Chapter 8

Propagation of Alfvén Wave Packet in an Anomalous Dispersion Plasma

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Abstract. Wave packet propagation in an anomalous dispersion medium is examined both theoretically and experimentally. A new propagation velocity is derived using the saddle point method. Alfvén-wave packet experiment at Shizuoka University reveals that the new propagation velocity well agrees with the measured propagation velocities. The center frequency shift due to the differential damping among Fourier components plays an essential role on the packet propagation in anomalous dispersion range of frequency, and is confirmed in this experiment.

8.1 Introduction

A common understanding of wave packet propagation is that the wave packet or its energy propagates with the group velocity, and that the group velocity does not exceed the light velocity as long as normal dispersion media are concerned [1]. However, in anomalous dispersion media, the group velocity

\[ v_g = \frac{c}{\text{Re} \left[ \frac{\partial n \omega}{\partial \omega} \right]} = \frac{c}{\text{Re} \left[ n + w \frac{\partial n}{\partial \omega} \right]} \]

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may exceed the light velocity or even become infinite when the frequency derivative of refractive index \(Re(\partial n/\partial \omega)\) is negative in the anomalous dispersion range of frequency [2,3]. The difficulty of group velocity was first pointed out by Sommerfeld and Brillouin in 1910’s [4,5]. They considered the signal propagation velocity with a step-like wave train, and showed that the fastest signal arrives at the light speed, indicating consistency with the theory of relativity. Hence, they concluded that group velocity loses its physical meaning in anomalous dispersion media. They also considered the propagation velocity of the main body of the signal (signal velocity), but, as written by themselves, "the definition of the signal velocity is somewhat arbitrary". The problem of characteristic velocity of a wave packet traveling in an anomalous dispersion medium still remains unsolved [6–9].

A localized absorption of dielectric medium in a certain range of frequency generally produces a negative slope (anomalous dispersion) on the real part of refractive index, which is related to the imaginary part through the Kramers-Kronig relation: the real and imaginary parts of dielectric function are related each other by the causality principle [1]. Hence, anomalous dispersion always accompanies absorption, and we should take both dispersive and absorptive effects into account in analyzing wave packet propagation in such media.

To describe a wave packet traveling in an anomalous dispersion medium, we applied the saddle point method [10] and derived a new propagation velocity [11, 12]. This velocity is different from the group velocity in the anomalous dispersion cases, and is identical to the group velocity in the normal dispersion limit. A distinct feature of the theoretical results is that the new velocity is not constant even in homogeneous media and this is attributable to the spectrum variation due to the differential damping among Fourier components (center frequency shift during the propagation).

Recently, it was experimentally found that the shear Alfvén wave was strongly influenced by the ion-neutral collision effect and that anomalous dispersion appeared around the resonant frequency of absorption [13]. Using this plasma, we carried out the packet propagation experiments, and determined the average propagation velocity of Alfvén-wave packet by measuring the traveling distance and the flight time of the peak amplitude. The observed propagation velocity well agrees with the theoretical prediction, and the center frequency shift has been confirmed in the experiments [14]. We also observed the split pulse propagation: a short pulse with a center frequency close to the resonant frequency of absorption propagates, after a certain traveling distance, as a superposition of two wave packets, one of which is a shear Alfvén mode and the other a compressional Alfvén mode. This phenomenon is also understood in terms of the differential damping, and is analyzed using the saddle point method.

In the following, the saddle point method in application to wave packet
propagation is briefly described in Section 8.2, and the Alfvén-wave packet experiments are presented in Section 8.3.

8.2 Description of Wave Packet Using The Saddle Point Method

8.2.1 The saddle point method

A wave packet traveling in a homogeneous and dispersive medium is considered using the saddle point method. The full description of the saddle point method in application to wave packet propagation is given in Ref. [11].

