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Tiling Properties of Drainage Basins and Their Physical Bases

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Abstract. Fractality of drainage basins is shown by their tiling properties with subbasins and interbasin areas. Drainage basin forms projected on the two-dimensional plane have mathematical properties similar to those of one-dimensional quasicrystals. Fractal drainage basins are divided into self-similar subbasins and interbasin areas to the infinitesimal limits of their sizes while one-dimensional quasicrystals are divided into segments of two lengths, namely, shorter ones and longer ones also to their infinitesimal limit. The law of stream numbers is expressed by a recurrence formula consisting of three terms. The total number of segments in the line of a quasicrystal is also given by a recurrence formula. In either case, one of the coefficients of them in the recurrence formula is given with the product of solutions of the quadratic equation and the other one with the sum the solutions. The stability of quasicrystals is discussed by using the concept of the Helmholtz free energy. Fractality of drainage basins is explained based on a statistical thermodynamics regarding potential energy expenditure of water in streams. The statistics to explain fractality of drainage basins are peculiar being different from BE, FD and MB statistics. It should be constructed as the nest of a most probable state in a most probable state in a most probable state and so forth.

Keywords: Fractality of Drainage Basins, Quasicrystal, Recurrence Formula, Statistical Thermodynamics, Nest of Most Probable States

INTRODUCTION

Projections of drainage networks on the two-dimensional plane have been recognized as possessing self-similar structures over a considerable range of scales. This self-similarity is an important basis on which drainage networks are regarded as fractals (Mandelbrot, 1977, 1983). There are numerous studies made by applying fractal geometry to analysis of drainage network composition. Such studies are synthesized by Rodoriguez-Iturbe and Rinaldo (1997). Many works are done by regarding Horton's law of stream numbers as expressing self-similarity of drainage networks (e.g. Tarboton *et al.*, 1988; La Barbera and Rosso, 1989; Marani *et al.*, 1991; Rosso *et al.*, 1991; Liu, 1992). However, it has been proved that Horton's law of stream numbers is available in strict sense to express only self-similarity of structurally Hortonian networks (Tokunaga, 1966, 1975; Smart, 1967). Further more Horton's laws (including laws of stream lengths and basin areas) are inadequate in that they do not admit space-filling networks as shown by Tarboton (1996). Nevertheless we can use Horton's law of stream numbers redefining it as the asymptotic law of self-similar networks as shown by

Peckham (1995).

The space-filling problem is also the tiling problem. The latter problem has been extensively studied in the field of crystallography (e.g. Takeuchi, 1992) while the former one discussed rather in that of drainage basin geomorphology in connection with fractal geometry (e.g. Nikora *et al.*, 1996; Rodoriguez-Iturbe and Rinaldo, 1997; Cui *et al.*, 1999; Tokunaga, 2000). The stability of quasicrystals is discussed by using the concept of the Helmholtz free energy (e.g. Takeuchi, 1992). Many trials to explain drainage basin structures by applying concepts of thermodynamics have been made since Murray (1926). Such studies are also reviewed by Rodoriguez-Iturbe and Rinaldo (1997). The author, however, considers that they are not completely successful.

We deal with problems much more complex than those in crystallography. We, however, find mathematical formulas with the expressions common to tiling properties of drainage basins and quasicrystals which satisfy fractality of these objects considered to have natures physically different to each other. This suggests that there is a possibility to settle a firm theoretical basis for explanation of structures of drainage basins. This paper will show the mathematical natures common to these at first. Then it will be shown that the stability of drainage networks, which have natures more complex than those of quasicrystals, is explained by introducing a peculiar statistics different from Bose-Einstein (BE), Fermi-Dirac (FD) and Maxwell-Boltzmann (MB) ones. This means that we have to advance thermodynamics itself to settle a theoretically firm basis to fractality of drainage basins.

RECURRENCE FORMULAS FOR QUASICRYSTALS AND SELF-SIMILAR DRAINAGE BASINS

A simple model is used to demonstrate self-similarity of quasicrystals in books of crystallography (e.g. Takeuchi, 1992). Each line constructed with segments of length S and that of L in Fig. 1 is considered to be a one-dimensional quasicrystal. Let denote the number of segments in the nth line from the uppermost one by N_n. Then N_n expresses the Fibonacci Numbers when $L/S = (1 + \sqrt{5})/2$. The Fibonacci Numbers are produced by the recurrence formula

$$N_{n+2} = (P+Q)N_{n+1} - PQN_n$$

when $P + Q = 1$ and $PQ = -1$, where $N_1 = 1$ and $N_2 = 1$. Then $P = (1 - \sqrt{5})/2$ and $Q = (1 + \sqrt{5})/2$. These P and Q are the solutions of a quadratic equation. Each line has a self-similar structure as shown by the five lower lines in Fig. 1. Then,

$$\frac{L}{S} = \frac{(L+S)}{L} = \frac{(L+S+L)}{(L+S)} = \frac{(L+S+L+L+S)}{(L+S+L)}$$
$$= \frac{(L+S+L+L+S+L+S+L)}{(L+S+L+L+S)} = \cdots = \frac{\left(1+\sqrt{5}\right)}{2} \tag{1}$$

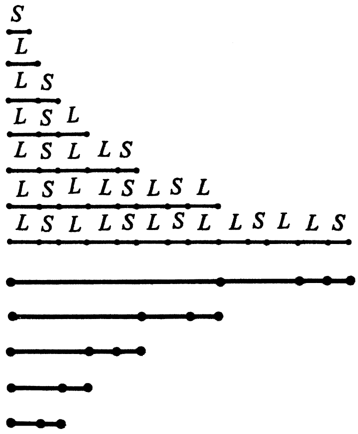

Fig. 1. One-dimensional quasicrystal and its self-similarity. The upper seven lines show the generating process of the Fibonacci sequence: $L/S = (1 + \sqrt{5})/2$. The lower five lines demonstrate self-similarity of the one-dimensional quasicrystal.

Here $(1 + \sqrt{5})/2:1$ is well known as the golden ratio.

