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Planar Organization of River Networks: A Hidden Gamma Law Structure

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Abstract. We consider the water path distribution from any point in a basin to the outlet through the river network. Using the Strahler ordering scheme (Strahler, 1957) we separate every hydraulic path length L as the sum $L = \sum_{i=1}^{n} l_i$ of the lengths of its different components l_i, where n is the basin's highest order. We obtain L and l_i ($\forall i$, $1 \le i \le n$) maps and probability distributions through a GIS-based analysis of the geographical and topological organization of river networks. It appears that the l_i probability distributions follow convex and rapidly decreasing shapes and have a strong scaling property, except for the highest orders. This scaling is consistent with river networks fractality (Rodriguez-Iturbe and Rinaldo, 1997) and its upper limit can be explained by the branched-out topology. Indeed, for a single basin, the highest order component l_n necessarily can not have a decreasing probability distribution. This is due to its particular position in the Strahler hierarchical network organization. For similar reasons, probability distributions for l_{n-1}, l_{n-2}, ... present a weaker scaling than for l_1, l_2, ... Thus, we consider the truncated hydraulic lengths $L' = \sum_{i=1}^{m<n} l_i$. The probability distributions of L' are negatively skewed and some of them can be described by a gamma law, whereas basin width functions are generally positively skewed (Rodriguez-Iturbe and Rinaldo, 1997). This shows how a gamma law underlies the river network self-similar organization, even if the shape does not obviously appear in the basin-scale geomorphological functions, due to hierarchical constraints.

Keywords: Hydro-Geomorphology, Scaling, Underlying Pattern, Branched-out Network, River Network Organization

INTRODUCTION

At the beginning of the nineteenth century, Playfair, quoted by Horton as an introduction to his famous article (Horton, 1945), noticed that <<every river appears to consist of a main trunk, fed from a variety of branches, each running in a valley proportionned to its size, and all of them together forming a system of valleys, communicating with one another, and having such a nice adjustment of their declivities that none of them join the principal valley either on too high or

too low level>>. Through this sentence Playfair expressed most of the fractal characteristics of the river network such as the similarity between the main trunk fed from branches and the branches fed from valleys', and the branching rules. Playfair's <<nice adjustment>> is in fact the network fractality. Indeed, since Mandelbrot's definition of the fractal geometry (Mandelbrot, 1982), some pioneer authors showed the fractal nature of river networks (La Barbera and Rosso, 1987; Tarboton *et al.*, 1988) and many works have been performed, which are well synthesized by Rodriguez-Iturbe and Rinaldo (1997) and Dodds and Rothman (1999).

The obvious characteristic of shape similarity between parts of networks was first studied, which led to the idea of self-similarity. Then many relations were found in terms of power laws, showing scaling properties, and expressions of the fractal dimension were proposed (La Barbera and Rosso, 1989; Liu, 1992). From their synthesis of the topic, Rodriguez-Iturbe and Rinaldo (1997) reached the conclusion that <<the search for invariance properties across scales as a basic hidden order in hydrologic phenomena is one of the main themes of hydrologic science>>. This means that this search is crucial for geomorphology in itself and also for geomorphology-related hydrology. For this reason, we focus on a geomorphological function and study how it is structured through the branched out drainage topology.

THE TOPOLOGICAL AND GEOMORPHOLOGICAL DESCRIPTION OF RIVER NETWORKS

Our work is motivated by two major aims. The first one is to describe the river network itself, seen as a fractal object within a basin considered as a two-dimensional planar object. Because this network is the particular topological network of all the lowest altitude points of the basin and because it is the result of complex geomorphological mechanisms, it appears to be a good indicator of the whole relief of the basin. More precisely, this network is itself oriented since there is one particular point of the lowest altitude, which is the outlet. The branched-out topology can thus be pointed out and used as a scale. The second aim is to lead a reasoning which can be fruitfull in hydrology, and we think as some other authors (Rodriguez-Iturbe and Valdès, 1979; Gupta *et al.*, 1980; Beven and Kirkby, 1993) that the hydrological response of a basin can be closely related to relevant basin-scale geomorphological functions.

