Chapter 2

FUNDAMENTALS OF SHOCK WAVE PROPAGATION

In this chapter we develop a mathematical concept of a shock wave, derive general equations of steady discontinuous flow in plane geometry, and examine their fundamental properties, using two basic models of solid response behavior. Response of porous and powder materials is treated separately in Chapters 3 and 4.

2.1 Hydrodynamic Jump Conditions and the Hugoniot Curve

Attempts to understand the shock phenomena in mathematical form go back to the late 19th century and the work of Stokes, Riemann, Rankine, Hugoniot, and Lord Rayleigh. A simple explanation for the appearance of a shock front is that it is formed by the balancing of the overtaking of the lower pressure portion of the pressure wave by the high pressure portions through dissipative processes like viscosity and heat conduction. The resulting shock-forming layer is usually extremely thin. In gases the thickness of a shock front is of the order of a few mean free paths. In solids the profile of a shock front is complex because of their shear strength, but the observed time of a strong shock may typically be less than 10 nsec. Hence an idealized shock in a continuous medium is represented by its wave front, across which quantities such as pressure, density, material velocity, and temperature change in a discontinuous manner.

A simple shock wave is a plane steady shock propagating in a semi-infinite medium. Such a shock can be generated by applying normal stress on the surface of a half-space as illustrated in Fig. 2-1. In this figure the shock wave moves from left to right with the velocity $U$ relative to stationary laboratory coordinates.

The nature of the discontinuity can be studied by applying the conservation of mass, linear momentum, and energy to a unit cross-sectional area of the flowing material relative to the observer who moves with the shock front.

The conservation of mass. The medium moves toward the front at the velocity $u-U$, where $u$ is the particle velocity of the medium and $\rho$ the density. Similarly it moves away towards the right at $u_0-U$ and $\rho_0$. Then the conservation
of mass, in terms of mass flux per unit cross-sectional area, is

$$\rho (u - U) = \rho_0 (u_0 - U) = m$$  \hspace{1cm} (1)

where $m$ is the mass flux across the shock front.

The conservation of linear momentum. As the materials cross the shock front (stationary to our observer), their velocity changes from $u$ to $u_0$. Then the rate of change of linear momentum of the system in unit time is $m(u_0 - u)$. Hence impulse relation per unit time yields

$$\sigma - \sigma_0 = m(u_0 - u)$$  \hspace{1cm} (2)

where $\sigma$ is compressive stress normal to the front.

The conservation of energy. The rate of doing work on a system containing the shock front is $\sigma (u - U) = \sigma_0 (u_0 - U)$ where $u - U$ and $u_0 - U$ are the displacement of the left and right boundary of the system. The corresponding change in energy of the system observed at the front is

$$m \left[ \frac{1}{2} \left( u_0 - U \right)^2 - \frac{1}{2} (u - U)^2 + \left( E_0 - E \right) \right]$$

where $E$ is internal energy of the material per unit mass. Then, the equation of energy conservation is given by
\[
\sigma(u - U) - \sigma_0(u_0 - U) = m \left[ \frac{1}{2} (u_0 - U)^2 - \frac{1}{2} (u - U)^2 + (E_0 - E) \right] \tag{3}
\]

Eliminating \(U\) and \(u\) from equations (1)–(3), one obtains the Rankine-Hugoniot equation

\[
E - E_0 = \frac{1}{2} (\sigma + \sigma_0)(V_0 - V) \tag{4}
\]

where \(V = 1/\rho\).