A wave packet propagating in a dispersive medium is expressed by the following Fourier integral:

\[ \phi(z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega A(\omega) \exp \left[ \frac{n(\omega)\omega}{c} z - \omega t \right] + C.C. \quad (1) \]

where \( \phi(z, t) \) is the field component of the wave packet, \( A(\omega) \) the Fourier component, \( c \) the light velocity and \( n(\omega) \) the refractive index of the medium. For simplicity, we consider a Gaussian wave packet \( \phi(0, t) = \exp \left[ -\frac{t^2}{2\Delta^2} \right] \cos(\omega_c t) \), and its Fourier spectrum is given by

\[ A(\omega) = \frac{\Delta}{2} \exp \left[ -\frac{(\omega - \omega_c)^2}{2} \right], \quad (2) \]

where the quantity \( \Delta \) is the pulse width and the center frequency (carrier frequency) is denoted by \( \omega_c \). To apply the saddle point method, we rewrite eq. (1) as

\[ \phi(z, t) = \frac{\Delta/2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \exp \left[ \frac{z}{z_0} P(\omega) \right] + C.C. \quad (3) \]

where \( P(\omega) \) is given by

\[ P(\omega) = i \frac{z_0}{z} \left[ \frac{n\omega}{c} z - \omega t \right] - \frac{z_0}{z} \frac{\Delta^2(\omega - \omega_c)^2}{2}. \quad (4) \]

The quantity \( z_0 \) is the characteristic scale length and is assumed to be \( z/z_0 \gg 1 \). To evaluate eq. (3), \( P(\omega) \) is analytically continued to the complex-\( \omega \) plane, and the integration path from \(-\infty\) to \( \infty \) on the real \( \omega \)-axis is modified to pass the stationary point (saddle point) of the exponent \( P(\omega) \), which is defined by

\[ \frac{\partial P(\omega_s)}{\partial \omega} = i \frac{z_0}{c} \left[ \left( \frac{\partial n\omega}{\partial \omega} \right)_{\omega_s} - \frac{ct}{z} \right] - \frac{z_0}{z} \frac{\Delta^2(\omega_s - \omega_c)^2}{2} = 0. \quad (5) \]
Then, the main contribution to the integration is that from the saddle points, and eq. (3) is calculated as the sum of those contributions. Performing the integration around the each saddle point on the modified integration path and collecting those contributions, we finally obtain

\[
\phi(z, t) = \sum_{\omega_s} \frac{\Delta/2}{\sqrt{\frac{z}{z_0} \frac{\partial^2 P(\omega_s)}{\partial \omega^2}}} \exp \left[ \frac{z}{z_0} P(\omega_s) \right] + C.C.,
\]

\[
= \sum_{\omega_s} \frac{A(\omega_s)}{\sqrt{\Delta^2 - i \frac{z}{c} \left( \frac{\partial^2 n\omega}{\partial \omega^2} \right)_{\omega_s}}} \times \exp \left[ i \left( \frac{n\omega_s}{c} x - \omega_s t \right) \right] + C.C. \tag{6}
\]

The saddle point frequency \( \omega_s \) is usually a complex quantity and is a function of \( z \) and \( t \) as seen in eq. (5). When we specify a set of certain values of \( z \) and \( t \), the corresponding saddle point is determined by eq. (5), and the real space amplitude \( \phi(z, t) \) is obtained by substituting the saddle point \( \omega_s \) into eq. (6). If there are more than one saddle points in the complex-\( \omega \) plane, we have to take all the contributions from the saddle points located on the modified integration path. Usually we need to draw the contour plot of \( \text{Re}[P(\omega)] \) on the complex-\( \omega \) plane to choose the saddle point to be taken into account. However, in many cases, only one saddle point is enough to evaluate the integral and the contributions from the other saddle points are negligibly small, which is the case we consider in the following.

The amplitude of the wave packet is determined by the real part of the exponent, \( \text{Re}[P(\omega_s)] \), in eq. (6) and the peak amplitude at a given traveling distance is determined by

\[
\text{Re} \left[ \frac{\partial P(\omega_s)}{\partial t} \right] = \text{Re} \left[ \frac{\partial P(\omega_s)}{\partial t} + \frac{\partial P(\omega_s)}{\partial \omega_s} \frac{\partial \omega_s}{\partial t} \right] = 0. \tag{7}
\]

Since the saddle point satisfies \( \partial P(\omega_s)/\partial \omega = 0 \), eq. (7) is identical to

\[
\text{Im}[\omega_1] = 0 \tag{8}
\]

where \( \omega_1 \) denotes the saddle point for the peak amplitude; i.e., the saddle point \( \omega_1 \) corresponding to the peak amplitude is always located on the real axis.

