an ellipse that is tilted by 45 degrees and passes through the origin; see Fig. 3 for which $R' = 1$. A sequence of six f_k values that satisfy the local equation can then be generated as follows.

(1) Choose any point, P_1, on the ellipse and let f_1 be its x-coordinate.

(2) Draw a horizontal line from P_1 to the other side of the ellipse to get a second point, P_2, and let f_2 be its y-coordinate.

(3) Draw a vertical line from P_2 to the other side of the ellipse to get P_3, and let f_3 be its x-coordinate.

(4) Continue this procedure, alternating between horizontal and vertical lines to get f_4, f_5 and f_6.

This procedure can also be summarized as follows: f_1, f_3 and f_5 are the ordered x-coordinates of the six points on the ellipse, and f_2, f_4 and f_6 are the ordered y-coordinates. Note that regardless of where you start, you will return to the starting point, P_1. Starting at certain special points, such as the points with the largest and smallest possible values of x and y or a point on the line $y = x$ yields high-symmetry solutions. In particular, the isolated peak and all monkey saddle solutions are in the class where $T_k = -2R'/3$. In general, however, the solutions of this method are asymmetric and tend to have only one node height that is positive.

The rules for moving around on the ellipse can be relaxed in various ways as long as you return to P_1 after six steps. For example, you can draw only four lines and then reverse direction and return to the starting point. Note that when the T_k values are not all equal, then the local equation has a geometric interpretation in terms of a closed path through six points on six ellipses.

A GENERAL SOLUTION METHOD FOR THE LOCAL EQUATION

A general method for generating solutions to the local equation when $\gamma = -1$ has also been found by the author. In the previous section it was noted that any solution to the $R' = 1$ case could be rescaled to give a solution for $R' > 0$ by dividing all six node heights by R'. Taking this idea further leads to the following general solution method. Begin by choosing any six numbers for the initial node heights. A random number generator can even be used. Compute the quantity $B = \sum_{k=1}^{6} T_k$ and let $R^* = -4/B$. Multiply each of the initial node heights by R^* to get a set of node heights that satisfy the local equation for $R' = 1$. As before, these node heights can be divided by R' to get a solution for any value of $R' > 0$. Realizing that $f_k = f_k(R')$, the scaling property can be stated as $R' f_k(R') = f_k(1)$. This method shows that most node height configurations that drain water can be vertically rescaled to give a solution that is valid for a given $R' > 0$. Initial node height configurations that do not drain, such as pits ($f_k > 0$, for all k) or flats ($f_k = 0$ and $f_{k+1} = 0$ for some k) do not yield valid solutions. Pits are converted to peaks by this method, and flats result in a division by zero. Another exceptional case is that of an inclined plane, e.g. $\{ f_k \} = \{a, a, 0, -a, -a, 0\}$, which is not a valid solution for $R' > 0$ and is converted by this method to $f_k = 0$ for all k. This method also shows how the "geomorphically effective rainrate," R can be estimated from any six node heights in a solution surface as $R = -4q_1/B$.

NUMERICAL SOLUTIONS

Laguerre's iteration method can be used to find the four roots to the "local equation" numerically. Starting from known values on top and side boundaries of a rectangular domain, this makes it possible, in principle, to iteratively solve for node heights at all interior nodes one-by-one, by marching across the domain. However, this simple numerical method is known to be unstable, even for the linear case of Poisson's equation ($\gamma = 1$). One must also be careful to consistently choose the real root that minimizes the integral of the functional over the domain. Figures 4(a) and (b) show two simple solution surfaces that were obtained by this method. Work is ongoing to find a stable numerical method for solving the coupled set of local equations simultaneously and efficiently for all values of γ.

CONCLUSIONS

In previous work (Peckham, 1999, 2003), a steady-state version of the standard fluvial landform evolution equation was analyzed as a model for understanding the relationship between function and form in highly-evolved or mature fluvial landscapes, with an emphasis on longitudinal profiles. The current paper establishes an important reformulation of this steady-state partial differential equation as a variational or "optimality" problem. Moreover, it was shown how this variational formulation could be exploited as the basis of a finite-element numerical solution strategy, following the well-known Ritz-Galerkin method. While the solution strategy has been derived in detail, only preliminary applications of it have been presented here. In particular, the resulting "local equation" was analyzed to demonstrate that solution surfaces can have many of the key at-a-point geometric properties that one wishes to reproduce and understand in a fluvial landscape model, such as peaks, ridges, saddles, hillslopes, forks and sources. In future work, the author hopes to develop a stable numerical method for simultaneously solving the nonlinear local equation at all lattice nodes, while ensuring proper root selection. A simple application of a global Newton method does not work due to the root-selection issue. It is expected on the basis of the at-a-point results presented here that the resulting numerical solutions will exhibit dendritic valley networks that conserve mass and have realistic branching patterns and longitudinal profiles.

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