In summary,

\[
\rho(u - U) = \rho_0(u_0 - U) = m
\]

\[
\sigma - \sigma_0 = m(u_0 - u)
\]

\[
E - E_0 = \frac{1}{2} (\sigma + \sigma_0)(V_0 - V)
\]

It may be seen from Eqs. (1)–(3) that those jump conditions are invariant with respect to the coordinate transformation under translation with a constant velocity in the direction of shock propagation, in accordance with the Galilean principle of relativity. If we let \(q\) be a velocity relative to any such coordinate, Eqs. (1)–(3) become

\[
[\rho q] = 0 \tag{5}
\]

\[
[\sigma + \rho q^2] = 0 \tag{6}
\]

\[
[\sigma q + \rho q \left( E + \frac{1}{2} q^2 \right)] = 0 \tag{7}
\]

where \([x]\) signifies the discontinuous change in \(x\) from one side of the front to the other. These equations simply denote the conservation of mass, momentum, and energy in the form of fluxes. Eqs. (1), (2) and (4) can be recovered by setting \(q_0 = U\) and \(q = U - u\).

We note here some of the useful formulas that result from Eqs. (4)–(6).
\begin{align}
q_0^2 &= V_0^2 \frac{\sigma - \sigma_0}{V_0 - V} \\
q^2 &= V^2 \frac{\sigma - \sigma_0}{V_0 - V}
\end{align}

$q_0 - q = \text{The particle velocity relative to the material ahead of the shock}$

\begin{equation}
\begin{aligned}
u &= \pm \left[ \left( \sigma - \sigma_0 \right) \left( V_0 - V \right) \right]^{\frac{1}{2}}
\end{aligned}
\end{equation}

The negative sign corresponds to the backward facing shock wave in which $q_0 > 0$ and $q > 0$.

It is clear from Eqs. (1)–(4) that if a supplementary relation between $\sigma$, $V$, and $E$ are known, say through experiments or an equation of state, these equations define a unique curve in the $\sigma$–$V$ plane, representing states that are attainable by a single jump from a particular initial state. Such a locus is called a Hugoniot curve. In the case of a fluid, $\sigma$ becomes the hydrodynamic pressure $p$ and $E$ may be specified as a function of pressure and specific volume. Even in solids, if the magnitude of pressure exceeds their shear strength greatly, many flow problems can be adequately treated by replacing normal stress by hydrostatic pressure. We will see specific examples in the next section. In this section, however, general features of shock transitions will be examined without specific information regarding material response behavior.

Figure 2-2 illustrates a schematic of a Hugoniot curve for a fluid having a concave upward $p$–$V$ relationship. The chord drawn from the initial state to state $p$, called the Rayleigh line, defines the shock velocity through the relationship

\begin{equation}
-U^2 = V_0^2 \frac{\Delta p}{\Delta V}
\end{equation}

where $\Delta$ signifies discontinuous changes in $p$ and $V$.

The inequality $p_2 > p_1$ immediately implies $V_2 > V_1$. It is emphasized that a Hugoniot curve does not represent a path followed during shock compression. It is a locus of states reached by a single shock passing through a particular initial state $(p_0, V_0)$.

For a single shock starting at zero pressure, it may be deduced from Eqs. (4)–(10) that the energy imparted by the shock is equally partitioned between the internal and kinetic energy. That is,
\[ \Delta E = \frac{1}{2} p (V_0 - V) = \frac{1}{2} u^2 \] (per unit mass) \hfill (12)

Although the shock transition did not involve heat transfer between the initial and the final state, it is not an isentropic process. As we will see below, some irreversible work must be dissipated through viscosity, plastic deformation, and other dissipative processes, to maintain the linear relation between \( p \) and \( V \) in the shock front.

Other unique features of Hugoniot curves come from their slopes. We start with the derivative of Eq. (4).

\[ dE = \frac{dp}{2} (V_0 - V) - \frac{p}{2} dV \] \hfill (13)

where it is assumed for simplicity that \( p_0 = 0 \). Since the internal energy change is related to entropy change by the second law of thermodynamics,

\[ dE = T dS - pdV \] \hfill (14)

where \( S \) signifies the entropy.

Also, expanding the entropy in terms of pressure and specific volume, one finds
\[ dS = \left( \frac{\partial S}{\partial p} \right)_V dp + \left( \frac{\partial S}{\partial V} \right)_p dV \] (15)

Now eliminating \( dp \) and \( dS \) among Eqs. (13)–(15), one obtains a differential equation for Hugoniot curves.

\[ \frac{dp}{dV} = \frac{\left( \frac{\partial p}{\partial V} \right)_S}{1 - \gamma(V_0 - V)/2V} \] (16)

where \( \gamma \) is the Grüneisen coefficient given by

\[ \gamma = V \left( \frac{\partial p}{\partial p} \right)_V = V \frac{\partial p}{\partial T} \] (17)

and \( C_V \) is the specific heat at constant volume.

