Chapter 11

Nonlinear Dynamical Studies of Global Magnetospheric Dynamics

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Abstract. The coupled solar wind-magnetosphere system exhibits a high level of complexity, a characteristic of nonlinear systems. Such nonlinearity has been suggested by many linear correlative studies using ground and spacecraft data, and coupled with its dissipative nature, the magnetosphere is expected to show the features of nonlinear dissipative systems such as low dimensionality and chaos. The techniques of state space reconstruction from time series data have been used recently to study such behavior. Among the most important aspects of nonlinear dynamical techniques is the possibility of reconstructing the dynamics from experimental data and thus gaining insight into the physics of complex nonlinear systems, independent of particular modeling assumptions. The correlation dimension, which gives the dimensionality of the dynamical system, has been computed using the auroral electrojet indices AE and AL and yield values close to 3 for the magnetosphere, indicating low dimensionality, and consequently deterministic dynamics. Based on this result a set of dynamical equations has been constructed from the AE time series data and yields good predictions for 1–4 hrs. The coupled solar wind-magnetosphere system has been modeled as an input-output system using the solar wind $vB_z$ as the input and the auroral index AL as the output, and such nonlinear prediction techniques have given good substorm predictions. In the case of magnetic storms, the

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lack of continuous solar wind data limits the scope of input-output studies. Current attempts to predict intense storms from the $Dst$ yield good magnetic storm predictions for 3–4 hours after the onset and can distinguish the very intense from the milder storms.

11.1 Introduction

Ground-based and spacecraft observations of the magnetosphere have shown a high degree of irregularity, and their interpretation using conventional tools such as linear prediction filters has led to interesting results but has been limited in its scope. Recent developments in the theory of nonlinear dynamical systems that exhibit such irregular behavior have led to new techniques based on the characterization of the intrinsic phase space structure using time series data. The key advantage of these techniques is that they yield the inherent dynamical characteristics of the system as represented in the observational data. The basis for this approach is the realization that seemingly irregular behavior does not necessarily require a complex dynamical or statistical description, but can arise from simple dynamical models because of nonlinearities. Thus a physical system that exhibits complex behavior can be a deterministic dynamical system with a few degrees of freedom. This has led to a variety of techniques aimed at characterizing nonlinear phenomena that exhibit complexity. Among the most important aspects of nonlinear dynamical techniques is the possibility of reconstructing the dynamics from experimental data and thus gaining insight into the physics of complex nonlinear systems, independent of particular modeling assumptions. This is done by examining the reconstructed phase space representations of a time series of one or a few variables.

The extensive data available from satellite and ground-based instruments provide an appropriate database for the nonlinear dynamical study of the magnetosphere. The data from many ground-based magnetometers around the globe have been recorded for many decades in the form of the indices such as the AE, AL, and $Dst$ [Allen and Kroehl, 1975; Mayaud, 1980]. Satellite data from the upstream satellites, notably IMP-8 and ISEE-3, and have been used for correlated analysis of the coupled solar wind-magnetosphere system [Akasofu, 1981; Baker, 1986; Baumjohann, 1986; Clauer, 1986]. The major features of the global magnetospheric dynamics are substorms and storms, and have been widely studied using a variety of techniques [Kan et al., 1991]. The essential global characteristics of substorms are presented in the auroral electrojet indices such as AE and AL, and the long time series data of these indices make it suitable for the phase space reconstruction techniques, which require large data sets to yield meaningful results. For the study of the geomagnetic storms, the $Dst$ index data yield similar advantages.
The complexity in the observational data from natural as well as laboratory systems is usually modeled in two ways. In the first approach, the physical system is described in terms of basic principles with a model, which usually consists of a set of ordinary differential equations. Such a model is deterministic in character and the system dynamics may be predicted for given initial conditions. The second is the statistical approach in which the randomness arises from the interaction of a large number of degrees of freedom. The statistical model predicts averages of physical variables, and an ensemble of initial conditions, rather than specific ones, are meaningful in this case. In general the deterministic models have dimensions smaller than that of the initial state space whereas statistical models are based on stochastic processes which are space-filling and thus are high dimensional.

The novelty of the nonlinear dynamical techniques is in their ability to distinguish the statistical and the dynamical characteristics of a system from time-series data. The next section is an overview of dynamical systems studies, in particular the techniques of the reconstruction of dynamics and recent applications. The characteristic quantities such as dimension, exponents, entropy, etc. of the magnetosphere computed from the time series data are discussed in Section 11.3. In Section 11.4 the nonlinear models, developed using the time series data are presented. The techniques of prediction using the nonlinear dynamical techniques and their applications to substorms and storms are described in Section 11.5. The concluding section summarizes the main achievements of nonlinear dynamical techniques and their future outlook.

11.2 Reconstruction of Phase Space from Time Series Data

The motivation for the study of nonlinear dynamics of systems that exhibit complexity is that a simple deterministic system can lead to such irregular behavior and if the system is chaotic, it could be modeled by a few nonlinearly coupled ordinary differential equations. This remarkable property arises from the inherent characteristic of a dissipative nonlinear system, viz. its phase space volume contracts as the system approaches its asymptotic state. This dynamical state is called an attractor and may generally be described by fewer variables than the original system. The attractor is characterized by its dimension, which is a lower bound to the number of independent variables necessary to describe the attractor. In nonlinear dissipative systems the dimension has fractional values and is called a fractal. This property is indicative of chaotic dynamics governing the motion on the attractor. The chaotic behavior of such a system can then be characterized by different quantities, such as Kolmogorov-Sinai entropy and Lyapunov exponents.