1. The branched-out topology and scaling

The geographic unit at the scale of which the river network is organized is the basin. In a basin, besides the outlet, which is the only downstream extremity, the river network is made up of particular points: upstream extremities, called sources, and junctions. The part contained between two such successive particular points is a link. The Strahler ordering scheme (Strahler, 1957) permits the description of a link position in a river network (Fig. 1a). According to this ordering method, the stream, which is a set of successive links of same order, is

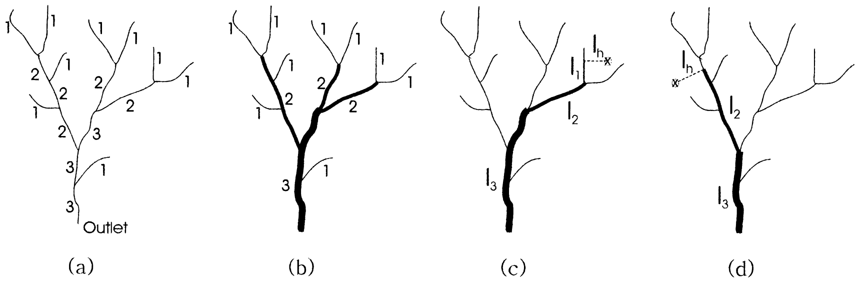

Fig. 1. Hypothetical network of order n = 3. (a) Strahler ordering of links. (b) Identification of streams of order 1 —, order 2 ▬ and order 3 ■. (c) and (d) Examples of paths run by a water drop from a point to the outlet. - - - is the path through the hillslope whose length is l_h. Identification of 1st (—), 2nd (▬) and 3rd (■) orders components whose lengths are respectively l_1, l_2 and l_3.

defined (Fig. 1b). The spectrum of orders represented in the network can be seen as a cascade, in the sense of Mandelbrot (1982), which is the first basement of fractality. Indeed, Horton laws (Horton, 1945) give the number of streams and the mean stream length through Strahler's ordering scheme (Schumm, 1956), on the basis of a bifurcation ratio $R_B = \frac{N_{i-1}}{N_i}$ and of a length ratio $R_L = \frac{\overline{L_i}}{\overline{L_{i-1}}}$—where N_i is the number and $\overline{L_i}$ the mean length of i-th order streams—, whose stabilities are statistically verified (Smart, 1968). According to Shreve (1966), Horton laws suggest that, if there is no geological control, the river network natural organization is ruled by general erosion processes and secondarily by local environmental factors. The Tokunaga law (Tokunaga, 1966) is similar, concerning the numbers $T_{i,j}$ of j-th order streams flowing into an i-th order stream from the side. These branching laws, as well as Hack's law (Hack, 1957) giving the basin mainstream length as a power function of its area, were the first implicit scalings found in a river network.

2. *The basin-scale geomorphological functions*

Then, in relation to the isochrone notion (Larrieu, 1957; Surkan, 1969), the area function (Foroud and Broughton, 1981; Snell and Sivapalan, 1994; Robinson *et al.*, 1995) and the width function (Kirkby, 1976; Gupta and Waymire, 1983; Snell and Sivapalan, 1994) were introduced. The area function is <<the frequency distribution of contributing area with respect to flow distance from the outlet>> (Snell and Sivapalan, 1994). This function is also called area-distance function by Rodriguez-Iturbe and Rinaldo (1997) in their synthetic publication. Many authors approached this function by the width function $W(x)$ which gives the number of links in the network at a flow distance x from the outlet. Rodriguez-Iturbe and Rinaldo (1997) justify this in a DEM-use context, because <<in a system where a link is associated with every unit area used to discretize the catchment surface, the width function and the area-distance function are equal.

When the necessary support is the unit area, the network is space-filling because every pixel is assigned a link and no holes are allowed in the distribution of pixels covering the total drainage area.>> We agree with Rodriguez-Iturbe and Rinaldo (1997) who see in the width function an operator that maps the area of the basin, a two-dimensional object, onto a one-dimensional support, and who also think that <<the width function contains important information about the mechanisms of development of the drainage network and incorporates the essential characters of the hydrologic response through its linkage with residence time distributions>>, and we think that this is true even more with the area function.

NEW OBJECTS AND ORGANIZATION EVIDENCES

We think that considering the link as the elementary object for which the flow path is computed—with no relation to its position, draining area or length—takes the width function away from the area function. Indeed if there is a use of a DEM, the equivalence between the pixel and the link breaks the exact nature of the link. Secondarily, the idea of elementary-surfaces associated to the links over-simplifies the relation between the hillslopes and the thalwegs, both on geometrical and hydrological points of view, and this leads to a space-filling network which does not fit to actual observations (La Barbera and Rosso, 1989). Thus we consider a new geomorphological function and study its organization through the self-similar branched-out structure.