The average propagation velocity for the peak amplitude is then obtained from eq. (5). Substituting \( \omega_1 \) into eq. (5) and noting that \( \omega_1 \) is
always a real quantity, we have the following equations:

\[ \frac{z}{t} = \frac{c}{\left[ \frac{\partial n\omega}{\partial \omega} \right]_{\omega_1}}, \]  

(9)

\[ \omega_1 - \omega_c = -\frac{z}{c\Delta^2} \left[ \frac{\partial n\omega}{\partial \omega} \right]''_{\omega_1}, \]  

(10)

where ' stands for the real part and '' for the imaginary part. Equation (9) is the average propagation velocity for the peak amplitude and eq. (10) determines the quantity \( \omega_1 \) for a given propagation distance. Although eq. (9) is in the same form as the conventional group velocity, the derivative is taken at a different frequency \( \omega_1 \). The physical meaning of the quantity \( \omega_1 \) will be clear in the later subsection. The quantity \( \omega_1 \) is a function of propagation distance, and, therefore, the average velocity given by eq. (9) changes with the traveling distance even the medium is homogeneous. When we evaluate the average propagation velocity at a certain distance \( z \), we first solve eq. (10) to find \( \omega_1 \) and then evaluate eq. (9) using the \( \omega_1 \).

In a normal dispersion medium (absorption free medium), the right-hand side of eq. (10) vanishes and then \( \omega_1 \) is always equal to the constant quantity \( \omega_c \). In this case, the average propagation velocity, eq. (9), is identical to the group velocity, and is constant as long as the medium is homogeneous.

The velocity \( \frac{dz}{dt} \), referred to as instantaneous velocity in the following, is generally different from the average velocity \( \frac{z}{t} \), and is derived from the variation of saddle point during the infinitesimal propagation distance. The saddle point for the peak amplitude, \( \omega_1 \), always satisfies

\[ \frac{\partial P(\omega_1, z, t)}{\partial \omega} = 0. \]  

(11)

After infinitesimal increment of time \( t \rightarrow t + \delta t \), the peak position and the corresponding saddle point change as \( z \rightarrow z + \delta z \), \( \omega_1 \rightarrow \omega_1 + \delta \omega_1 \) to satisfy eq. (11). Expanding this equation in powers of \( \delta \) and dividing by \( \delta t \), we have

\[ \frac{\partial^2 P}{\partial z \partial \omega} \frac{dz}{dt} + \frac{\partial^2 P}{\partial \omega^2} \frac{d\omega_1}{dt} = -\frac{\partial^2 P}{\partial t \partial \omega}. \]  

(12)

Noting that \( \omega_1 \) is always a real quantity, we have the following equations from the real and imaginary parts of eq. (12),

\[ \frac{dz}{dt} = \frac{c}{(\frac{\partial n\omega}{\partial \omega})' + (\frac{\partial n\omega}{\partial \omega})'' \left( \omega_1 - \omega_c \right) / \left[ (\frac{\partial n\omega}{\partial \omega})'' - (\frac{\partial^2 n\omega}{\partial \omega^2})'' \left( \omega_1 - \omega_c \right) \right]}, \]

(13)
\[
\frac{d\omega_1}{dt} = \frac{-\frac{1}{\Delta^2} \left( \frac{\partial n\omega}{\partial \omega} \right)''^2}{\left( \frac{\partial n\omega}{\partial \omega} \right)' \left( \frac{\partial n\omega}{\partial \omega} \right)'' + \left[ \left( \frac{\partial n\omega}{\partial \omega} \right)''' - \left( \frac{\partial n\omega}{\partial \omega} \right)' \left( \frac{\partial^2 n\omega}{\partial \omega^2} \right)'' \right] (\omega_1 - \omega_c)},
\]
where all the derivatives with respect to \( \omega \) is taken at \( \omega_1 \). The full derivation of the above equations is given in Ref. [12]. In absorption free media (normal dispersion media), eq. (12) gives

\[
\frac{dz}{dt} = \frac{c}{\left[ \frac{\partial n\omega}{\partial \omega} \right]_1' },
\]

\[
\frac{d\omega_1}{dt} = 0,
\]
the latter of which tells us that \( \omega_1 \) does not change during the propagation and, therefore, is equal to the initial quantity \( \omega_c \). Then, eq. (15) is identical to the conventional group velocity, showing the consistency with the previous result of the average propagation velocity. It is also verified that when the propagation distance is infinitesimal, eqs. (13) and (14) are identical to eqs. (9) and (10), respectively.