Another interesting relation may be obtained by equating Eqs. (13) and (14). That is,

\[ \left( \frac{dS}{dV} \right) = \frac{1}{2T} \left[ p + (V_0 - V) \frac{dp}{dV} \right] \] (18)

At the initial state this equation becomes

\[ \left( \frac{dS}{dV} \right)_{V=V_0, p=0} = 0 \] (19)

Differentiation of Eq. (18) with respect to \( V \), yields

\[ \left( \frac{d^2 S}{dV^2} \right) = \frac{1}{2T} \left[ (V_0 - V) \frac{d^2 p}{dV^2} \right] + \left[ p + (V_0 - V) \frac{dp}{dV} \right] \frac{d}{dV} \left( \frac{1}{2T} \right) \]

Hence, we have again the vanishing of the second derivative at the initial state

\[ \left( \frac{d^2 S}{dV^2} \right)_{V=V_0, p=0} = 0 \] (20)
Repeating the differentiation once more, one obtains

\[
\left( \frac{d^3 S}{dV^3} \right)_{V=V_0, p=0} = -\frac{1}{2T} \left( \frac{d^2 p}{dV^2} \right)
\]  
(21)

Combining Eqs. (19)–(21) with a Taylor series expansion of entropy in terms of \( V \), one finds

\[
S(V) - S(V_0) = \frac{1}{12T} (V_0 - V)^3 \left( \frac{d^2 p}{dV^2} \right)
\]  
(22)

This means that the Hugoniot makes a second order contact with the adiabat passing through \( p_0, V_0 \). This change in entropy across a shock increases with increasing shock strength, but is a function only of third order terms in shock strength \( \Delta \Phi \) where \( \Phi \) is any flow variable.

If the entropy change along a Hugoniot curve is positive, Eq. (18) implies that since \( T > 0 \),

\[
-\frac{dp}{dV} > \frac{p}{V_0 - V}
\]  
(23)

where it is assumed that \( p_0 = 0 \).

Still another interesting relation may be obtained through interchanging the role of \( p \) and \( V \) in Eq. (18).

\[
TdS = \frac{1}{2} (V_0 - V) \left[ dp + \frac{p}{V_0 - V} dV \right] = \frac{1}{2} (V_0 - V) \left[ 1 - \frac{p}{V_0 - V} \left( \frac{V^2}{C^2} \right) \right] dp
\]  
(24)

where it is defined that

\[
C^2 = -V^2 \left( \frac{dp}{dV} \right)
\]  
(25)

Also from Eqs. (1) and (2)

\[
(U - u)^2 = V^2 \frac{p}{V_0 - V}
\]  
(26)
Upon substitution of Eqs. (26) into (24), one finds

\[ \frac{dS}{dp} = \frac{V_0 - V}{2T} \left[ 1 - \left( \frac{U - u}{C} \right)^2 \right] \]  

(27)

If entropy increases under shock compression, Eq. (27) implies that

\[ \left( \frac{U - u}{C} \right)^2 < 1 \]  

(28)

Since the quantity \((U - u)/C\) is the ratio of the slopes of the Rayleigh line and the tangent to the Hugoniot, the above inequality equation can be used as a criterion for the stability of a shock wave. However, a more physically transparent criterion, known as the supersonic-subsonic conditions, can be stated in terms of the speed of the shock wave. That is, a shock wave is said to be stable if it is supersonic with respect to the material ahead and subsonic behind. Without the supersonic condition, the shock front cannot maintain its steepness. Without the subsonic condition, its strength cannot be maintained. Mathematically, this is written as

\[ C_s + u > U > C_{s0} \]  

(29)

where \(C_s\) is the isentropic sound speed and “0” signifies an initial state. This also means that unloading cannot take place in a single step. It will be a continuous process creating a dispersed pressure profile. Figure 2-3 shows a schematic summary of interrelationships of the Hugoniot curve, Rayleigh line, and intersecting adiabats.