1. *Original objects*

From the flow path of a given point of the basin, which is the path followed by a water drop to reach the outlet, we consider separately the hillslope path and the hydraulic path through the river network. Within the hydraulic path, we call the "i-th order component" the part run through successive channels of the same order i (Figs. 1c and 1d). This definition has two important implications.

1) The i-th order component of a given hydraulic path is a set of successive links of the same order i—and eventually of a part of a link, for the first one—, which is not necessarily a whole i-order stream. For instance the 3rd order component on Fig. 1d is made up of two links of order 3, whereas the 3rd order component on Fig. 1c is the whole stream made up of three links. Moreover Figs. 1c and 1d show that the first link constituting the first component of the path (resp. 1st and 2nd orders) is partially run.

2) Because of the branching structure, a hydraulic path can present less than n components—n being the basin order—since the i-th order component flows into a j-th order component ($j > i$), which does not necessarily mean $j = i + 1$. For instance the hydraulic path drawn on Fig. 1c is made up of $n = 3$ components, whereas the hydraulic path drawn on Fig. 1d does not have any 1st order component.

Then we focus on the length of the hydraulic path, called the hydraulic length L. This appears to be an interesting variable characterizing the river network seen as a fractal object within the two-dimensional planar basin as well as a media determining the drainage from hillslope and rapid dynamic transfer from up to downstream. The probability density function of this hydraulic length is equivalent to the area-hydraulic length function of the basin, which can be seen as a new concept derived from the one of area-distance function. Moreover, for any geographic point of the basin, the hydraulic length is the sum $L = \sum_{i=1}^{n} l_i$ where n is the basin order and l_i the length of the i-th order component.

We assume that the network scaling property applies to the components, of course to their mean lengths, but more generally to their probability density functions. Thus, besides the stream length ratio R_L (Horton, 1945), we define the component length ratio $r_l = \frac{\overline{l_i}}{\overline{l_{i-1}}}$ where $\overline{l_i}$ is the mean length of the i-th components, and propose to check its steadiness by a GIS-based analysis.

2. *A GIS-based analysis*

A georeferenced vector image of the river network is obtained from a human interpretation of 1/25000 scale maps, with the software ©ERDAS. This leads to realistic round shaped meanders and conjunctions, which are crucial for our issue. Then a topological analysis of the relationships between the vectors allows to build a binary branched-out tree. The interpretation of this tree leads to identify the links and their Strahler orders, the streams, and to estimate statistics of these constituents (Cudennec, 2000). Furthermore a square grid is used as a set of points to sample the basin territory, and for each of these points the hydraulic path is identified across the hillslope and the network, which means also across objects of increasing Strahler orders. Then one can evaluate the lengths of all the components l_i for each point and build georeferenced raster images of the obtained values: one image for each value $1 \le i \le n$. This multi-layer raster mapping of all the components allows then by image processing 1) to extract statistics and probability distributions of the lengths of components l_i; and 2) to study sums of lengths of components $L' = \sum_{i=1}^{m \le n} l_i$, by combining raster images.

As the grid is square and represents a sampling of the basin surface, the actual normalized distribution of the hydraulic length is the area-hydraulic length function of the basin, defined above. But as far as a flow path enters the network at a given point, there is no artificial link assigned to an elementary surface like for the width function. Then the topological basic element is displaced from the network link or source to the surface sampling grid point and among the infinity of points along the network, all of them are not equivalent, but have a variable weight according to the proportion of flow paths entering the network through them.

3. *Experimental results*

The actual shapes, statistics and scalings of the components and of the area-

Fig. 2. Map of the Yvel river network for the whole basin and four sub-basins.