### 8.2.2 Numerical example

To elucidate the underlying physics in packet propagation in anomalous dispersion media, a numerical example is presented here. An electron Lorentz gas with a single absorption line is considered as a model medium, and the refractive index is given by

\[
n = \sqrt{\varepsilon(\omega)} = \sqrt{1 + \frac{\omega^2_{pe}}{\omega_0^2 - \omega^2 - 2i\omega}},
\]

where \( \omega_{pe} \) is the plasma frequency, \( \omega_0 \) the resonant frequency of absorption and \( \rho \) the collision frequency. Figure 1 shows the real and the imaginary part of refractive index. As seen in this figure, the localized absorption is present at the resonant frequency \( \omega_0 \), and the medium exhibits anomalous dispersion in this frequency range.

Using eq. (5) and (17), we see the motion of saddle point as a function of time (see Fig. 2). Two cases are presented in the figure, one is a normal dispersion case (\( \omega_c \ll \omega_0 \)) and the other an anomalous dispersion case (\( \omega_c \simeq \omega_0 \)). According to the functional dependence of \( n(\omega) \), there are a pole and a zero in the lower half plane, and these are connected by the branch cut (hatched line in the figure). Three saddle points are present in the complex-\( \omega \) plane, and two of them make no contribution to integration. Thus, the
motion of the primary saddle point is shown in the figure. The saddle point moves from the upper half plane to the lower one as time elapses (normal dispersion case) or is trapped near the branch cut (anomalous dispersion case). The saddle points indicated by 1, 2 and 3 in the figure correspond to the leading half maximum, the peak, and the rear half maximum of the packet amplitude, respectively.

The wave packet $\phi(z,t)$ is numerically obtained using eqs. (2), (6) and (17). In the calculations, the space coordinate $z$ is first fixed, and the saddle point is determined by eq. (5) as a function of time $t$. Repeating these procedures by replacing $z$ with $z + \delta z$, we obtain the wave packet $\phi(z,t)$ in the whole range of $z$ and $t$. Figure 3 shows the amplitude of the wave packet at different propagation distances. Figure 3(a) is for a normal dispersion case and Fig. 3(b) for an anomalous dispersion case. The peak amplitudes are normalized to unity for both cases. As seen in the figure, the trajectory of the peak amplitude coincides with the group velocity trajectory in the normal dispersion case. However, the group velocity trajectory and the peak trajectory are completely different in the anomalous dispersion case, and the latter is not a straight line even the medium is homogeneous. This means that the propagation velocity changes with the traveling distance.

The physical mechanism of velocity variation during the propagation is understood by examining the Fourier spectra of the wave packet at the different traveling distances, which are shown in Fig. 4. Figure 4(a) is for a normal dispersion case and Fig. 4(b) for an anomalous dispersion
case. The dashed line indicates the initial profile of the Fourier spectrum (Fig. 4(b)). In the normal dispersion case, the spectrum does not change its initial profile after 300 wavelength traveling distance, while, in the anomalous dispersion case, the frequency spectrum shifts away from the resonant frequency of absorption as the wave packet travels. This is attributable to the differential damping among the Fourier components; the imaginary part of refractive index \((\propto\) damping rate) is a function of frequency as seen in Fig. 1, and the Fourier components with frequencies closer to the resonant frequency of absorption attenuate faster than the rest. Then, after a certain propagation distance, those components disappear, and the resultant frequency spectrum is modified such that the center frequency of the spectrum moves away from the resonant frequency of absorption. This mechanism always acts on the wave packet and the center frequency shift continues during the propagation. Therefore, the propagation velocity of the packet changes with the traveling distance, and is not constant even the medium is homogeneous.

The physical meaning of the quantity \(\omega_1\) is the center frequency of
Figure 3: Amplitudes of a wave packet observed at different propagation distances for a normal dispersion case (a) and an anomalous dispersion case. The peak amplitudes are normalized to unity for both cases, and the propagation distances normalized by the wavelength are indicated in the right. $\omega_c/\omega_0 = 0.6$ for (a) and 1.0 for (b). $\omega_{pe}/\omega_0 = 0.25$, $\rho/\omega_0 = 0.02$. 
Figure 4: Fourier spectra of a wave packet observed at different propagation distances. The dashed lines are the initial spectrum profile. The peak amplitudes are normalized to unity and the traveling distances normalized by the wavelength are indicated in the right. (a): normal dispersion case, (b) anomalous dispersion case.

the spectrum at a given traveling distance, and it changes in accordance with eq. (10). Since the frequency shift is caused by the differential damping among the Fourier components, we expect it to be proportional to the traveling distance $z$ and the differential damping rate $Im[\partial k/\partial \omega] = [(\partial n/\partial \omega)^n/c]$, which is exactly the same functional dependence as in eq. (10).