An analytical demonstration of the subsonic condition can be easily obtained by the substitution of (16) into (23). The supersonic condition can be found by expanding Eq. (11) in a Taylor series.

\[ U^2 = V_0^2 \left[ -\left( \frac{dp}{dV} \right)_{V = V_0} - \frac{1}{2} \left( \frac{d^2 p}{dV^2} \right)_{V = V_0} (V_0 - V) + \ldots \right] \]  

(30)

Since the Hugoniot makes a second order contact through the initial state \((p = 0, V = V_0)\), Eq. (30) leads to the inequality

\[ U^2 \geq C_{s0}^2 \]
provided that the entropy increase along the Hugoniot is positive.

A more general analysis of the subsonic-supersonic condition for shocks is given by Fowles (1975).

Figure 2-4 shows an experimentally observed multi-wave structure in Armco iron. The observed instability is attributed to an elastic-plastic transition as well as a phase change at 130 kb from bcc to hexagonal structure (Morris,
1982).

2.2 Shock Transition in Hydrodynamic Solids

If stress anisotropy is ignored, then there are five variables \( U, u, p, V, \) and \( E \) in the Hugoniot equations that describe a shock transition. Therefore, a fourth equation is needed to determine features of the transition in terms of, e.g., \( V \). Since Eq. (4) is a relationship between thermodynamic variables, \( p \) as a function of \( V \) and \( E \) will define a unique Hugoniot curve in the \( p-V \) plane. Ideally, such a function may be provided if we knew a potential function of thermodynamics such as Helmholtz free energy in terms of temperature and volume. However, for solids such a potential is rarely available because of their variability in the nature of bonding. Alternatively, one can determine the flow variables, \( U \) and \( u \), experimentally. Again this will define a unique Hugoniot curve in the \( p-V \) plane. In fact, there are two compendium of shock data for hundreds of materials (Marsh, 1980; Hgvan Thiel, 1977) and they show that the \( U-u \) relationships are linear up to several hundred kilobars for most materials listed.

In this section we will consider a combination of these two approaches using the Grüneisen equation of state,

\[
p(V, E) = p_L(V) + p_T(V, E_T)
\]

(31)

where \( p_L \) is the lattice pressure due to the lattice potential energy and \( p_T \) due to the thermal energy \( E_T \) of atomic vibrations about their mean positions. \( E \) is given as the sum of \( E_L \) and \( E_T \)

\[
E_T = E - E_L
\]

(32)

An appealing feature of Eqs. (31) and (32) is that they can be explained using atomistic models of solids and statistical mechanics. At very high temperatures a third term can be added to deal with thermal contributions from electrons. References for more complex models are found in Davison and Graham (1979). Effects of shear strength will be considered in the subsequent sections.

Equation (31) is rooted in the thermodynamic expression of the Grüneisen parameter

\[
\gamma = V \left( \frac{\partial p}{\partial E} \right)_V
\]

(33)

Then assuming \( \gamma \) to be a function of \( V \) alone and integrating the above equation once, one finds
\[ p = p_r(V) + \frac{\gamma}{V}(E - E_r(V)) \] (34)

where \( p_r \) and \( E_r \) are the reference pressure and energy respectively. If \( p_r \) and \( E_r \) are identified as those associated with the ideal lattice, a particular form of Eq. (31) may be given by

\[ p = p_L + \frac{\gamma(V)}{V}(E - E_L) \] (35)

Since the initial conditions for shock experiments are often those of standard conditions (\( p_0 = 0, T = \text{room temperature} \)), it is useful to measure \( E \) from with respect to that state, \( E_0 \). Then, the above equation becomes

\[ p = p_L + \frac{\gamma}{V}(E + E_0 - E_L) \]

or

\[ p = \frac{\gamma}{V}E + p_L + \frac{\gamma}{V}(E_0 - E_L) \] (36)