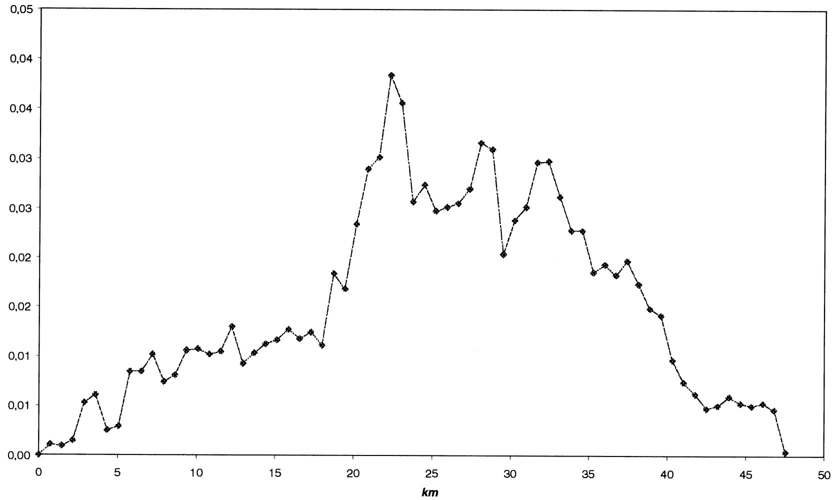

Fig. 3. Actual probability distribution of the hydraulic length L of the whole Yvel basin.

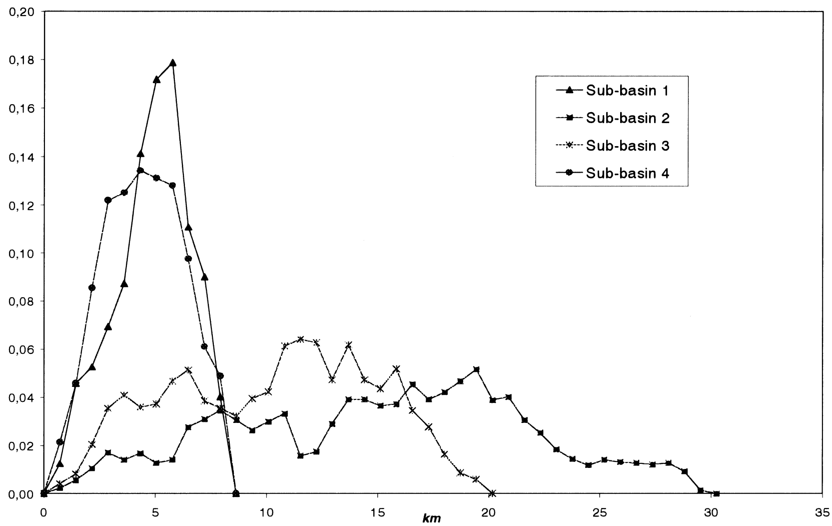

Fig. 4. Actual probability distributions of the hydraulic lengths L of the four sub-basins.

hydraulic length function are studied for the particular basin of the Yvel river (Lambert coordinates of the outlet: 248979/1043554; 302 km^2) whose river network is shown on Fig. 2. Figure 3 shows the probability distribution of the hydraulic length of the whole basin, and Fig. 4 the ones of the four sub-basins identified on Fig. 2. Despite big differences between the basins, in terms of size

Table 1. Horton's ratios and component length ratio of the whole Yvel basin.

Order i	2	3	4	5	6	7	Selected value	R^2
R_B	4.55	5.32	4.65	3.78	3	3	3.72	0.99
R_L	1.19	2.37	3.05	1.32	3.03	1.48	1.93	0.97
r_l	1.2	2.35	2.98	1.16	4.28	1.68	1.97	0.96

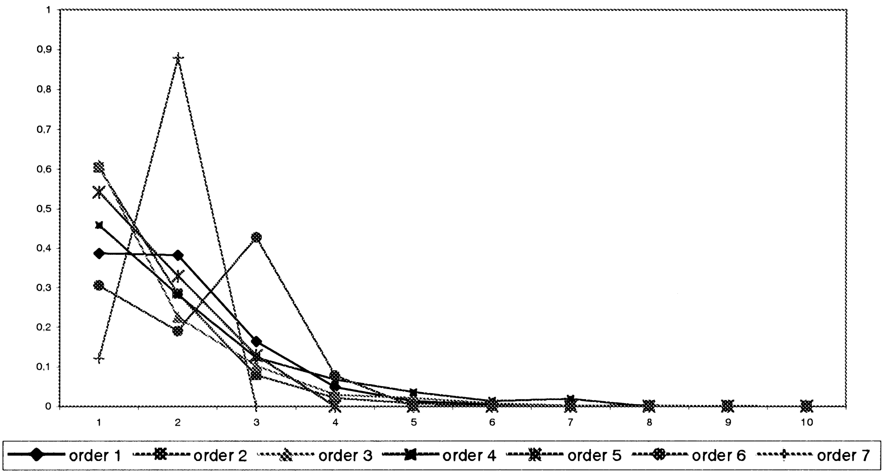

Fig. 5. Actual probability distributions of the reduced lengths of components $\frac{l_i}{r_l^{(i-1)}}$ of the whole Yvel basin, with $r_l = 1.97$. The results are grouped in 200 meters wide classes.