Let denote a segment of length S by $\bar{S}$ and that of length L by $\bar{L}$, then the lines with self-similar structure are produced by the replacements $\bar{L} \to \bar{L} + \bar{S}$ and $\bar{S} \to \bar{L}$. We can also form a self-similar set of segments of two different lengths, namely longer one and shorter one, dividing line $\bar{L}_k$ of the finite length L_k by the following replacements

$$\bar{L}_k \to \bar{L}_{k-1} + \bar{S}_{k-1}, \quad \bar{S}_{k-1} \to \bar{L}_{k-2}$$

Therefore

$$\bar{L}_k \to \bar{L}_{k-1} + \bar{S}_{k-1} \to \bar{L}_{k-1} + \bar{L}_{k-2}, \quad \bar{L}_{k-1} \to \bar{L}_{k-2} + \bar{S}_{k-2} \to \bar{L}_{k-1} + \bar{L}_{k-3} \quad (2)$$

where $S_{k-j} = L_{k-j-1}$ for $j = 1, 2, ..., (k-2)$. When $L_{k-j}/S_{k-j} = (1 + \sqrt{5})/2$ for $j = 1, 2, ..., (k-1)$, we can divide the line into segments which provide the Fibonacci Numbers. The reputation of replacements shown by Replacement (2) is also the process to form a one-dimensional Sierpinski space.

Recently Sakamoto (2000) showed that we can define lines with self-similar structure using $N_{n+2} = (P + Q)N_{n+1} - PQN_n$ for another conjugate values of P and Q (e.g. $P = 1 - \sqrt{2}$, $Q = 1 + \sqrt{2}$; $P = 2 - \sqrt{3}$, $Q = 2 + \sqrt{3}$).

The law of stream numbers for self-similar drainage basins is also expressed by using a recurrence formula. First we denote the number of side tributaries of order j entering into a stream of order k by $E_{k,j}$. When $E_{k,j}$ takes the same value for a certain value of $(k - j)$, independent of the individual values of k and j, we can

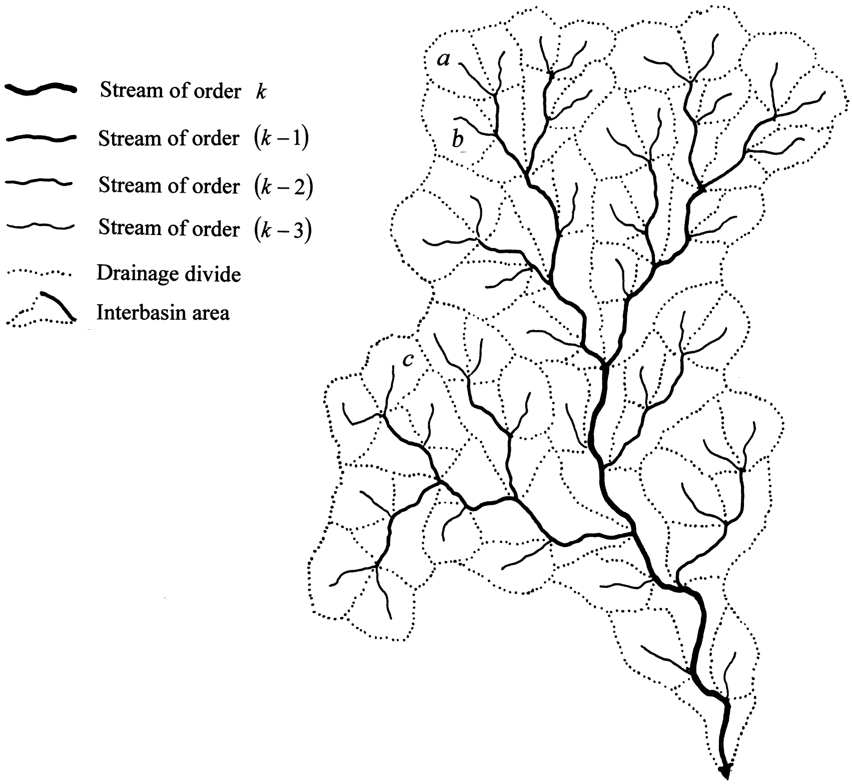

Fig. 2. Hypothetical drainage basin with $E_1 = 1$ and $K = 2$, after Tokunaga (2000). Streams of orders lower than $(k - 3)$ are ignored. Basin names, *a*, *b* and *c*, are used for explanation of relation of the law of stream fall by Yang (1971) to the most probable state of potential energy expenditure of running water.

put $E_{k-j} = E_{k,j}$. Then we can derive the recurrence formula for the number $N_{k,j}$ of streams of order j in a basin of order k when E_{k-j} satisfies $E_2/E_1 = E_3/E_2 = \cdots = E_{k-j}/E_{k-j-1} = K$ where K is constant. That is

$$N_{k,j} = (P+Q)N_{k,j+1} - PQN_{k,j+2} \tag{3}$$

where P and Q are given as follows (Tokunaga, 1966, 1978, 1994, 1998, 2000):

$$P = \frac{\left[2 + E_1 + K - \sqrt{(2 + E_1 + K)^2 - 8K}\right]}{2}$$

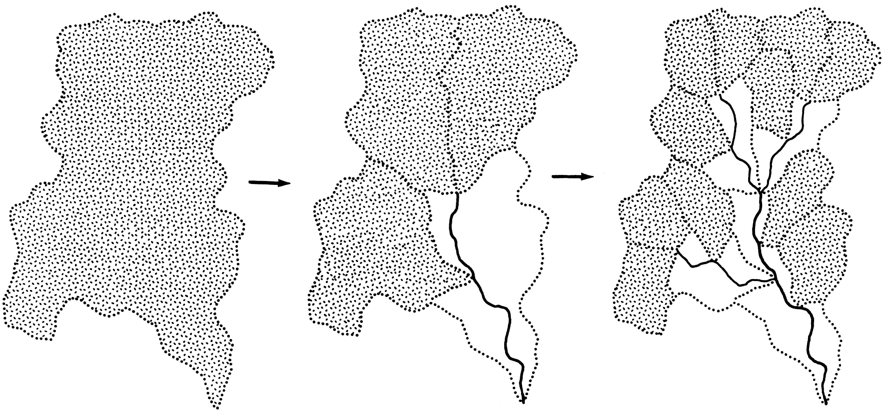

Fig. 3. A process to form a self-similar drainage basin with $E_1 = 1$ and $K = 2$ by replacements. A basin, dotted area, is replaced by three subbasins and three interbasin areas. A white area composed of three interbasin areas is replaced by two subbasins and five interbasin areas. The figures also show a process to form a two dimensional Sierpinski space.

$$Q = \frac{\left[2 + E_1 + K + \sqrt{(2 + E_1 + K)^2 - 8K}\right]}{2}$$

Here it should be noted that P and Q are the solutions of a quadratic equation. This type of recurrence formula can be transformed into a continued fraction (Tokunaga, 1994).