In general, quantities such as dimension, exponents and entropy are
Reconstruction of 3 dimensional space with $\tau = 2$ from $x(t)$

\[
t_1 \quad t_2 \quad t_3
\]
\[
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots
\]
\[
\bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg|
\]
\[
x_1(t_1) \quad x_2(t_1) \quad x_3(t_1) \quad \rightarrow \quad X_1
\]
\[
\bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg|
\]
\[
x_1(t_2) \quad x_2(t_2) \quad x_3(t_2) \quad \rightarrow \quad X_2
\]
\[
\bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg| \quad \bigg|
\]
\[
x_1(t_3) \quad x_2(t_3) \quad x_3(t_3) \quad \rightarrow \quad X_3
\]

Figure 1: The construction of the time series of a multi-dimensional vector from a single variable $x(t)$ for $m = 3$ and $\tau = 2$. The dots mark the times at which $x(t)$ are given and the vertical lines mark the values chosen to construct the vector $x_i$ at $t_i$.

determined from the system’s evolution in phase space, but most measure-ments are of one or few variables and thus the details of the phase space required for computing these characteristic quantities are apparently lack-ing. However, this difficulty can be overcome if the system variables are sufficiently nonlinearly coupled [Packard et al., 1980; Ruelle, 1989; Takens, 1981]. In such cases the time delay embedding technique is an appropriate method for using time series data to reconstruct the phase space and obtain its characteristic quantities. In this technique we construct a $m$-component “state” or “phase” vector $X_i$ from a time series $x(t)$ as

\[
X_i = \{x_1(t_i), x_2(t_i), \ldots, x_m(t_i)\},
\]

where $x_k(t_i) = x(t_i + (k - 1)\tau)$ and $\tau$ is an appropriate time delay (of the order of characteristic physical time scales). A simple case of the construction of the state or phase space from the time series $x(t)$ is shown in Fig. 1. The time series $x(t)$ is given at $t_1, t_2, t_3, \ldots$ marked by the dots and are equally spaced. Choosing a time delay of $\tau = 2$ and an embedding dimension of $m = 3$, the vertical lines marks the values of $x(t)$ that constitute the vector $X_i$ at time $t_i$.

11.2.1 Correlation dimension

Attractors are sets of points in the state or phase space defined by the evolution of the system variables and their dimensions are closely related to the corresponding numbers of degrees of freedom. In this reconstructed
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phase space the distribution of state vectors is directly related to the
dimension, which is usually defined from a scaling behavior. The dimension
of a set of points is defined by the Hausdorff dimension [Falconer, 1990;
Wiggins, 1990], but it cannot be computed readily from the observational
data. The dimension can be obtained by defining a suitable quantity that
depends on the distribution of points in the phase space and by examin-
ing its scaling with phase spatial distance. An appropriate function for
this analysis is the correlation sum or integral [Grassberger and Procaccia,
1983a; Eckmann and Ruelle, 1985] defined for \( N \) vectors distributed in an
\( m \)-dimensional space as a function of distance \( r \) as

\[
C(r; m) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \Theta(r - |X_i - X_j|),
\]

where \( \Theta \) is the Heavyside step function. If the number of points is large
enough this distribution will exhibit a power-law scaling with \( r \) for small \( r \) as

\[
C(r, m) \sim r^\nu,
\]

where \( \nu \) is the correlation dimension, and is defined as

\[
\nu = \lim_{r \to 0} \frac{\log C(r, m)}{\log r},
\]

As the embedding dimension \( m \) is increased, the correlation dimension
converges to its true value for an attractor and \( \nu \) is less than the embedding
dimension \( m \) beyond this value. However, in the absence of an attractor,
the phase space is filled and the computed dimension will be the dimension
of the embedded space, e.g., in the case of random processes. For an
attractor, the minimum degrees of freedom is the nearest integer above \( \nu \).

It should be noted that dimensions are geometrical characteristics and
reflect the way the points are distributed. However, they are connected
with the evolution of a deterministic dynamical system in which nonlineari-
ity couples the multitude of available degrees of freedom and dissipation
leads to a contraction of the available phase space, thus yielding a small
dimension. This transition from a system of many degrees of freedom to
one with a few is a form of self-organization.

11.2.2 Kolmogorov entropy and Lyapunov exponents

In a chaotic dynamical system, nearby initial states evolve into states
of very different behavior, thus leading to mixing or loss of coherence. The
rate of change of the ability to specify the microscopic state of a dynamical
system is characterized by the Kolmogorov-Sinai (KS) entropy [Lichtenberg
and Lieberman, 1983], which is defined in terms of the invariant measures of the system and is thus well defined for cases with known dynamics. As in the case of the correlation dimension the formally defined KS entropy may not be readily computed when the time series data in just one or a few variables are available. However another quantity, the $K_2$ entropy, which is directly related to the K-S entropy [Grassberger and Procaccia, 1973b], can be computed from the observational data. The $K_2$ entropy is a lower bound of the KS entropy and may be computed from the correlation integrals $C(r, m)$ as

$$K_2 = \lim_{r \to 0} \frac{1}{T} \ln \frac{C(r, m)}{C(r, m + 1)},$$

where $T$ is the sampling time. When $K_2$ is nonzero, the system is chaotic; if the entropy is infinite the system is random (nondeterministic).

Although a fractional correlation dimension is an indication of the chaotic nature of a dynamical system, it is not a sufficient condition. Further, although a finite value of the $K_2$ entropy does indicate its chaotic nature, but it is computed from the same correlation integrals $C(r, m)$ as the correlation dimension and thus may not be an independent quantity. It has been shown that colored noise, viz. noise whose frequency spectrum has a power law $f^{-k}$ with index $k$, can yield a finite fractional correlation dimension [Osborne and Provenzale, 1989] even though the signal actually has infinite number of degrees of freedom. Although this result has not been extended to other types of random signals, it underscores the need for an independent confirmation of the chaotic dynamics indicated by the fractional dimension. The Lyapunov exponents provide this confirmation of chaos.