and Strahler order ($n = 7, 5, 6, 6$ and 5 respectively for the whole basin and for sub-basins 1, 2, 3 and 4), the five probability distributions on Figs. 3 and 4 are irregular and broadly positively skewed, which is relevant with the universal skewness of width functions (Rodriguez-Iturbe and Rinaldo, 1997).

More precisely, we then observe how the probability density function of the hydraulic length of the whole basin is structured and organized in terms of components. Table 1 presents the observed values of the Horton bifurcation and length ratios, and the values of the newly defined component length ratio r_l. The global values are obtained by logarithmic regressions, whose R^2 coefficients are indicated. It appears that the three ratios are steady, which shows that this particular basin presents a strong self-similarity of streams identified within the Strahler ordering scheme, and that this self-similarity is verified with components. Moreover one can notice that r_l and R_L values are close to each other.

Figure 5 presents the probability distributions of the lengths of the seven components, reduced by the scaling factor r_l identified in Table 1: $\frac{l_i}{r_l^{(i-1)}}$. The five

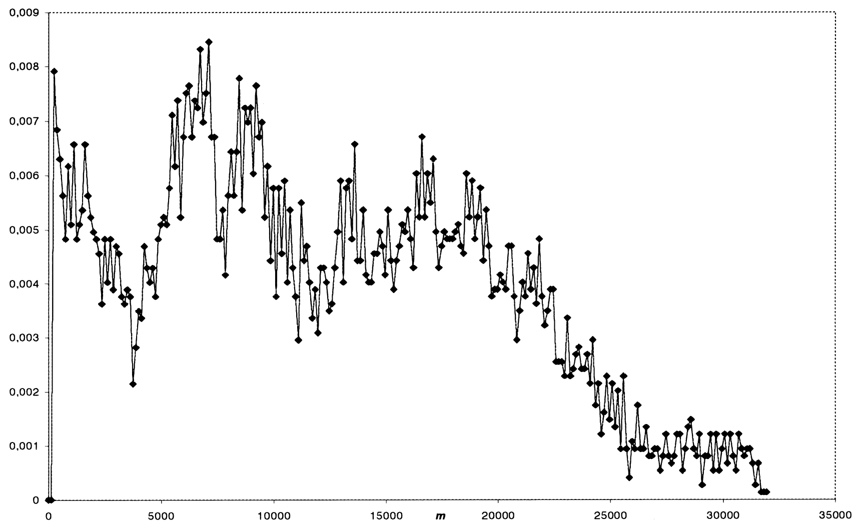

Fig. 6. Actual probability distribution of the truncated hydraulic length $L' = \sum_{i=1}^{m} l_i$ of the whole Yvel basin, with m = 6.

first curves (orders one to five) are very close together, convex and rapidly decreasing. This shows that the scaling of the components is not only verified in terms of geometric averages but also in terms of distribution. The two last curves are different since they are not decreasing and out of the envelop of the others. This shows a difference in the role and the representativity of the highest orders components, due to their particular hierarchical positions. Indeed the last order n is unavoidable and the (n–1)-th one is nearly unavoidable for hydraulic paths from anywhere in the basin to the unique outlet. Thus the probability distributions can necessarily not be decreasing, which would mean that avoiding is majority. This particular role of upper orders objects, that we call "hierarchy constraint", led us to consider the sub-system of the m ($m < n$) first Strahler levels, by defining the sum $L' = \sum_{i=1}^{m<n} l_i$, corresponding to the truncation of the highest orders components.

Experimental probability distributions of L', for any value of m, can be obtained through the GIS-based analysis. Figures 6, 7, 8, 9 and 10 show the actual probability distributions of L' for decreasing values of m, respectively 6, 5, 4, 3 and 2. It appears that the skewness is obviously becoming negative and that a regular shape is appearing. This is already true for m = 6 and m = 5, whereas the probability distributions of L for sub-basins of orders 6 and 5 (Fig. 4) are positively skewed. Nevertheless, we noticed and can here add that the truncated lengths L' present the same distribution shapes for any sub-basin.