Put $T_{k,j} = T_{k-j} = E_1 K^{k-j-1}$. Then $T_{k,j}$ is called the tree generator (Veitzer and Gupta, 2000). We can derive an equation, which expresses $N_{k,j}$ in the form of the sum of a series (Tokunaga, 1966, 1978, 1994, 1998, 2000). Then

$$N_{k,j} = \frac{2 + E_1 - P}{Q - P} Q^{k-j} + \frac{2 + E_1 - Q}{P - Q} P^{k-j} \tag{4}$$

The drainage system expressed by Eq. (3) or (4) has been called Branching System I (Tokunaga, 1994, 1998, 2000). The law of basin areas of Branching System I is given by

$$A_k = A_j Q^{k-j} \tag{5}$$

where A_k is the area of a basin of order k and A_j that of order j. This equation is derived on the assumption that a basin of a given order is divided into subbasins and interbasin areas of infinitesimal sizes in the ultimate. A hypothetical drainage basin with $E_1 = 1$ and $K = 2$ is illustrated in Fig. 2. Here let $N_{\mathrm{I},k,j}$ be the number of interbasin areas adjoining a stream of order k and $B_{k,j}$ be the area of such an interbasin area when streams of orders lower than j are ignored. Then the following replacements hold for Branching System I.

$$[A_k] \to [(2 + E_1)A_{k-1} + N_{\mathrm{I},k,k-1}B_{k,k-1}] \tag{6}$$

$$[N_{\mathrm{I},k,j+1}B_{k,j+1}] \to [E_1 K^{k-j-1} A_j + N_{\mathrm{I},k,j}B_{k,j}] \tag{7}$$

where $N_{\mathrm{I},k,j} = 2 + E_1 + E_1K(K^{k-j-1} - 1)/(K - 1)$ and this relation is called the law of numbers of interbasin areas (Tokunaga, 1975, 1978, 2000). The values of E_1, K, $N_{\mathrm{I},k,j}$, A_j and $B_{k,j}$ are given by the statistical values, for examples, the average values for actual drainage basins (Tokunaga, 1966, 1975, 1978; Onda and Tokunaga, 1987; Peckham, 1995; Tarboton, 1996; Jämtnäs, 1999; Peckham and Gupta, 1999; Veitzer and Gupta, 2000).

Replacements (6) and (7) can be demonstrated by illustrations. Let the order of the three basins with $E_1 = 1$ and $K = 2$ in Fig. 3 be k, then the transition from Fig. 3(a) to Fig. 3(b) means Replacement (6). That from Fig. 3(b) to Fig. 3(c) means two replacements. One is Replacement (7) for $j = k - 2$ in the area occupied by interbasin areas in Fig. 3(b). The other is $[A_{k-1}] \to [(2 + E_1)A_{k-2} + N_{\mathrm{I},k-1,k-2}B_{k-1,k-2}]$ in each of the basins of order $(k - 1)$. This relation is obtained by substituting $(k - 1)$ into k in Replacement (6).

Moussa (1997) used at first a Sierpinski space to describe a fractal property of drainage basins. Figure 3 also shows that a two-dimensional Sierpinski space is generated by repetitions of a successive shift of indices in Replacements (6) and (7).

We can also demonstrate the replacements to form the Sierpinski space using the Peano basin. Natural drainage networks are modeled on a statistical basis as a system with branching number 2 (Peckham, 1995), binary tree, whereas streams form deterministically a regular triadic tree, system with branching number 3 (Peckham, 1995), in the Peano basin. Nevertheless the Peano basin is a good tool to explain the self-similar natures of drainage basins. If we regard a trifurcating point to consist of two bifurcating points, which are dislocated with an infinitesimal distance from each other, Strahler's ordering method is available for it and its stream network. Then $E_1 = 1$ and $K = 2$, $A_{k-1} = A_k/4$, and $B_{k,k-1} = A_k/8$ in the Peano basin. The law of numbers of interbasin areas in a basin with branching number 3 is given by $N_{\mathrm{I},k,j} = 1 + E_1 + E_1K(K^{k-j-1} - 1)/(K - 1)$ (Tokunaga, 2000). We can show the process to form the Sierpinski space substituting these values and relations into Replacements (6) and (7). The transition from Fig. 4(a) to Fig. 4(b) shows the replacement, $[A_k] \to [3A_{k-1} + 2B_{k,k-1}\ (=A_{k-1})]$. The transition of the area occupied by interbasin area in Fig. 4(c) to the corresponding area in Fig. 4(d) shows the replacement, $[2B_{k,k-1}\ (=A_{k-1})] \to [2A_{k-2} + 4B_{k,k-2}\ (=2A_{k-2})]$. These rules

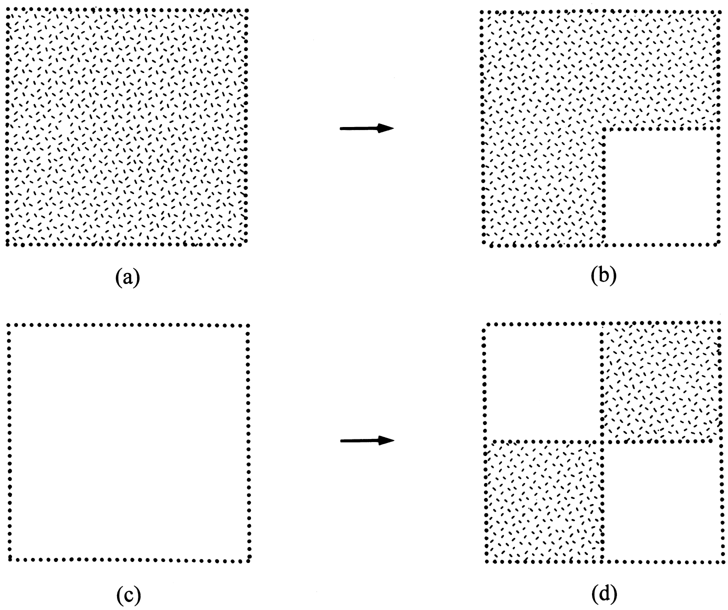

Fig. 4. The two rules to generate a Sierpinski space in a Peano basin. A dotted square is regarded as a basin and a white square as to be occupied by two interbasin areas.

to form the Sierpinski space, however, are entirely different from those demonstrated by Moussa (1997).