Noticing that the second term on the right-hand side is still a function of volume only, one can put Eq. (36) into the form

\[ p = \frac{\gamma}{V}E + f(V) \] (37)

There are two commonly referred models for evaluating the variation of \( \gamma \) with volume (Eliezer et al., 1986). They are

\[ \gamma = -\frac{V}{2} \frac{p''_L}{p'_L} - \frac{2}{3} \] (Slater) (38)

\[ \gamma = -\frac{V}{2} \frac{(p_L V^{2/3})''}{(p_L V^{2/3})'} - \frac{1}{3} \] (Dugdale & MacDonald) (39)
where the prime indicates differentiation with respect to volume. Empirically, it is known (McQueen & Rice, 1958) that for many materials over a wide range of pressure

$$\frac{\gamma}{V} = \text{constant to a reasonable approximation} \quad (40)$$

Calculations of $E_L$ from basic physical models are available, but because they were done for a very limited class of materials such as alkali metals (e.g., Bardeen, 1938), they are rarely used to calculate Hugoniot states. Instead, experimental data on the $U-u$ relationship are used to determine the lattice energy semi-empirically (Eliezer, 1986).

As stated earlier it is found that over limited conditions to a reasonable approximation

$$U = C_0 + su \quad (41)$$

where $C_0$ is theoretically the bulk sound at $p = 0$ as may be seen from Eqs. (11) and (19). Then combining Eq. (41) with Eqs (1), (2), and (4), one obtains an analytical expression for the Hugoniot curve

$$p_H = \frac{\rho_0 C_0 \epsilon}{[1 - s \epsilon]^2} \quad (42)$$

where $\epsilon = 1 - V/V_0$ and $\rho_0 = 1/V_0$.

Taylor series expansion of $p_H$ in $\epsilon$ gives

$$p_H = \rho_0 C_0^2 [\epsilon + 2s \epsilon + 3s^2 \epsilon + ..] \quad (43)$$

The corresponding internal energy change is

$$E_H = \frac{p_H}{2} (V_0 - V) = \frac{1}{2} \left( \frac{C_0 \epsilon}{1 - s \epsilon} \right)^2 \quad (44)$$

Figure 2-5 shows representative Hugoniot curves based on the $U-u$ relation. Temperature calculations will be considered later.

Since the Hugoniot states are assumed to be thermodynamic equilibrium states, the substitution of Eqs. (42) and (44) into Eq. (37) yields
Fig. 2-5. Hugoniot curves of some representative materials based on their linear $U-u$ relationships.

\[ f(V) = p_H - \frac{\gamma}{V} E_H = \frac{\rho_0 C_0^2}{(1 - s \varepsilon)^2} \left[ 1 - \frac{\gamma}{2V} (V_0 - V) \right] \]

Hence, the off-Hugoniot equation of state is given by

\[ p = \frac{\gamma}{V} E + \frac{\rho_0 C_0^2}{(1 - s \varepsilon)^2} \left[ 1 - \frac{\gamma}{2V} (V_0 - V) \right] . \tag{45} \]

Since the Hugoniot curve is a line on the thermodynamic surface, the validity of the above equation cannot be extended too far.

The value of $\gamma$ can be evaluated by making several assumptions. The first is the empirical observation that is based on more rigorous calculations for many metals (McQueen & Rice, 1958).

\[ \gamma = \frac{V}{V_0} \gamma_0 \tag{46} \]

where $\gamma_0$ is the thermodynamic value at standard conditions given by
\[ \gamma_0 = \frac{K_s \alpha}{P_0 C_p} = \frac{K_T \alpha}{P_0 C_v} \]

where

\[ \alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p, \quad C_V = \left( \frac{\partial E}{\partial T} \right)_V, \quad C_p = \left( \frac{\partial H}{\partial T} \right)_p, \quad \text{and} \quad H = E + pV \]