The time evolution of a deterministic dynamical system is given by a trajectory in the state space and its stability is quantified by the Lyapunov exponents. A trajectory is stable if two neighboring points on the trajectory remain in the small neighborhood. On the other hand a trajectory is unstable if initially nearby points on it diverge from each other. If the initial points are at a distance $\Delta x(0)$ from each other and at time $t$ this distance evolves to $\Delta x(t)$, an average rate of divergence or convergence can be estimated by the largest Lyapunov exponent defined as

$$\lambda \equiv \lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{\Delta x(t)}{\Delta x(0)} \right) = \lim_{t \to \infty} \frac{1}{t} \sum_i \ln \left( \frac{\Delta x(t_i)}{\Delta x(t_i - 1)} \right).$$

As is evident from this relation, $\lambda$ is a time averaged quantity. There are many algorithms for computing these exponents from time series data [Abarbanel et al., 1993]. The simple algorithm of Wolf et al. (1985) starts
with the construction of a trajectory in the phase space by the method of time-delay and embedding. The algorithm considers two points on the trajectory at time \( t \), separated by a distance \( \Delta x(t) \), and then looks for the locations of these two points at time \( t + 1 \). The ratio of the new distance \( \Delta x(t + 1) \) to the original one gives a contribution to the sum in the above equation. The process is repeated by advancing in time until all points on the trajectory are exhausted. With a suitable normalization, the sum of all such contributions forms the largest Lyapunov exponent. The existence of a positive exponent implies that neighboring trajectories diverge exponentially in a locally defined direction in phase space. In the same time, a dissipative system will have at least one negative Lyapunov exponent and the trajectory will collapse exponentially in another local direction. This simultaneous expansion and contraction leads to chaotic dynamics.

A Lyapunov exponent has two important implications. The first is its value, the inverse of which is the characteristic time of chaos in the system. The second is the fact that a positive exponent is the sufficient condition for a system to be chaotic. This second point is particularly important because of the fact that while a fractional dimension generally indicates chaotic dynamics, systems which are not chaotic have been shown to have fractional dimensions [Osborne and Provenzale, 1989].

### 11.2.3 Singular spectrum analysis

While the correlation dimension and Lyapunov exponents characterize the dynamical system, from the modeling point of view it is crucial to determine the number of actual degrees of freedom or variables. A salient feature of chaotic dynamical system is that it is described by a small number of coupled nonlinear ordinary differential equations. The number of variables required to describe the system however may be between the nearest integer above \( \nu \) and \( 2\nu + 1 \) according Takens' theorem [Takens, 1981]. An estimate of the actual number of variables can be obtained from singular spectrum analysis [Broomhead and King, 1986; Mees et al., 1987; Vautard and Ghil, 1989]. From the \( m \)-dimensional state vectors \( X_i \) defined earlier, a trajectory matrix can be constructed as

\[
X = N^{-1/2} \begin{bmatrix}
    x_1(t_1) & x_2(t_1) & \cdots & x_m(t_1) \\
    x_1(t_2) & x_2(t_2) & \cdots & x_m(t_2) \\
    \vdots & \vdots & \ddots & \vdots \\
    x_1(t_N) & x_2(t_N) & \cdots & x_m(t_N)
\end{bmatrix}
\]

where \( N \) is the number of vectors (usually close to the number of data points), \( t_i = i\Delta t \), and \( \Delta t \) is the sampling time. This \( N \times m \) matrix contains
all the dynamical features of the system embodied in the data. However the essential features of the dynamics may be described by a smaller number of linearly independent vectors. The number of such nonsingular vectors and the vectors themselves may be obtained from a singular spectrum analysis of the $N \times N$ matrix $XX^T$. Further, this analysis shows that the relevant eigenvalues and eigenvectors may be obtained from the $m \times m$ matrix $X^TX$ [Broomhead and King, 1986]. This is a major advantage because the number of vectors $N$ may be many orders of magnitude larger than the number of components $m$ of each vector. For example, in the analysis of time series data of dynamical systems the data size $N$ is typically in thousands and the embedding dimension $m$ is about 10. The number of non-zero eigenvalues of $X^TX$ then gives the number of variables needed to model the system and the eigenvectors give the directions of these variables in the embedded space.

11.3 Dynamical Studies Using Geomagnetic Time Series Data

The irregularity in the solar wind-magnetosphere coupling is evident in many observational data such as the indices AL, AE and $Dst$ and many spacecraft measurements. The earliest indication that this irregular behavior can be linked to the nonlinearity in the magnetospheric activity came from the linear prediction filter analysis of AL index with respect to the product $VB_s$ of the solar wind flow speed $V$ and the southward component $B_s$ of the interplanetary magnetic field [Bargatze et al., 1985; Clauer, 1986]. These studies showed that the magnetospheric response to the solar wind input is not linear and also there are two time scales corresponding to directly driven and loading-unloading processes. It is now recognized from the study of nonlinear dynamical systems that such irregular behavior can arise from a deterministic system that is described by a small number of variables, in contrast to a stochastic system which has a large number of degrees of freedom. This recognition motivated the modeling of magnetospheric substorm dynamics as a dripping faucet that is low-dimensional and chaotic [Baker et al., 1990].

Considering the solar wind-magnetosphere interaction to be a complex system, its dynamical properties may be obtained from the time series data of one of its variables. The auroral electrojet (AE) index is a convenient and well documented measure of magnetospheric activity. The 1-min averaged time series AE index for the period 1–20 January 1983 is shown in Fig. 2. Analysis of time series AE index data using the time delay embedding and correlation sum techniques described above has shown that magnetospheric activity is characterized by a correlation dimension of $\sim 3.5$ [Vassiliadis et al., 1990]. Figure 3 shows the convergence of the correlation dimension $\nu$ with increasing embedding dimension. The straight line in this figure shows
the expected linear increase of $\nu$ in the case of random noise. A detailed analysis for different magnetospheric activity levels and with additive random noise, shows that the correlation dimension has values in the range 3.5–4 for all activity levels and that the noise in the data does not affect these results significantly. This indicates that magnetospheric activity may behave as a low-dimensional chaotic system.