The average propagation velocity as a function of center frequency of the initial wave packet is shown in Fig. 5, where open circles are determined by the numerical results, the dashed line is the conventional group velocity and the solid line is the average velocity obtained by eqs. (9) and (10).

8.3 Alfvén-Wave Packet Experiments

We here review the experimental procedure and the main results concerning the wave-packet propagation of Alfvén waves in an anomalous dispersion range of frequency [13] [14].
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![Graph showing the relationship between \( \omega_k/\omega_o \) and \( (x/k)/\omega \). The graph illustrates the behavior of the phase velocity \( \theta \omega / \theta k \) as a function of \( \omega_k/\omega_o \). The solid line represents the numerical results, while the dashed line indicates the group velocity. The open circles are from the experiment, with \( \omega_{pe}/\omega_0 = 0.25 \) and \( \rho/\omega_0 = 0.02 \).](image)

Figure 5: Average propagation velocity as a function of initial carrier wave frequency. Traveling distance normalized by the wavelength is 120. The open circles are obtained in the numerical experiments by measuring the flight time of the peak amplitude. The solid line is determined by using eqs. (9) and (10), and the dashed line indicates group velocity. \( \omega_{pe}/\omega_0 = 0.25 \), \( \rho/\omega_0 = 0.02 \).

8.3.1 Experimental procedure and Alfvén wave excitation

The experiment was conducted in a linear device TPH (Test Plasma of High density, 15 cm diam and 200 cm long) at Shizuoka University with a maximum density of \( 4 \times 10^{20} \text{ m}^{-3} \) of a singly charged helium plasma and an ion temperature \( T_i \leq 20 \text{ eV} \) and an electron temperature \( T_e \sim 5 \text{ eV} \) around the plasma center (see Fig. 6). The ionization rate of the pulsed plasma (duration time \( \sim 2 \text{ ms} \)) was not determined or controlled with accuracy, but we reduced it to about 60% near the plasma center by controlling the gas-puff pressure or changing the timing of the discharge operation [13]. The axial magnetic field, \( B_0 = 0.3 \text{ T} \), gives an ion-cyclotron frequency \( \omega_{ci} = 7.2 \times 10^6 \text{ rad/s} \). Then, the strong absorption was found to be localized around the critical frequency \( \omega^* \):

\[
\omega^* \approx \omega_{ci} \left[ 1 + \left( \frac{\rho_n}{\rho_0} \right) \right]^{-1/2} = \omega_{ci} \sqrt{R} \quad (18)
\]

\[
\approx (4.5 - 5) \times 10^6 \text{ rad/s.}
\]

where \( \rho_0 \) and \( \rho_n \) are, respectively, the mass densities of the ion and the neutral, and \( R \) is the ionization rate.
We designed a small antenna (see Fig. 7) for the experiment in order to directly excite the shear Alfvén wave (SAW) of the poloidal mode number $m = 0$ near the plasma center. Antenna excitation was achieved by discharging capacitors charged up to $\sim 10$ kV or using a 1 MW signal amplifier. According to Borg et al. [15], the Green function of the azimuthal magnetic field component of the SAW for the delta function current along the external magnetic field ($z$ direction) is, for our experimental parameters and $\omega/\omega_{ci} \ll 1$, approximately given as follows:

$$G_\theta \sim \frac{1}{4\pi V_A r} \exp \left[ i \frac{\omega}{V_A} z \right], \quad (19)$$

where $V_A$ is the Alfvén speed and $r$ the radial position. This suggests the possibility of direct excitation of the $m = 0$ SAW, for at least $\omega/\omega_{ci} \ll 1$ in our experiment, by an external current along the magnetic field in the plasma center. Considering that there is a small deviation of the poloidal mode from $m = 0$ and that the $m = 0$ compressional Alfvén wave (CAW) is cutoff below $\omega_{ci}$ when the boundary conditions exist, it may be possible to excite the $m = +1$ (right-hand rotation) CAW near the ion-cyclotron frequency; the dispersion relation of the $m = +1$ CAW is known to be influenced by the plasma density profile in the radial direction and the boundary conditions, but not to have a cutoff as long as a vacuum annulus exists between the plasma and the conducting wall [16–18].