Now we find the terms, namely, recurrence formula, solutions of a quadratic equation, and Sierpinski space, common to the one-dimensional quasicrystal and the self-similar drainage basin. We can regard the one-dimensional quasicrystals as the one-dimensional drainage basins although such basins never exist in the actual world. Crystallographers regard one-dimensional quasicrystals as being quasi-cyclic (e.g. Takeuchi, 1992; Sakamoto, 2000) whereas geomorphologists use the term "cyclic" or "cyclicity" for the property of drainage basins expressed by Eq. (4) (e.g. Tokunaga, 1978; Tarboton, 1996; Veneziano *et al.*, 1997; Perera and Willgoose, 1998; Cui *et al.*, 1999). The author considers one-dimensional quasicrystals or one-dimensional drainage basins to be cyclic in geomorphological sense.

POTENTIAL ENERGY EXPENDITURE OF RUNNING WATER IN ALL STREAM CHANNELS IN A BASIN

The law of stream fall by Yang (1971) states that the ratio of the average stream fall between any two different order streams in the same basin is unity. Numerable data taken from actual drainage basins show their favorable conformity to the law (Yang, 1971; Yang and Stall, 1973; Shimano, 1978). Yang (1971) tried to explain the physical basis of the law using the analogy of entropy in thermodynamics. His explanation is valid at least partly. Let denote the fall of a stream of order j by H_j and suppose water that flows into the stream of order $(k - 3)$ at its uppermost reach in basin a in Fig. 2. The most probable state of potential energy expenditure per unit mass of water is certainly sustained when $H_k = H_{k-1} = H_{k-2} = H_{k-3}$, as shown by Yang (1971). We cannot, however, specify the condition to keep the most probable state using such an equation for water flowing into the stream of order $(k - 3)$ in basin b within the traveling root from its inflow point to the uppermost reach of the stream of order $(k - 1)$ whereas the relation $H_k = H_{k-1}$ sustains the most probable state of potential energy expenditure of a unit mass of water after flowing into the stream of order $(k - 1)$. The relation $H_{k-1} = H_{k-2} = H_{k-3}$ also sustains the most probable state for a unit mass of water flowing into the stream of order $(k - 3)$ in basin c at its uppermost reach up to the confluence point of the streams of order $(k - 1)$ and order k. We cannot, however, specify the condition to keep the most probable state in the reach downward from the confluence point by using the relation regarding stream falls. Nevertheless let us regard provisionally the following relation as a necessary condition to sustain the most probable state of potential energy expenditure per unit mass of water supplied to a drainage basin of order k.

$$H_k = H_{k-1} = \cdots = H_j = \cdots = H \tag{8}$$

Here let $L_{k,h}$ be the length of a stream of order k measured by a ruler with length l_h and $L_{j,h}$ be that of order j, then the law of stream lengths for a self-similar network is expressed as follows:

$$L_{k,h} = L_{j,h} Q^{D_S(k-j)/D_B} \tag{9}$$

where D_S is the fractal dimension of individual streams, D_B that of individual basins, and l_h the straight-line length of a stream of order h for $j \geq h$ (Tokunaga, 1994, 1998, 2000). Then usually $D_B = 2$ for actual drainage basins. Combination of Eqs. (8) and (9) provides a self-affine stream network in three-dimensional space (Tokunaga, 1998). We can meditate on the most probable state of potential energy expenditure per unit mass of water in all stream channel of in a basin of a given order, which satisfies the condition of self-affinity of its steam network, using Eqs. (4), (5), and (8).

Let denote the quantity of water supplied as precipitation to a basin of order j in a given period of time by W_j. Then we may postulate the relation $W_j = \alpha A_j$ where α is constant regardless of the value of j. Let e_j^* be the potential energy expenditure per unit mass of water supplied to a basin of order j in the stream of order j. Then the following relation can be assumed between e_j^* and H.

$$e_k^* = e_{k-1}^* = \cdots = e_j^* = \cdots = \beta H \tag{10}$$

Table 1. Potential energy expenditure, e_{j-1}, per unit mass of water in all streams in a basin of order $(j-1)$. Streams of orders lower than i are ignored. In the basin, $E_1 = 1$ and $K = 2$. The third terms in the braces show the rapid convergence of them for decrease of i value.

	$i = j-2$	$i = j-3$	$i = j-4$
e_{j-1}	$\{(2/3)2+(4/9)-(1/36)\}\beta H$	$\{(2/3)3+(4/9)-(1/144)\}\beta H$	$\{(2/3)4+(4/9)-(1/576)\}\beta H$

where β is constant. Here let e_k be the potential energy expenditure per unit mass of water supplied to a basin of order k in all streams in the basin. When we ignore streams of orders lower than i. Then we can give e_k by

$$e_k = \left[\sum_{j=i}^{k} e_j^* W_j N_{k,j}\right] \Big/ W_k = \left[\sum_{j=i}^{k} e_j^* A_j N_{k,j}\right] \Big/ A_k \tag{11}$$

From Eqs. (4), (5), (10), and (11),

$$e_k = \beta H \sum_{j=i}^{k} \left[C_1 + C_2 (P/Q)^{k-j}\right] = \beta H C_1 (k-i+1) + \beta H C_2 \frac{\left[Q - Q(P/Q)^{k-i+1}\right]}{(Q-P)} \tag{12}$$

where $C_1 = (2 + E_1 - P)/(Q - P)$ and $C_2 = (2 + E_1 - Q)/(P - Q)$. For a basin of order $(j-1)$, we have

$$e_{j-1} = \beta H C_1 (j-i) + \beta H C_2 \left[Q - Q(P/Q)^{j-i}\right] \Big/ (Q-P) \tag{13}$$

The average state of infinite topologically random channel networks satisfies Eq. (4) with $E_1 = 1$ and $K = 2$ (Shreve, 1969; Tokunaga, 1978). Therefore $P = 1$ and $Q = 4$. The values of e_{j-1} for $i = j-2, j-3$, and $j-4$ for $E_1 = 1$ and $K = 2$ are shown in Table 1. The values show that absolute value of the second term of the right side of Eq. (13) rapidly approaches $(4/9)\beta H$ with decrease of i. Therefore we can use

$$e_{j-1} \approx \beta H C_1 (j-i) + \beta H C_2 Q / (Q-P) \tag{14}$$

instead of Eq. (13) for a large value of $(j-i)$. Put $e_\infty = e_{j-1}$, then we derive the following equation for

$$e_k = \beta H C_1 (k-j+1) + e_\infty \tag{15}$$

We can regard e_∞ as constant for a very large value of $(j - i)$. Then Eq. (15) shows that e_k, e_{k-1}, ..., e_{j+1}, and e_j distribute equidistantly, increasing their values as the index value increases. This means that the amount of potential energy expenditure per unit mass of water in all streams in a basin of a given order increases equidistantly as the order increases when streams of orders lower than a certain value are ignored. Such a situation occurs in self-similar drainage basins when the law of stream fall is kept in them.