The AL index has been used similarly to compute the correlation dimension and values in the same range as for the AE index have been obtained [Roberts et al., 1991]. For the 1–20 January 1983 AE data (1–min, 28,800 points) and the Bargatze et al. (1985) AL data (2.5–min, ~40,000 points), the computed dimensions have values between 3 and 4 [Vassiliadis et al., 1990; Roberts et al., 1991]. For the 1–5 April 1983 AE data (1–min, 7,200 points), smaller values of the correlation dimension 2.4, have been obtained [Shan et al., 1991a]. While the difference in the dimension values need not be due to the different data lengths, the data length used in these computations is an important issue. Estimates of the data size required for these techniques to be meaningful are obtained by considering the statistical error
Figure 3: The function $\ln C / \ln r$ as a function of the embedding dimension for time series AE index of January 1983. Data sets consisting of 5000 points with time delay 10–50 min and embedding dimension from 1 to 10 were used. The saturation for embedding dimension beyond 5 is apparent and gives a correlation dimension $\sim 3.5$. The straight line corresponds to the case of random noise, which has an infinite number of degrees of freedom (dimension) and thus shows no saturation [Vassiliadis et al., 1990].

in the computation of the slope of the correlation sum $C(r, m)$, especially in the region of small $r$ [Ruelle, 1990; Theiler, 1990; Abarbanel et al., 1993]. An estimate of the number of data points required is in the range $10^d - 30^d$, where $d$ is the nearest integer above $\nu$ [Wolf et al., 1985]. It should, however, be noted that $\nu$ is determined by the distribution of $C(r, m)$ for small $r$; and the relevant quantity is the number of $C(r, m)$ values, and not the data size directly. From $N_t$ data points, $N = N_t - (m - 1)\tau$ vectors are constructed and this yields $N_c \simeq N^2/2$ values of $C(r, m)$. For small data sizes, $N_t \sim N$ only for small $m\tau$, otherwise $N$ can be significantly less than $N_t$.

An important issue in the computation of correlation dimension is the time delay $\tau$ which should be chosen on the basis of physical considerations such as the auto-correlation time [Liebert and Schuster, 1989] and mutual-information time scale [Fraser and Swinney, 1986]. The auto-correlation
time of AE and AL indices are 100 min [Shan et al., 1991a; Roberts, 1991] and the mutual information time is \( \sim 10 \) min [Prichard and Price, 1992]. However variations of the time delay over the range 1–1000 min are found to yield \( \nu \) in the same range, viz., 3.5–4, for a data size of 30,000.

The reconstruction of the phase space from the time series data can yield the dynamical features in the original system, as provided by Takens’ theorem [Takens, 1981]. However, the computation of correlation dimensions from time series data has limitations due to the finite length of data, the presence of noise, the choice of the time delay, the autocorrelation, the non-stationarity, etc. The roles of these factors in the low-dimensionality of the magnetosphere have been studied in a number of papers following that of Vassiliadis et al. (1990) and reviewed by Roberts (1991) and by Sharma (1995). The effect of autocorrelation was studied by Shan et al. (1991a and 1991b) by computing the correlation dimension for \( \tau \) values comparable to the autocorrelation time of the AE data (1–5 April 1983). The computed values of the dimension was 3.5 for \( \tau = 1 \) min and 2.4 for \( \tau > 30 \) min. Also they drew attention to the fact that a randomly phased power law spectrum with an index of 2 gives a dimension 2.1, and thus the low-dimensionality exhibited by the AE/AL data could be due to a noise like behavior with a power law spectrum with an appropriate index. Many features of the AE data have indeed been shown to be reproduced by a bicolored noise spectrum [Takalo et al., 1993, 1994]. Further, it was shown that the power spectrum of AE data has modulations with a period of 24 hrs, which could be the cause of the increase of the dimension value by unity. Based on these it was concluded that AE time series yields a dimension between 2 and 3 [Shan et al., 1991a, 1991b]. Another study of the correlation dimension by Roberts et al. (1991) used the AL data set of 40,000 values at 2.5 min resolution constructed by concatenating subsets corresponding to varying activity levels [Bargatze et al., 1985]. For delay times in the range 5–125 min, the correlation dimension was found to be close to 4.

In the reconstructed time-delay embedded space the dynamical system evolves along a trajectory. The correlation among different parts of the trajectory that intersects a small but specified region of the phase space yields the correlation dimension. However this correlation given by \( C(r) \) includes not only the correlation between different passes of the trajectory but also those along it. In the case of long autocorrelation times \( C(r) \) may be dominated by the latter and consequently the dimension computed from \( C(r) \) may be spurious. This undesirable effect can be eliminated by introducing a cut-off parameter \( W \), thus excluding the points within a specified distance from the reference point [Theiler, 1986, 1990]. When this technique was used the convergence in the computation of the correlation dimension from many segments of AE and AL data were lost, except for
the April 1983 data, in which case the dimension had a value of 3.1.

The AL index time series has been used to compute the largest Lyapunov exponent and it yields values in the range 0.06–0.17 min⁻¹ with a typical value of ~0.1 min⁻¹ [Vassiliadis et al., 1991]. Within the limitations of the algorithms used this confirms the chaotic nature of magnetospheric activity [Eckmann and Ruelle, 1992]. Furthermore, this gives a characteristic time scale for magnetospheric activity and for a growth by one order of magnitude, corresponds to a time scale of ~ ln(10/λ), i.e., 14–38 min. It may be noted that the time scales of ~60 min [Bargatze et al., 1985] obtained from the linear prediction filter analysis has been interpreted as the time scale for the onset of chaos in magnetospheric activity [Baker et al., 1990] and thus the time scales are roughly consistent.