The poloidal mode number $m$ was measured by using small magnetic probes azimuthally separated by $90^\circ$ at $r = 1$ cm and $z = 23$ cm downstream from the antenna. The results are shown in Fig. 8(a), which suggests
Figure 7: An antenna to directly excite $m = 0$ shear Alfvén wave at the plasma center.

Figure 8: (a): Measurement of poloidal mode number $m$ as a function of frequency $\omega$ at a radial position $r=1$ cm from the center. (b): Polarization measured with $b_\theta$ and $b_r$ probes, showing left-hand polarization of the $m = 0$ mode at $\omega = 3.5 \times 10^6$ rad/s (left) and right-hand polarization of the $m = +1$ mode at $\omega = 5.5 \times 10^6$ rad/s (right).

The azimuthal mode is changing from the $m = 0$ to $m = +1$ at $\omega \sim \omega^*$. The polarization of the $m = 0$ mode measured with $b_\theta$ and $b_r$ probes at $r = 1$ cm is left handed with the $b_\theta$ component much larger than the $b_r$ component, while the $m = +1$ mode is right handed with almost circular polarization (see Fig. 8(b)). These features in addition to the dispersion relation, which will be shown in Fig. 9, lead to the conclusion that the former is a SAW and the latter a CAW.
Figure 9: Measured real values of dispersion relation of $m = 0$ shear Alfvén wave (SAW) and $m = +1$ compressional Alfvén wave (CAW). The solid lines indicate eq. (20) with ionization rates 60% and 80%.

### 8.3.2 Kramers-Kronig relation

Figure 9 shows the measured frequency and the real wavenumber parallel to the magnetic field, which were determined from the cross-power spectra of the $b_e$ signals detected by two probes with 20 cm separation along the magnetic field. The $m = 0$ mode is obviously in good agreement with the best fit curve of SAW with $\rho_n/\rho_0 = 2/3$ corresponding to 60% ionization ($\nu_{in} \leq \omega^*$) and a plasma density of $4 \times 10^{20}$ m$^{-3}$; the curves in the figure are calculated from the dispersion relation of the SAW including ion-neutral collisions, which is written as

$$S \frac{\omega^2}{k_{||}^2 V_A^2} - \left(1 - S^2 \frac{\omega^2}{\omega_{ci}^2}\right) = 0. \quad (20)$$

The complex factor $S$ is given by

$$S = 1 + \frac{\rho_n}{\rho_0} \frac{\omega}{1 - i \frac{\rho_n}{\rho_0} \frac{\omega}{\nu_{in}}} \quad (21)$$
where $\nu_{in}$ is the ion-neutral collision frequency. On the other hand, the $m = +1$ mode in Fig. 9 follows the dispersion relation of CAW [16] that is little subject to the ion-neutral collision.

One can see in Fig. 9 an anomalous nature of the dispersion curve at $\omega \sim \omega^* (4.5 - 5 \times 10^6 \text{ rad/s})$, and the wave packet observed at the point $z = 23 \text{ cm}$ downstream from the antenna exhibits heavy damping and deformation with anomalous time delay (see Fig. 10) as expected from the theory [11, 12]. The complex refractive index $n_{\parallel} = n_r + i n_i$ in the $z$ direction is calculated from both the dispersion relation shown in Fig. 9
and the damping rate of the wave packet propagating along the $z$ direction. Figure 11 shows the complex refractive index as a function of frequency, from which one can obtain the complex dielectric function, $\epsilon_\parallel = \epsilon_r + i\epsilon_i$, using the relations $\epsilon_r = n_r^2 - n_i^2$ and $\epsilon_i = 2n_r n_i$. Figure 12(a) represents the real and imaginary parts of the dielectric function as a function of real $\omega$. 

Figure 11: Complex refractive index as a function of frequency. The closed circles are obtained from the evolution of phase and amplitude of the launched wave along the $z$-direction. The solid lines are the fitting functions to the experimental data. (a): real part, (b): imaginary part.
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Even though the number of data might not be enough to numerically apply the Kramers-Kronig relation [1], let us try to estimate the imaginary part of the dielectric function from the real one given in Fig. 12(a), by calculating the principal integral of the Kramers-Kronig relation. The result, that is shown in Fig. 12(b), indicates a fairly good agreement with experimental result shown in Fig. 12(a), except for the frequency width in the absorption region.