MOST PROBABLE STATE OF POTENTIAL ENERGY EXPENDITURE OF WATER IN SELF-SIMILAR DRAINAGE BASINS

Two streams of a given order join to form a stream of the next higher order in Strahler's ordering system. This designation inevitably divides the basin into two areas with topographically different natures. We can clearly demonstrate it using the example in Fig. 2. The drainage basin of order k is divided into the areas occupied by the two basins of order $(k - 1)$, which feed the streams entering into the stream of order k at its uppermost point, and the area occupied by the basins, which feed the side tributaries entering into the stream of order k, and the interbasin areas adjoining the stream of order k. We refer to the latter one a side area of the stream of order k and designate its area by A_k^*. Then $A_k^* = A_k - 2A_{k-1}$. This relation defines the side area of a stream of a given order. Therefore $A_k^* = A_k - 2A_{k-1}$, $A_{k-1}^* = A_{k-1} - 2A_{k-2}$, ..., $A_{j+1}^* = A_{j+1} - 2A_j$. Basins of order $(k - 1)$ are always structurally similar to that of order k in a self-similar drainage basin. Therefore, how to divide the side area to the stream of order k into subbasins of various orders decides the composition of the drainage network of order k. We can derive an equation for A_k^*, A_{k-1}, ..., A_j, and $B_{k,j}$ as follows:

$$A_k^* = E_1 \sum_{h=j}^{k-1} K^{k-h-1} A_h + N_{\mathrm{I},k,j} B_{k,j} \tag{16}$$

where $N_{\mathrm{I},k,j} = 2 + E_1 + E_1 K(K^{k-j-1} - 1)/(K - 1)$ as mentioned before. Let denote the potential energy expenditure of water supplied as precipitation to the side area of a stream of order k in all streams in the side area by U_k. Then U_k is given

$$U_k = E_1 \sum_{h=j}^{k-1} e_h K^{k-h-1} W_h = \alpha\beta C_1 H E_1 \sum_{h=j}^{k-1} K^{k-h-1}(h - j + 1) A_h + \alpha e_\infty E_1 \sum_{h=j}^{k-1} K^{k-h-1} A_h \tag{17}$$

when Eq. (15) is satisfied in all basins in the side area. We can eliminate the second term of the right side of Eq. (16) for a considerably large value of $(k - j)$ because of the small value of $(N_{\mathrm{I},k,j} B_{k,j} / A_k^*)$. Thus we obtain

$$A_k^* \approx E_1 \sum_{h=j}^{k-1} K^{k-h-1} A_h \tag{18}$$

Then we can also derive

$$U_k - \alpha e_\infty A_k^* \approx \alpha\beta C_1 H E_1 \sum_{h=j}^{k-1} K^{k-h-1} (h-j+1) A_h \tag{19}$$

from Eqs. (17) and (18).

Here let us replace the approximation signs in Eqs. (18) and (19) with equality signs respectively. Then we can derive the following equations for constant values of A_k^* and U_k.

$$A_k^* = E_1 \sum_{h=j}^{k-1} K^{k-h-1} A_h \tag{20}$$

$$D = H E_1 \sum_{h=j}^{k-1} K^{k-h-1} (h-j+1) A_h \tag{21}$$

where $D = (U_k - \alpha e_\infty A_k^*)/\alpha\beta C_1$ and therefore D is regarded as constant for a given value of $(j-i)$ in Eq. (14). Equation (20) means that the quantity of water supplied to the side area of the stream of order k in the given period is constant because $W_j = \alpha A_j$, and Eq. (21) that the amount of potential energy expenditure of water supplied to the side area in the given period is also constant, where the unit of mass of water can be given arbitrarily.

Let us regard a unit mass of water supplied to a unit area in the given period as a body and postulate that the amount of potential energy expenditure of each body differs from those of the other bodies in a basin in the side area. Then infinitesimally small difference in potential energy expenditure is postulated to be distinguishable. This means that all the bodies can be numbered by the amounts of their potential energy expenditure and therefore compose a numerable set in the basin. Let us divide a basin in the side area into two-dimensional cells of the unit area and postulate the state that all the cells can be numbered according to their positions in the basin. The number of all different arrangements of the numbered cells in a basin of order j is $A_j!$, where A_j is regarded as a natural number. Let denote the product of the numbers of all different arrangements of cells in respective basins in a side area by g. Then

$$g = (A_{k-1}!)^{E_1} (A_{k-2}!)^{E_1 K} (A_{k-3}!)^{E_1 K^2} \cdots (A_h!)^{E_1 K^{k-h-1}} \cdots (A_j!)^{E_1 K^{k-j-1}} \tag{22}$$

Let us decrease water supplied to a basin of order $(k-2)$ by two bodies and assume that the amounts of potential energy expenditure of these bodies in all streams in the basin approximate to the average value, e_{k-2}, respectively. And let us increase water supplied to a basin of order $(k-1)$ by one body, the amount of potential energy expenditure of which approximates to e_{k-1}, and that supplied to a basin of order $(k-3)$ by one, the amount of potential energy expenditure of which approximates to e_{k-3}. This procedure leaves U_k as well as A_k^* unchanged because

$$\begin{aligned}&e_{k-1}-2e_{k-2}+e_{k-3}\\&=\beta C_1 H(k-j)+e_\infty-2\beta C_1 H(k-j-1)-2e_\infty+\beta C_1 H(k-j-2)+e_\infty=0\end{aligned}$$

The product g' of numbers of all different arrangements of cells in the respective basins in the side area after the procedure is expressed by

$$\begin{aligned}g'=&\left(A_{k-1}!\right)^{E_1-1}\left(A_{k-1}+1\right)!\left(A_{k-2}!\right)^{E_1K-1}\left(A_{k-2}-2\right)!\left(A_{k-3}!\right)^{E_1K^2-1}\\&\times\left(A_{k-3}+1\right)!\left(A_{k-4}!\right)^{E_1K^3}\cdots\left(A_j!\right)^{E_1K^{k-j-1}}\end{aligned} \quad (23)$$

The ratio g'/g is given by

$$\frac{g'}{g}=\frac{\left(A_{k-1}+1\right)!\left(A_{k-2}-2\right)!\left(A_{k-3}+1\right)!}{A_{k-1}!A_{k-2}!A_{k-3}!}=\frac{\left(A_{k-1}+1\right)\left(A_{k-3}+1\right)}{\left(A_{k-2}-1\right)A_{k-2}}$$