The state space reconstructed by time-delay embedding is quite noisy, mainly due to the randomness of the solar wind driver. This noise can be removed significantly by the technique of singular spectrum analysis discussed in the previous section. The AE index data has been used to compute the singular spectrum of the auroral electrojet indices and it yields 3 dominant eigenvalues, as shown in Fig. 4. The higher eigenvalues are not zero, but beyond the first three, their values are within a “noise floor” and thus are not significant. The corresponding eigenvectors define the principal directions and thus the principal coordinates in the embedding space, and

![Figure 4: The normalized singular spectra computed from the first 4000 min of the 1 min averaged AE data beginning January 1, 1983 with τ = 40 min and m = 5, 10, 15, 20 and 25 as labelled. The eigenvalues beyond the first 3 are close to each other and define the noise floor [Sharma et al., 1993].](image-url)
Figure 5: The phase space of magnetospheric activity reconstructed from the AE data. The upper plot is from a time delay embedding $= x(t)$ versus $x(t + \tau)$, with $\tau = 40$ min and the lower one is a plot of the variables $y_2$ and $y_3$ obtained by projections along the eigenvectors [Sharma et al., 1993].
the time series of these principal variables may be obtained by projecting the given time series onto these directions [Sharma 1993, 1994; Sharma et al., 1993]. The projected variables $x_i$ may then be used to describe and reconstruct the dynamics. A comparison of the reconstructed phase space from the time delay embedding and the singular spectrum analysis, brings out many interesting features, as shown in Fig. 5.

Both the reconstructions show clear evidence of a dynamical trajectory that follow a certain pattern. The time delay reconstruction yields very noisy trajectory with widespread randomness superimposed on it. In the case of AE index, this could be due mainly to the turbulence in the solar wind which drives the magnetospheric dynamics. The AE index measures the response of the magnetosphere to the solar wind magnetic field variations. This response consists of a component with randomness, due probably to the turbulent solar wind and another component due to the internal magnetospheric dynamics [Rostoker et al., 1980, 1987]. The singular spectrum analysis removes the turbulent or random effects and yields the deterministic dynamical features. This not only reveals the dynamical aspects, but also yields a new technique for the analysis of time series data.

The variables $\{y_1, y_2, y_3\}$ obtained from the singular spectrum analysis have been used to compute the correlation dimension by constructing a new

![Figure 6: The slopes of the correlation sum $C(r)$ obtained from the principal components shown in Fig. 5 as a function of $\ln r$, for $m = 1$ to $10$. The slopes converge to 2.5 on the average and is independent of the autocorrelation effects for up to $W = 100$.](image-url)
state space. The correlation sum \( C(r) \) is readily defined from these vectors and it yielded a dimension of 2.3, as shown in Fig. 6, and was found to be independent of the autocorrelation effect for the AE data of January 1983 [Sharma et al., 1993]. Since the techniques of state space reconstruction from the time series data have many possible pitfalls it is essential to use as many independent techniques as possible to determine the dynamical behavior. Prichard and Price [1993] used two other measures of dimension, namely the Takens estimator [Takens, 1985] and the BDS statistic [Brock, 1988] and found no evidence for low dimensional behavior in the AE data of January 1993. There are other techniques such as the scaling of a suitably defined structure function that support the low dimensionality of magnetosphere. Thus, many techniques have been used to study the magnetospheric dynamics using the time series data of AE/AL index from different intervals. In some of these intervals the low-dimensionality is not clear cut, while some others, e.g., the Bargatze et al. data set (1985) the nonlinear behavior and hence the low-dimensionality is clear [Sharma, 1995].

11.4 Modeling of the Global Behavior

The solar wind-magnetosphere system is a natural input-output system and can be modeled as a RLC circuit, in which the resistance \( R \), the inductance \( L \) and the capacitance \( C \) are obtained by optimizing the output (AL index) for given input, viz. the interplanetary magnetic field (IMF). It is found that for many intervals of IMF/AL, these parameters have values within a narrow range. Using these optimized values and a specified length of IMF, the AL index is predicted [Vassiliadis et al., 1993]. For the Bargatze et al. (1985), data set the correlation between the predicted and actual values of the AL index varies in a wide range and has values close to those found with linear prediction filters.

In the low dimensional solar wind-magnetosphere system, time series data can be used to construct the dynamical equations [Sharma, 1993, 1994; Sharma et al., 1994]. A procedure for the construction of dynamical equations may be developed along the following lines. First, the time series of the variables are obtained by projecting the given time series data along the directions of the orthogonal eigenvectors given by the singular system analysis. Second, a set of coupled first order ordinary differential equations are written with chosen couplings among the variables. Third, the coefficients of the coupling terms in the equations are determined by an appropriate fit to the time series of the variables and their derivatives.