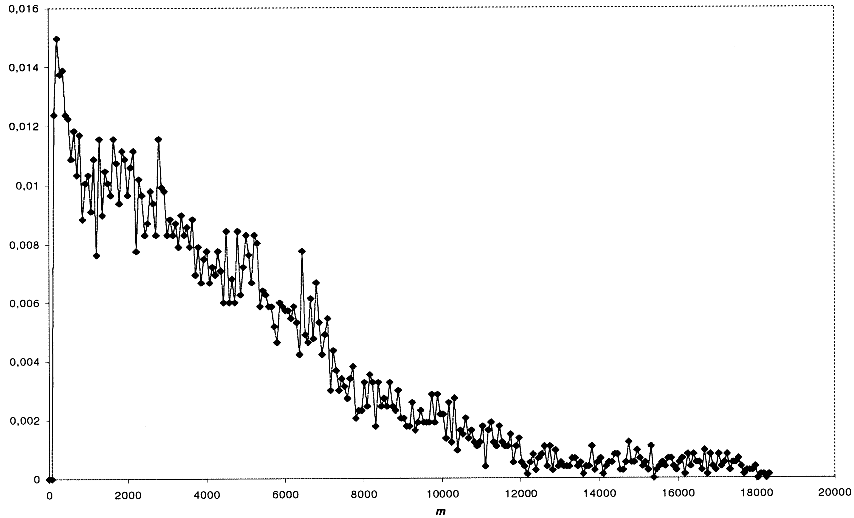

Fig. 7. Actual probability distribution of the truncated hydraulic length $L' = \sum_{i=1}^{m} l_i$ of the whole Yvel basin, with $m = 5$.

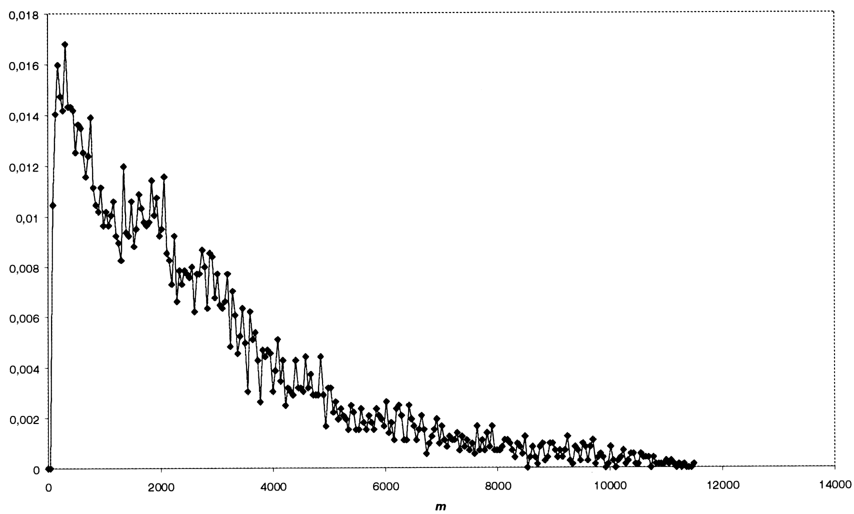

Fig. 8. Actual probability distribution of the truncated hydraulic length $L' = \sum_{i=1}^{m} l_i$ of the whole Yvel basin, with $m = 4$.