### 8.3.3 Average propagation velocity of Alfvén wave packets

Let us see examples of spatial evolution of the magnetic component $b_\theta$ of the wave packet propagating along the $z$ direction ($r = 1$ cm). Figure 13 shows that the received signals at different propagation distance and the
Figure 13: Observed wave packets and auto-power spectra of the magnetic field $b_\theta$ of SAW. $\omega_c = 4.33 \times 10^6$ rad/s.

Figure 14: Observed wave packets and auto-power spectra of the magnetic field $b_\theta$ of CAW. $\omega_c = 5.88 \times 10^6$ rad/s.
Figure 15: Variation of center frequency of the spectrum as a function of traveling distance. The closed circles indicate the experimental values, and the vertical bar the half peak width at half height of the auto-power spectrum. The solid line is the results calculated from eq. (10). $\Delta = 3.6 \mu s$, $\omega_c = 4.48 \times 10^6$ rad/s.

Fourier spectra (right column). As seen in the figures, the wave packet propagates with a decrease in central frequency of the spectrum, but the packet profile looks self-similar except for the change in amplitude. The dispersion relation of this mode has been identified as that of the SAW with ion-neutral collisions [13].

Figure 14 shows a CAW packet, whose initial central-frequency is higher than the critical frequency $\omega^*$. This mode has been identified as the $m = +1$ first radial eigenmode of CAW [13], and its dispersion relation is, unlike the shear-Alfvén continuous spectrum, determined by the density and electric resistivity profiles, and the boundary conditions in the radial direction [16]. In this case, the CAW packet propagates with less absorption than the SAW packet, and the central frequency does not shift within the experimental errors because the CAW in this range of frequency is little subject to the ion-neutral collision.

Figure 15 demonstrates that the measured central-frequency of the shear-Alfvén-wave packet is a function of traveling distance. The solid line in Fig. 15 indicates the theoretical curve calculated from eq. (10); since eq. (10) is in a implicit form with respect to $\omega_1$, we should numerically solve it to find the root $\omega_1$. In order to evaluate the frequency derivative
Figure 16: Average propagation velocity as a function of initial central frequency \( \omega_c \). The horizontal bar indicates the half peak width at half height of the auto-power spectrum, and the vertical bar the measurement error. The solid line represents the velocity calculated from eqs. (9) and (10), and the dashed line the group velocity. \( \Delta = 3.6 \) \( \mu s \) and \( z = 20 \) cm.

\[ \frac{\partial n}{\partial \omega} \]", we first fit a linear combination of analytic functions to the experimental data of \( n_i(\omega) \) shown in Fig.11 (b). Differentiating the fitting function and substituting the result into eq. (10), we numerically search the solutions \( \omega_1 \) (those procedures are easily followed by means of mathematical software Mathematica). Figure 15 clearly shows that a close agreement has been obtained between the experimental shift of the central frequency and the theoretical one. The conventional group velocity is based on the assumption that the central frequency remains unchanged and is equal to the initial one, but it does change as has been predicted by the theory [11, 12].

We next estimate the average propagation velocities of the wave packet from the time difference of the peak amplitudes received by two probes at an intervals of 20 cm along the magnetic field: a spline interpolation was applied to the received signals to calculate the wave envelope. The measured average velocity of the wave packet is shown in Fig. 16 as a function of the initial central-frequency \( \omega_c \). The solid curve in this figure represents the velocity calculated from eqs. (9) and (10). In the frequency range far from \( \omega^* \) (normal dispersion region), it is experimentally hard to distinguish between the two velocities, i.e. eq. (9) and the conventional group velocity,
while in the neighborhood of the anomalous dispersion region for the SAW, the measured velocities never go towards infinity as the conventional group velocity shown by the dashed line, but follow the velocity given by eq. (9).

### 8.3.4 Split propagation of short wave packets

When a short wave packet with the carrier wave frequency close to $\omega^*$ was excited in the plasma, we observed, in the downstream region, a signal composed of SAW and CAW packets (referred to as split propagation in the following). Such an example is shown in Fig. 17, where figures in the left column are the received signals at different propagation distances, and the low-frequency and high-frequency components extracted from these signals are indicated in the right figures. It is clearly seen that the observed signals are composed of low-frequency and high-frequency wave packets. The low-frequency wave packet is confirmed to be a SAW, and the high-frequency one a CAW. It was also observed that the center frequency of the SAW packet decreased with the propagation distance, while that of the CAW packet slightly increased with the propagation distance, and at $z = 34$ cm those frequencies were respectively $3.7 \times 10^6$ rad/sec and $5.7 \times 10^6$ rad/sec.