The main step in the construction of equations is the choice of the nonlinear coupling among the variables. Considering a system with 3 variables, its dynamics may be modeled by the set of coupled nonlinear ordinary dif-
ferential equations
\[ \frac{dy_i}{dt} = f_i(y_1, y_2, y_3), \]
where \( i = 1, 2 \) and \( 3 \); and the nonlinear functions \( f_i \)'s are expressed as polynomials with the highest nonlinear term to be cubic. The unknown coefficients in the 3 equations were determined by many techniques, such as \( \chi^2 \) minimization, singular value decomposition, etc. From the time derivatives \( y_i'(t_j) \) at \( t_j \) and the values of \( f_i \)'s, the \( \chi^2 \) can be written as
\[ \chi^2 = \sum_{j=1}^{N} [y_i'(t_j) - f_i]^2. \]

The minimization of \( \chi^2 \) with respect to the fitting parameters then yields the coefficients. These equations can be used to predict geomagnetic activity. The AE data of January 1983 were used to obtain these coefficients and then used for prediction. The predictability may be quantified by comparing the prediction with the observations. If \( x(t) \) is the observational data and \( y(t) \) is the prediction from the model, the predictability of the model can be quantified in terms of the normalized cross-correlation function defined as [Kravtsev, 1989]
\[ D(\tau) = \frac{\langle x(t)y(t) \rangle}{\langle x^2(t) \rangle^{1/2}\langle y^2(t) \rangle^{1/2}}, t = t_o + \tau. \]
Initially \( x(t_o) = y(t_o) \) and \( \langle \rangle \) indicates averaging over an ensemble of initial conditions. We always have \( |D(\tau)| \leq 1 \) and for a partially predictable system \( 0 < |D| < 1 \). A predictability time scale \( \tau_{det} \) may be defined by the condition \( D(\tau_{det}) = 0.5 \), so that the behavior of the system may be considered predictable within this time scale. The correlation function for the AE data is shown in Fig. 7, and yields predictability close to 90% for periods of 1–3 hrs [Sharma et al., 1994].

The low-dimensionality of the magnetosphere has the obvious implication that models with only a few variables are required to describe its dynamics. Along with the studies of low-dimensionality from the series data, simple nonlinear models of magnetospheric dynamics have been developed. The first such model is a mechanical analog model of a dripping faucet [Baker et al., 1990] and has been extended to a plasma physical model based on the change of magnetic flux through a Faraday loop in the magnetotail [Baker et al., 1991; Klimas et al., 1991a, 1991b, 1992]. This model has been used extensively to study the directly driven and the loading-unloading components of substorms. Using the solar wind \( VB_s \) as the input, the AL index obtained from the model compares very well with the actual AL for the data set from Bargatze et al. (1985).
Figure 7: The correlation function $D(t)$ between the actual AE index and that obtained by integrating the dynamical equations. The three cases correspond to different periods and the predictability time scale in these cases are in the range 60–180 min [Sharma et al., 1994].
11.5 Prediction of Magnetospheric Behavior from Time Series Data

The empirical aspects of the solar wind-magnetosphere coupling have been studied extensively using linear prediction filters [Clauer, 1986]. The application of this technique to the interplanetary input such as $V B_{z}$ and the auroral electrojet indices AE and AL as the output has yielded many useful characteristics of the coupling. For example, the coupling functions or the prediction filters have been found to depend on the level of the geomagnetic activity and also they exhibit two peaks corresponding to two time scales in the system. In the following the coupling in the case of magnetic storms is analyzed.

The solar wind-magnetosphere coupling is enhanced when the IMF, convected by the solar wind to the front of the magnetosphere, is southward. A convenient measure of storms is the $D_{st}$ index obtained from the variations of the horizontal component $H$ of the magnetic field on the ground at low latitudes. Typically a $D_{st}$ below $-50$ nT is defined as a storm [Gonzales et al., 1987, 1994]. There is a clear correlation in the structure of the $D_{st}$ and the interplanetary variables during a storm as shown in Fig. 8 for the storm of March 10, 1979. The vertical component $B_{z}$ of the solar wind magnetic field (top panel) turns southwards at about 10 hrs, triggering the storm onset. The solar wind flow speed (second panel from the top) is nearly constant and has the features of a slow speed stream. The solar wind $v B_{z}$ (middle panel) essentially follows $B_{z}$ and the $D_{st}$ (second panel from the bottom) gives the storm and the AE (bottom panel) the substorm activity. This storm starts with a sudden increase of the $D_{st}$ index marking the onset of the storm, followed by a large decrease during the main phase of the storm as energetic particles are injected into the ring current. The minimum in the $D_{st}$ marks the beginning of the recovery phase during which the ring current loses the energetic particles. This phase is not directly correlated with the solar wind induced dawn-to-dusk electric field $V B_{z}$ which recovers on a faster time scale.

The $D_{st}$ index is most widely used to study magnetic storms and it has the advantage of having been monitored continuously over many decades. However storms do not occur very often and in fact, there are only 140 storms with $D_{st}$ value below $-100$ nT in the National Geophysical Data Center (NGDC) data base for 1964–1990. During this period there were only 14 storms for which simultaneous solar wind data were available. It is thus crucial to analyze the predictability of storms from the $D_{st}$ index alone.

The correlation of the southward turning of the interplanetary magnetic field with the storm onset is well established [Gonzalez et al., 1994; Tsurutani et al., 1988]. However, the relationship between the $D_{st}$ and solar
Figure 8: The geomagnetic storm of March 10, 1979. The top panel is the $B_z$ component of IMF, the second panel from the top is the solar wind flow speed, the middle panel is the solar wind induced electric field $V B_z$, the second panel from the bottom is the $Dst$ showing the storm and the bottom panel is the AE index showing substorm activity. All data are 1 hr averaged.
wind parameters and either the duration of a storm event or its magnitude is not clear. For the 14 storms during 1964–1990 with simultaneous $Dst$ and solar wind data a number of possible correlations between the integrated input for southward IMF during the storm, the storm duration, the storm magnitudes, the minimum value of the input were examined [Valdivia et al., 1995]. Among the different variables only $V B_z$ and the storm duration show significant correlation.