When a basin of order $(k-3)$ contains a very large number of cells, this expression may be replaced by

$$\frac{g'}{g}=\frac{A_{k-1}A_{k-3}}{A_{k-2}{}^2} \quad (24)$$

When g takes a maximum value, a small variation in the arrangements should leave it unchanged, then $g'/g = 1$ and thus $A_{k-1}/A_{k-2} = A_{k-2}/A_{k-3}$. A similar relationship applies to A_{k-2}, A_{k-3}, and A_{k-4}; ...; A_{j+2}, A_{j+1}, and A_j. Therefore the condition for the maximum value of g is

$$A_{k-1}/A_{k-2}=A_{k-2}/A_{k-3}=\cdots=A_{j+1}/A_j=Q \quad (25)$$

This relation is the law of basin areas itself. Let denote the numbers of water bodies supplied to all basins of order j in the side area of the stream of order k by W_j'. Then

$$W'_{k-1} / W'_{k-2} = W'_{k-2} / W'_{k-3} = \cdots = W'_{j+1} / W'_j = Q / K \tag{26}$$

because $W'_j = \alpha E_1 K^{k-j-1} A_j$. The relation of K to Q in Eq. (3) evidently shows that $Q/K > 1$.

A similar method is taken to obtain a most probable state under the restrictive conditions that the number of particles or oscillators and the total energy possessed by them are constant respectively in statistical thermodynamics (e.g. Reif, 1965). Equations (20) and (21) are apparently the restrictive conditions to sustain a most probable state of potential energy expenditure of water bodies in the side area of the stream of order k.

Statistics for potential energy expenditure of water bodies, however, differs from BE, FD, and MB statistics in some points. The number of quanta or particles decreases exponentially as their energy or energy level increases in BE, FD, and MB statistics. On the other hand, water bodies are clustered in respective basins and each clustered water bodies has the average value of their potential energy expenditure. The number of water bodies increases exponentially as the average value of their potential energy expenditure increases equidistantly as shown by Eqs. (15) and (26). The probability that a water body belong to the clusters, which produce an average value of potential energy expenditure, increases as the average value increases. This provides a distribution reverse to the canonical distribution, which expresses the relation between probability and energy in thermodynamics. We may refer to the distribution as the reverse canonical distribution for clustered bodies.

Equations (22) and (23) allow us to assume the state that in each basin the cells of unit area with different relative heights above its outlet distribute at random. This means that basins of higher orders have very rugged surfaces on which water does not flow at least smoothly. Random arrangements of the cells of different relative heights seldom form drainage divides. Conversely the marginal part of a cluster of cells ought to be occupied by ones relatively higher than those in the inner part to form a drainage network. Then the random process seems to be contradictive to the forming process of drainage basins. Altitudes of confluence points of side tributaries to a stream of a given order differ from each other in the side area of the stream. Consequently water bodies supplied to a basin have a base level different from those to the other basins with regard to potential energy expenditure in the side area of the stream, whereas the base level is given absolutely in BE, FD, and MB statistics. We need some additional settings to explain physical bases of the self-similarity of drainage basins.

NEST OF MOST PROBABLE STATES

A key concept to create the statistics with physical bases for drainage basin composition is a nest of most probable states. It will be shown in this chapter by using the Peano basin. As mentioned afore, $E_1 = 1$, $K = 2$, $P = 1$, and $Q = 4$ in it.

Let the lower half of the basin in Fig. 5(a) be the side area of the stream of

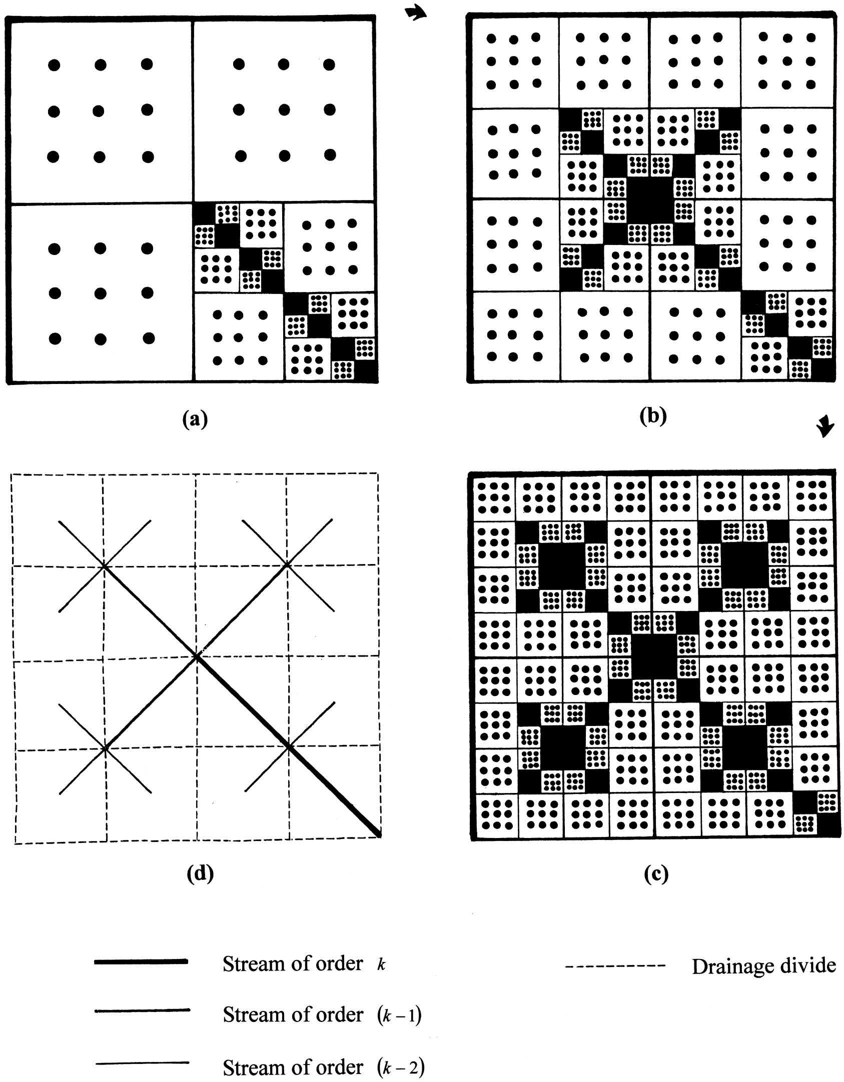

Fig. 5. The nest of most probable states of potential energy expenditure of water bodies in a Peano basin. Diameters of black circles show average values of potential energy expenditure per unit mass of water in respective basins qualitatively. The smallest right square consists of two interbasin areas with the shape of the right triangle. Streams of orders lower than $(k-2)$ are not drawn.