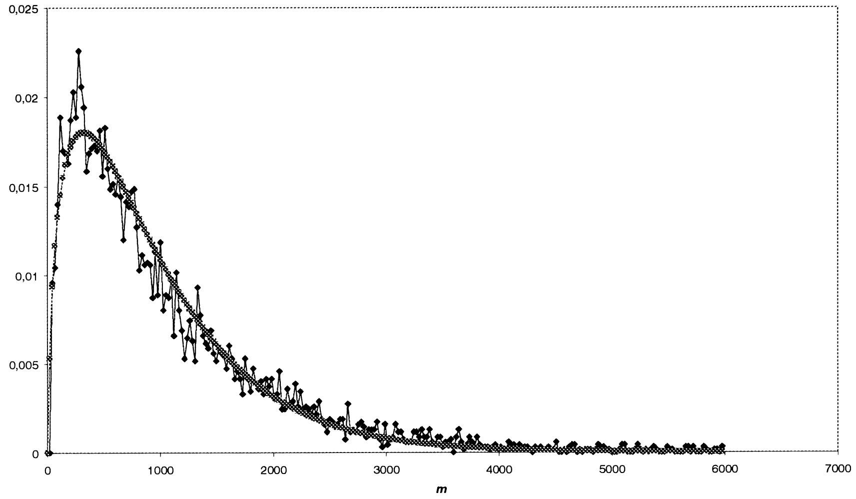

Fig. 9. Actual and theoretical $\left[g(L') = \left(\frac{m}{2 \cdot \overline{L'}}\right)^{\frac{m}{2}} \cdot \frac{1}{\Gamma\left(\frac{m}{2}\right)} \cdot L'^{\frac{m}{2}-1} \cdot e^{-\frac{m \cdot L'}{2 \cdot \overline{L'}}} \right]$ probability distributions of the truncated hydraulic length $L' = \sum_{i=1}^{m} l_i$ of the whole Yvel basin, with $m = 3$.

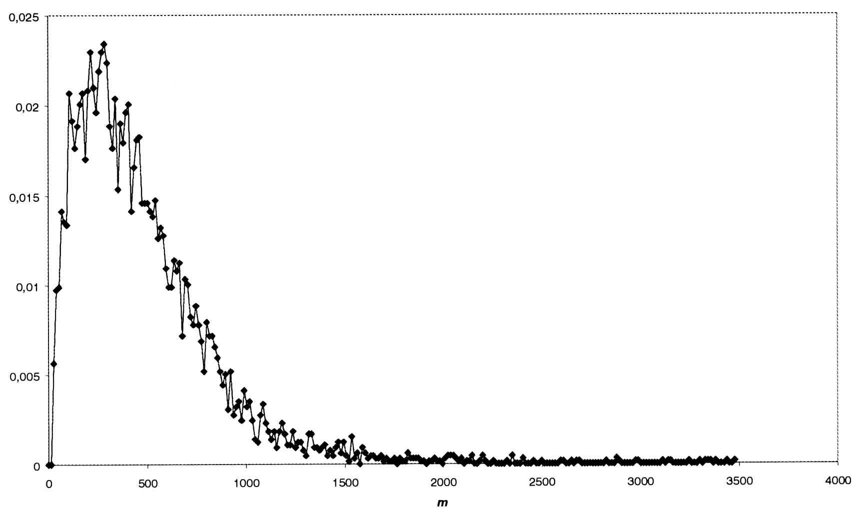

Fig. 10. Actual probability distribution of the truncated hydraulic length $L' = \sum_{i=1}^{m} l_i$ of the whole Yvel basin, with $m = 2$.

Moreover Fig. 9 shows how a gamma law fits very well to the actual distribution of L', for $m = 3$. This gamma law is not a statistical fitting but a theoretical proposal (Cudennec *et al.*, 2003):

$$g(L') = \left(\frac{m}{2\overline{L'}}\right)^{\frac{m}{2}} \cdot \frac{1}{\Gamma\left(\frac{m}{2}\right)} \cdot L'^{\frac{m}{2}-1} \cdot e^{-\frac{mL'}{2\overline{L'}}}$$

—where $\overline{L'}$ is the actual mean value of L'—, which has been obtained from a statistical physics reasoning based on a Strahler-components self-similar space identification. This confirms the idea of the hierarchy constraint and shows that a gamma law structure is underlying in the organization of the considered river network.

CONCLUSION

We consider new geomorphological objects within the river network of a basin, based on the Strahler ordering scheme: the hydraulic path and its components. The length of the hydraulic path L is thus the sum of the lengths of its n different components l_i, where n is the basin's highest order. It appears that the l_i probability distributions follow convex and rapidly decreasing shapes and have a strong scaling property, except for the highest orders. This scaling is consistent with river networks fractality and its upper limit can be explained by the branched-out topology. Indeed, for a single basin, the highest order component l_n necessarily can not have a decreasing probability distribution. This is due to its particular position in the Strahler hierarchical network organization. For similar reasons, probability density functions for l_{n-1}, l_{n-2}, ... present a weaker scaling than for l_1, l_2, ... The probability distributions of the truncated hydraulic lengths L' are negatively skewed and some of them can be described by a gamma law, whereas the probability distributions of L are positively skewed. This shows how a gamma law underlies the river network self-similar organization, even if the shape does not obviously appear in the basin-scale geomorphological functions, due to hierarchical constraints.

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