The primary cause of a storm is the southward IMF [Gonzalez and Tsurutani, 1987; Gonzalez et al., 1994]. As a simple input–output linear system the solar wind input (dusk to dawn electric field) $V B_z$ can be linked to the $Dst$ index through the kernel or filter $K(\tau)$ defined by the relationship:

$$\text{Output}[t] = \sum_{\tau=1}^{\tau_{\text{max}}} K_i(\tau) \text{Input}[t - \tau + 1] + \sum_{\tau=1}^{\tau_{\text{max}}} K_o(\tau) \text{Output}[t - \tau + 1].$$

The coupling functions or kernels $K_i(\tau)$ and $K_o(\tau)$ have been computed for many epochs [Fay et al., 1986; Murayama et al., 1986]. These functions typically have peaks one or two hours after the IMF turns south. The filter with only ten elements fits the $Dst$ very well, before the recovery phase, as shown in Fig. 9. These linear filter studies have shown that the kernel changes from storm to storm, indicating that either the system is nonlinear or that more variables are needed to predict the $Dst$ evolution, or both [Valdivia et al., 1996]. This limits the utility of the linear filters as a forecasting tool.

### 11.5.1 Prediction using reconstructed phase space

In a low-dimensional dynamical system the trajectories are well defined, although over long periods they may be chaotic. This makes short term predictions of the dynamics in such systems reasonable as well as practical. In the case of systems whose governing equations are known, predictions can be made for given initial conditions by integrating these equations. However, in the case of a system specified by the observational data of one of its variables, the reconstructed phase space can be used to make predictions [Farmer and Sidorowich, 1987, 1988]. The basic principle is illustrated in Fig. 10, showing the trajectories in the neighborhood of the point at time $t$. The filled circles within the neighborhood (the bigger circle) denote the locations of the trajectories and those outside denote the known locations of these points at the next time step. Knowing how the neighboring trajectories evolve, the location of the current state $x(t)$ at time $t + \tau$ can be predicted. Prediction can be made without the model equations by local approximation [Farmer and Sidorowich, 1988]. In this method the data is first embedded in a $m$-dimensional state space and in
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![Graphs showing input, Dst, input filter, and output filter.]

Figure 9: The linear filters relating the interplanetary $V_Bz$ to $Dst$. The predicted (fitted) $Dst$ is obtained from the filters, $V_Bz$ and $Dst$.

In this space we assume a general relation between the vector $y_n$ at time $t = n$ and its future after $k$ time steps, $y_{n+k}$:

$$y_{n+k} = F_k(y_n).$$

Then $F_k$ is expanded in a $m$-dimensional Taylor series in the neighborhood of the point of interest $y_n$, and orders of the expansion retained define the order of the technique.
11.5.2 Prediction of magnetospheric substorms

In the first application of the dynamical prediction method to the AL time series, predictions up to 40 steps (2.5 min), i.e., $\sim 100$ min, were made with errors of $\sim 45$ nT [Vassiliadis et al., 1992]. This indicates that AE or AL index data alone may not be suitable for long time prediction. The input-output nature of the solar wind-magnetosphere interaction can be incorporated into the local linear prediction techniques by including the solar wind input along with the geomagnetic response [Vassiliadis et al., 1994 and 1995]. In this formulation the function $F_k$ becomes a function of the geomagnetic response $x_n$ as well as the solar wind input $u_n$ so that $F_k(x_n, u_n)$. This scheme yields correlation coefficient of 90% compared with the linear prediction values of 60% for the Bargatze data set (1985). Further, the number of filters, which reflect the dimesionality, is in the range 3–6, in agreement with the values of the correlation dimension. However, a similar input-output analysis of the period 30–31 October 1978 [Price et al., 1994] found only limited support of the nonlinear behavior. The discrepancy in the results needs to be studied in detail with respect to the data sets used, the details of the algorithms, etc. [Sharma, 1995]. The neural net technique was used to forecast geomagnetic activity using state space nonlinear filters with the $VB_z$ and AL data and yielded predictions comparable to a linear stochastic model [Hernandez et al., 1993].

11.5.3 Prediction of magnetic storms

Most of the studies using nonlinear dynamical techniques have been of the substorms, based on AE/AL and $VB_z$ data. From the point of view of space weather, geomagnetic storms are clearly more relevant and the prediction technique has been used to analyze the $Dst$ data. Since storms
occur much less frequently than substorms and are usually far apart, it is not practical to predict the onset of storms from $Dst$ alone. However, the important issue is whether a storm in its initial phase will develop to a high intensity, with potentially damaging effects on ground and space systems [Siscoe et al., 1994].

We reconstruct the phase space for the evolution of the $Dst$ index using the embedding technique described in Section 11.2. Thus we construct the vector

$$\vec{X}(t_i) = [Dst(t_i), Dst(t_i - 1), \ldots, Dst(t_i - (m - 1))]$$

with the time delay of 1 hour. A singular system analysis (Fig. 11a) of the $Dst$ data (Fig. 11b) shows that the reconstructed phase space with 2 or 3 variables can adequately describe the dominant features. Therefore the irrelevant directions that are not populated (or populated by noise) can be separated from the deterministic components by projecting the embedded vectors onto the few main orthonormal directions. The time delay plot is shown in Fig. 11c. Figure 11d shows the evolution of the reconstructed first component and Fig. 11e shows the phase space reconstruction with dimension $m = 2$. It is readily seen that the projection preserves the leading features of the time series. This is a strong indication that the storm evolution is governed by a few variables and prediction should be possible. Also the projected dynamics are smoother, suggesting the separation of the deterministic aspect of the evolution from the noise. In this phase space first we construct a map function of the form

$$Dst[t + 1] = F[\vec{X}[t]],$$

that evolves the $Dst$ forward in time. If $F$ depends only on the position in phase space then it can be Taylor expanded around each point and the coefficients of the expansion obtained from its nearest neighbors and fitting an $m$-dimensional polynomial of given degree. This local fitting process is repeated at each point as the predicted $Dst$ is evolved forward in time, then compared with the actual $Dst$. Polynomials of different degrees were tried ranging from 1 to 5, but the one that gave us the best predictive results was the local linear case [Valdivia et al., 1996].