order k. Then there exist one basin of order $(k-1)$, two of order $(k-2)$, four of order $(k-3)$, eight of $(k-4)$, and sixteen interbasin areas in the side area when streams of order lower than $(k-4)$ are ignored. The number of interbasin areas adjoining a stream of a given order in a basin with branching number 3 is one less than that in a basin with branching number 2 as mentioned before. A black right square in Fig. 5(a) consists of two right triangles and each one is an interbasin area adjoining the stream of order *k*. Let us assume the state that numbered cells with different heights above its outlet distribute at random regarding their positions in each basin. Water bodies supplied to the cells in a basin of order $(k-1)$ provides e_{k-1}, those to the cells in a basin of order $(k-2)$ provides e_{k-2}, and so forth. We may also assume that the relief of the basin of order $(k-1)$ is larger than that of the basin of order $(k-2)$, and then the relief of the basin decreases as the basin order becomes lower. This state should satisfy Eqs. (20), (21), (22), (23), (24), and (25). The diameter of black circles in Fig. 5 shows the size of the relief of each basin qualitatively.

We can depict a most probable state in each basin of order $(k-1)$ in a similar manner as to obtain Fig. 5(a). The result is shown as Fig. 5(b). The basins of order $(k-3)$ and of order $(k-4)$, and interbasin areas with relief smaller than that of the basin of order $(k-2)$ distribute alongside the stream of order *k* and those of order $(k-1)$, whereas the area adjoining the margin of the basin of order *k* except the part near its outlet is occupied by eleven basins of order $(k-2)$ with relatively large relief. Drainage divides can exist only in areas with relatively large relief.

We can also depict a most probable state in each basin of order $(k-2)$ in Fig. 5(b) as shown by Fig. 5(c). Then a stream of order $(k-2)$ appears with four interbasin areas in each of the basins of order $(k-2)$. All the streams are mostly surrounded by basins of order $(k-4)$ and interbasin areas with relatively small relief. The areas adjoining the margins of the basins of order *k*, order $(k-1)$ and order $(k-2)$ are occupied by the basins of order $(k-3)$ with relatively large relief except the part of their outlets. The surface of the basin designated by Fig. 5(c) is much smoother than that by Fig. 5(a). The three-dimensional form of the former basin is more systematic than latter one.

We can assume basins of $(k-5)$ in the basin shown by Fig. 5(c). Then an interbasin area is divided into one basin of order $(k-5)$ and two interbasin areas. This procedure makes the surface of the whole basin smoother and its three-dimensional form more systematic. Here let denote a system, in which A_k cells of a unit area with different heights distribute at random and the heights of the cells produce the average value corresponding to e_k, by S_k. We can assume a system at a most probable state in which cells of unit area with different heights above the outlet distribute at random in each of basins of orders lower than *k*. Then the heights of A_{k-1} cells in each of basins of order $(k-1)$ produce the average value corresponding to e_{k-1}, those in each of basins of order $(k-2)$ the average value corresponding to e_{k-2}, ..., and those in each of basins of order *j* the average value corresponding to e_j. Let us denote the system by $S_{k,k-1}$. Similarly we can define a system at a most probable state denoted by $S_{k,j}$ with a general index *j* in which the cells with different heights distribute at random in each of basins of orders lower than *j*. The following relation is derived for S_k, $S_{k,k-1}$, ..., $S_{k,j}$.

$$S_k \supset S_{k,k-1} \supset \cdots \supset S_{k,j} \tag{27}$$

This relation shows a nest structure of most probable states: a most probable state in a most probable state in a most probable state and so forth. Such a structure for the Peano basin is demonstrated by using Fig. 5. The basin illustrated by Fig. 5(c) exists as a most probable state in that by Fig. 5(b), which exists as a most probable state in that by Fig. 5(a). The basin illustrated by Fig. 5(a) is a most probable state of a regular square with the area same to that of the basins of order k in Fig. 5 in which cells with different heights distribute at random.

A mountain landscape is formed as a set of basins and interbasin areas. It looks to have a smooth surface cover completely different from those of rugged figures with Brownian relief by Mandelbrot (1983). The basin forming process is also one to form the mountain landscape with a smooth surface.

The nest of most probable states proposes a general idea that randomness can produce convergence. The principle that a form is produced as the nest of most probable states seems to be applicable to not only self-similar branching systems including diffusion limited aggregation (DLA), lightning, neural networks etc. but also to another systems with stable forms.

VALUES OF E_1 AND K

The nest of most probable states is defined by using Eqs. (20), (21), (22), (23), (24), (25), (26) and (27). There are no mathematical restrictions as to the values of E_1 and K themselves except $E_1 > 0$ and $K > 0$ when we treat them as statistical values. These equations are so comprehensive regarding the values of E_1 and K that we can define a self-similar system as the nest of most probable states for any large value of them. On the other hand it has been proved that Shreve's (1966, 1969) random topological model satisfies Eq. (4) with $E_1 = 1$ and $K = 2$ as a law which expresses the average state (Tokunaga, 1975, 1978). Empirical data, however, show a general tendency that E_1 and K as the average values are larger than 1 and 2 but not far from these values respectively in low order parts which provide reliable values because of large populations of streams in them (Tokunaga, 1966, 1978; Onda and Tokunaga, 1987; Peckham, 1995; Tarboton, 1996; Jämtnäs, 1999; Peckham and Gupta, 1999; Veitzer and Gupta, 2000).

There are some discussions on discrepancies between the empirical data and the random topological model on the values of E_1 and K (e.g. Tokunaga, 1978; Peckham, 1995; Cui *et al.*, 1999; Veitzer and Gupta, 2000). Tokunaga (1978) has stated that an equilibrium state of a drainage network is kept on the balance of randomness and non-random force. He also regarded the network, which satisfies Eqs. (4) and (5), as to be at the equilibrium state encompassing that of the maximum entropy with $E_1 = 1$ and $K = 2$ (Tokunaga, 1978). His consideration was made at that time without the physical bases demonstrated in this paper.