![Diagram](image)

*Figure 17: Observed wave packet at different propagation distances (left column), and the low-frequency and high-frequency component (right column), showing that the observed signal is a superposition of two wave packet. $\Delta = 1.5 \mu s$ and $\omega_c = 5.6 \times 10^6$ rad/s.*
Figure 18: Dielectric function $\varepsilon(\omega)$ as a function of frequency. The closed circles indicate the experimental results, and the solid line the functional approximation (eq. (22)).

Since the resonant frequency of absorption $\omega^*$ ($4.5 - 5.0 \times 10^6$ rad/sec) lies between these frequencies, the Fourier components in the absorption region which is the main part of the spectrum, are strongly damped than that outside the absorption band. Then, the resultant spectrum after a certain traveling distance is expected to consist of two side-band remnants surviving the absorption. The low frequency remnant propagates as a SAW and the high frequency one a CAW. According to the differential damping
mechanism mentioned in the preceding section, the center frequencies of those wave packets shift away from the resonant frequency of absorption \( \omega^* \) and this behavior is consistent with the experimental observation.

So far, we consider the wave packet described by single saddle point (see eq. (6)). The present case is exceptional, and more than one saddle points contribute to the integration in the same order of magnitude. Here, we see the locations of saddle points by making the contour plot of \( \text{Re}[P(\omega)] \) in the complex-\( \omega \) plane. For this purpose, an analytical functional form for the experimentally-observed dielectric function \( \epsilon(\omega) \) should be obtained. We approximate \( \epsilon(\omega) \) by the following function:

\[
\frac{\epsilon(\omega)}{10^6} = 4.5 - 0.5i + \frac{15i}{\omega + 0.01i} - \frac{3 + 1.5i}{\omega - 4.65 + 0.4i},
\]

which is shown in Fig. 18 by the solid line. The contour plot of \( \text{Re}[P(\omega)] \) determined by eq. (22) is shown in Fig. 19. There are two saddle points near the real axis (each saddle point is indicated by \( S_1 \) and \( S_2 \)). The
low frequency one is just located on the real axis, and this corresponds to the arrival of the maximum amplitude of the low-frequency wave packet (SAW), and the location (on the real axis) gives the center frequency $\omega_1$ at this propagation distance. After a certain time, the saddle point in the high frequency side reaches the real axis, and this corresponds to the arrival of the maximum amplitude of CAW packet. The location of this saddle gives the center frequency of CAW packet.

Making the contour plots for different propagation distances and finding the location of saddle points on the real axis, we determined the variation of center frequency $\omega_1$ as a function of propagation distance, which is shown in Fig. 20 (solid line). As is expected, the center frequency of the CAW packet (high-frequency wave packet) increases, and that of the SAW packet (low-frequency wave packet) decreases with the propagation distance. There is a fairly good agreement between the theoretical predictions and the experimental results (closed circles). Consequently, the split propagation of short wave packet as seen in Fig. 17 can be also understood in terms of the differential damping of Fourier components.
8.4 Conclusions

In the theoretical work, we have found that the wave packet in the anomalous dispersion medium does not propagate with the group velocity, but with the new velocity given by eq. (9). The spectrum variation due to the differential damping among the Fourier component causes the center frequency shift during the propagation.

The TPH experiment using the SAW packet reveals that the center frequency shift actually occurs in an anomalous dispersion plasma, showing a good agreement with the theoretical prediction. The measured average propagation velocities also agree with the theoretical values calculated from eqs. (9) and (10). This is the first experimental verification of the validity of the new velocity.

So far, many works have been done on the problem of group velocity in anomalous dispersion media, but they mainly concerned with the dispersive effect, because the dispersive effect is strong in the anomalous dispersion region as seen in Fig. 1, and the center frequency of the packet was assumed to remain constant. The present work shows that the absorptive effect (differential damping) plays the essential role on the packet propagation in the anomalous dispersion medium, and, in this respect, the determination of the center frequency $\omega_1$ at a given traveling distance, i.e. eq. (10), is of primary importance. Then, the wave packet can be still describable by a characteristic velocity given by eq. (9) even in the anomalous dispersion region. In conclusion, the difficulty of group velocity in anomalous dispersion media is overcome.

References