From the 140 storm events in our database, 20 cases that range in magnitude from about $-100$ nT to about $-600$ nT were chosen at random. For the case plotted in Fig. 11a, the number of components $m$ was varied from 1 to 5, and the best prediction was obtained for $m = 2$, which agrees with our conjecture from the singular spectrum analysis. The predictions for the storm time $Dst$ are shown in Fig. 12. The vertical lines mark the starting times and are moved by 1 hr. The convergence of the predictions to the actual, especially prediction of the peak magnitude, is very good. Thus the peak of the storm and its magnitude can be predicted about 1–4
Figure 11: The singular spectrum analysis of $Dst$ data. The singular values, the represent how populated each direction is, for different embedding dimensions $m = 5, 10, 15, 20, 25$ (a). It clearly indicates that there are 2 or 3 relevant directions. The actual $Dst$ (b), and the phase portrait of a two component reconstruction (c). The first projected component (d) and phase space representation of the 2nd and 3rd projected components (e). They clearly show the dynamical features.
hours before it occurs with this technique. This procedure was repeated for other storms and similar results were obtained.

In the case of intense storms, this technique does not yield good predictions because of the small sample, as the procedure tries to find neighbors that are too far apart. Since most of the storms have magnitudes close to $-100$ nT the procedure will tend to predict lower magnitudes for large storms. For example, the big storm of March 1989 [Allen et al., 1990] could not be predicted with any reasonable confidence. This implies that these techniques can distinguish intense storms from moderate ones from the convergence (moderate storms) or the lack of convergence (intense storms) of the predictions [Valdivia et al., 1996].

The input-output model can be used to predict magnetic storms using the solar wind and $Dst$ data. The input $I[t]$, is chosen to be the dusk to dawn electric field $VB_z$ and we take the output as $O[t] = Dst[t]$. Embedded in an $m = m_{in} + m_{out}$ dimensional space, the trajectory is defined by

$$\vec{X}(t_i) = [O(t_i), \ldots, O(t_i - (m_{out} - 1)\tau_{out}), I(t_i), \ldots, I(t_i(m_{in} - 1)\tau_{in})],$$

where the $m_i$'s are the respective embedding dimension and the $\tau_i$'s are the delays. We chose $\tau = 1$ (hour) for the simple reason that it is a long compared to the solar wind transit time across the magnetosphere of about 5 min and thus, the evolution could be treated as a map [Abarbanel et al., 1993]. This is essentially projecting a low dimensional dynamics in a high dimensional space onto this $m$ dimensional subspace and still preserving the information necessary to predict the dynamics, at least for a significant length of time. The optimal embedded dimension will have to be determined empirically as the dimension that gives the best predictability. As $m$ is increased the predictability does not necessarily increase and in fact may decrease for large enough $m$, as the number of nearest neighbors in the higher dimensional space is likely to decrease for the same data size. The input-output approach is successful in predicting the storm onset and part of its evolution, but the current data set for 1964–1990 have simultaneous solar wind and $Dst$ values only in 14 cases out of $\sim 140$ storms with $Dst < -100$ nT. This limits the input-output modeling with the currently available data.

11.6 Summary

The complexity of global magnetospheric dynamics can be modeled as arising from a low dimensional dynamical system. The observational data can be used to construct dynamical models with the use of techniques developed in dynamical systems theory. The salient feature of these methods is the ability to extract the essential details of the system from the time
Figure 12: The prediction of $Dst$ during a magnetic storm. The actual and predicted values are shown in all the panels, which correspond to different starting times of the predictions.
series data of one or a few variables. In space physics satellite and ground based measurements yield data on a few variables such as electric and magnetic fields, plasma density and flow velocities, etc. First principle models are usually developed to account for one or a few physical processes that may be dominant at appropriate scale lengths and times. With the dynamical techniques, models which are independent of assumptions about physical processes and which include all the inherent features contained the data may be constructed. This can lead not only to new and powerful forecasting tools but also provide the scientific basis for geomagnetic predictions. The global coherence in magnetospheric dynamics is due to the electrodynamic nature of the interactions [Siscoe, 1991], because of which the different parts can interact very efficiently within a short time scale. This is the underlying reason for the low dimensional behavior of the magnetosphere, the only known case in large-scale natural systems.

For magnetic storms, the magnitude and duration can be predicted a few hours in advance for moderate storms from the Dst time series alone. This analysis also indicates internal dynamics in the Dst time series that might be used for reliable storm predictions. From the dynamical point of view the response of the ring current to a large particle injection during a long lasting southward component of the IMF seems to behave as 2–3 dimensional system. This low dimensionality needs to be studied in more detail for predicting storms, since it is clear that the solar wind alone is not responsible for the time evolution of storms. Also, the nonlinear dynamical approach is based on firm mathematical basis and can uncover relationships that have not been revealed with other techniques [Casdagli and Eubank, 1992; Weigand and Gershenfeld, 1994]. In another viewpoint the magnetosphere is considered to be a nonlinear stochastic system with a large number of degrees of freedom and infinite number of parameters [Chang, 1992]. Such a system will exhibit low dimensional behavior near criticality where long range correlations among the random fields develop.

The nonlinear dynamical techniques discussed in the above studies use time series data such as AE and Dst which represent global magnetospheric features. While this has given many interesting results, the next level of studies need to incorporate spatial structures and this can be done by using the ideas of spatio-temporal chaos [Kaneko, 1989]. With these latter techniques, many of which are being developed currently [Abarbanel et al., 1993; Cross and Hohenberg, 1993], the data from the different ground magnetometers as well as satellites may be used to study magnetospheric dynamics, including the spatial structures.

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