DISCUSSION

Theoretical studies on the values of E_1 and K have also advanced since the proposal of concept of self-similarity by Mandebrot (1977, 1983). Perera and

Willgoose (1998) examined the behavior of the cumulative area distribution based on the model defined by E_1 and K setting two different types of zeroth order hill slope flow patterns. Then they showed by simulation that the value of K is strongly related to the scaling exponent in the region of the catchment dominated by fluvial erosion. Cui *et al.* (1999) have considered the values of E_1 and K as representing the effects of regional controls and pointed out importance of a space-filling constraint to explain these values using their model, stochastic Tokunaga model. Peckham and Gupta (1999) discussed the value of $T_{k,j} = T_{k-j} = E_1K^{k-j-1}$ presenting a reformation of Horton's laws on the basis of statistical self-similarity. Veitzer and Gupta (2000) analyzed this value and its variability introducing a new class of random self-similar networks. These studies have widened the horizon of study on drainage basin geomorphology. Further development of these studies might result in providing some statistical bases in connection with physical quantities. We can discuss some about the values of E_1 and K relating the nest of most probable states of potential energy expenditure of water bodies, Shreve's (1966, 1969) random topological model, and empirical data to each other.

Empirical data exhibit a general tendency that the value of E_1 is not far from 1 and that of K from 2 as mentioned in the previous chapter. This implies that actual drainage networks strongly controlled by randomness. Any constraints are not imposed on direction of streams in Shreve's (1966, 1969) random topological model. Streams merge each other completely at random in it. This means that no directionally systematic inclinations of streams are postulated. Water never flows in networks in which streams have no systematic inclinations. There is, however, no reason for us to abandon Shreve's (1966, 1969) random topological model. It should be considered to be an asymptotic base for theoretical consideraton of drainage network composition.

Water is drained from a basin through systematically inclined streams. Directionally systematic inclinations should impose a non-random bias on confluences of streams. Then the non-random bias probably acts as a force to raise the values of E_1 and K above 1 and 2 respectively. A stream with larger slope provides a stronger non-random bias. If so, the tree generator $T_{2,1}$ must tend to be larger than $T_{3,2}$ in actual drainage basins because Horton's (1945) law of stream slopes shows that the average slope of streams decreases geometrically as the order of streams increases.

Jämtnäs (1999) obtained tree generator matrices for 48 drainage basins in the United States, orders of which are equal to or higher than 6, after Peckham's (1995) study. Figures of the generators are shown to the two places of decimals. The values of $T_{2,1}$ and $T_{3,2}$ are regarded as relatively stable for their large numbers of samples. Then $T_{2,1} > T_{3,2}$ in 29 basins, $T_{2,1} = T_{3,2}$ in 4 basins, and $T_{2,1} < T_{3,2}$ in 15 basins. The difference ($T_{2,1} - T_{3,2}$) is evaluated small in all basins. The result is favourable to the conjecture mentioned above in statistical sense. The comparison of the values $T_{2,1}$ and $T_{3,2}$ in a basin, however, is almost meaningless because these values have wide ranges of distribution due to their stochastic property and also more or less influenced by regional difference of geologic controls, etc. It must be still more difficult to prove that there exists the firm relation, $T_{k,k-w} < T_{k-1,k-w-1} < \cdots < T_{j+1,j-w+1}$ with a small difference between successive terms for $1 \leq w \leq j$, only using empirical data. If the existence of this relation is theoretically explained, the self-similar model defined by the constant values of E_1 and K will result in having a clear meaning as an asymptotic model with physical bases. Then drainage basins, which satisfy the relation mentioned above, are regarded as to be dislocated slightly from the self-similarity and at a quasi-equilibrium state.

Another asymptotic postulation was also used on the process to derive the nest of most probable states of potential energy expenditure of water bodies. We needed the assumption that a basin of a given order is divided into subbasins and interbasin areas of infinitesimal sizes to derive the law of basin area expressed by Eq. (5) (Tokunaga, 1978, 1994, 1998, 2000). The law of stream lengths expressed by Eq. (9) was derived from Eq. (5) (Tokunaga, 1994, 1998, 2000). Therefore the stream in a basin of an infinitesimal size should have the corresponding, namely infinitesimal, length. A stream with infinitesimal length and a finite fall given by Eq. (8) should be vertical. Such a stream never appears in nature. The assumption of subbasins and interbasin areas of infinitesimal sizes shows the asymptotic direction. We can presumably say that we need some asymptotic postulations, even if they are contradictive to each other in the ultimate, for modeling of complex systems such as drainage networks. Equations (20) and (21) are also derived by using the asymptotic property mentioned above.

FINAL REMARKS

Drainage basin forms projected on the two-dimensional plane have mathematical properties similar to those of one-dimensional quasicrystals. These properties are expressed by recurrence formulas consisting of three terms as well as by Sierpinski spaces. One of coefficients in the recurrence formulas is given by the product of the solutions of a quadratic equation and the other one by the sum of them. These expressions common to the quasicrystals and drainage basins describe their common properties, namely, cyclicity and self-similarity although the term, quasi-cyclic, is used in the field of crystallography.

The Helmholtz free energy should concern the stability of quasicrystals. The stability of self-similar drainage basins is explained by using the nest of most probable states of potential energy expenditure of water bodies. This must be considered an important physical base. The nest of most probable states never impose any constraints on the values E_1 and K except $E_1 > 0$ and $K > 0$. Empirical data, however, shows the general tendency that $E_1 > 1$ and $K > 2$ but these values are not so far from 1 and 2 respectively. This implies that confluences are fairly influenced by topological randomness in addition to any other forces than it. The author considers that stochastic theories combined with three-dimensional forms of basins and some explicit physical quantities in connection with them will explain the values of there are E_1 and K concretely. He also expects that the introduction of probabilistic theories into analysis of self-similarity of drainage basins (e.g. Cui *et al.*, 1998; Peckham and Gupta, 1999; Veitzer and Gupta, 2000) will lead to discovery of such quantities. If there exists the relation, $T_{k,k-w} < T_{k-1,k-w-1} < \cdots < T_{j+1,j-w+1}$, for $1 \leq w \leq j$ with a small difference between successive terms in a basin of order k when streams of orders lower than j are ignored, the basin is regarded as to be at quasi-equilibrium state.

Any thermodynamic functions are not yet defined for the nest of most probable states. Established statistics for thermodynamics are defined in a unitary space. This study clarified that statistics in nested spaces is needed to define thermodynamic functions for self-similar or self-affine branching systems. The author feels some limitations to established principles of physics in investigations of complex systems. It is rather expected that new principles of physics will be discovered in objects of geomorphology and hydrology.

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