Another method is to equate \( p_L \) in Eqs. (38) and (39) assuming that at low pressures and room temperature, thermal contributions to \( E \) are small compared with \( E_T \). Then applying Eq. (42) into Eqs. (38) and (39), one finds at \( p = 0 \),

\[ \gamma = 2s - \frac{2}{3} \quad \text{(Slater)} \quad (47) \]

\[ \gamma = 2s - 1 \quad \text{(Dugdale & MacDonald)} \quad (48) \]

Figure 2-6 shows a comparison of the two measures of \( \gamma \), with \( s \) in the linear \( U-u \) relationship (Skidmore, 1965). A slightly better agreement is observed with the Dugdale-MacDonald relation.

If \( \gamma \) is known as a function of volume, then there is an alternative method of calculating Hugoniot curves without relying on shock experiments. This method uses sound speed measurement at high pressures. Let \( p_r \) and \( E_r \) in Eq. (34) to be \( p_S \) and \( E_S \) of isentropic compression. Then

\[ p_H = p_S + \frac{\gamma}{V} (E_H - E_S) \quad (49) \]

where

\[ E_H = \frac{1}{2} p_H (V_0 - V) \quad (50) \]

\[ E_S = -\int_{V_0}^{V} p_S dV \quad (51) \]

Putting Eqs. (50) and (51) into (49) yield
Fig. 2-6. A comparison of measured and calculated Grüneisen parameters for metals.

\[
p_H \left[ 1 - \frac{\gamma}{2V} (V_0 - V) \right] = p_s + \frac{\gamma}{V} \int_0^V p_s dV \tag{52}
\]

Of many formulas for \( p_s \), we mention two known as Birch and Murnaghan equations (Anderson, 1966).

\[
p_s(\text{Birch}) = \frac{3}{2} K_0 \left\{ \left( \frac{V_0}{V} \right)^{7/3} - \left( \frac{V_0}{V_0} \right)^{5/3} \right\} \left\{ 1 + \frac{3}{4} (n - 4) \left[ \left( \frac{V_0}{V} \right)^{7/3} - 1 \right] \right\} \tag{53}
\]

\[
p_s(\text{Murnaghan}) = \frac{K_0}{n} \left\{ \left( \frac{V_0}{V} \right)^n - 1 \right\} \tag{54}
\]

where

\[
K_0 = K_s(p = 0) = \rho_0 C_0^2
\]
\[ n = \frac{dK_s}{dp} = 2\gamma + 1 \text{ (Dugdale and MacDonald Relation)} \quad (55) \]

\[ = 4s - 1 \text{ (Ruoff, 1967)} \]

Both equations are known to be equal to a relative compression \( V/V_0 \) of about 0.85 (Anderson, 1966). Anderson has shown that the difference between Eq. (43) and Eq. (54) is only a few percent over the pressure range of a few hundred kilobars. This result is anticipated because of the fact that the Hugoniot curve makes a second order contact with the adiabat. The mathematical demonstration is straightforward. Using, for instance, the simpler expression, Eq. (54), one finds

\[ p_H[1 - \gamma_0 \varepsilon] = \frac{K_0}{n} \left[(1 - \varepsilon)^{-n} - 1\right] + \frac{\gamma_0 K_0}{n} \left[\frac{1}{1-n} (1 - \varepsilon)^{-n+1} - (1 - \varepsilon) - \frac{n}{1-n} \right] \quad (56) \]

where Eq. (45) is substituted for \( \gamma \). Then the Taylor series expansion of this equation yields

\[ p_H/K_0 = \varepsilon + \frac{1}{2} (n + 1) \varepsilon^2 + \left[\frac{(n+1)(n+2)}{6} - \frac{n+1}{12} \gamma + \left(\frac{\gamma}{2}\right)^2\right] \varepsilon^3 + \ldots \quad (57) \]

This equation is identical to Eq. (43) up to the second order in \( \varepsilon \), provided that \( n \) is given by Eq. (55). The third order coefficient in terms of \( s \) is

\[ \frac{20}{6} s^2 - \frac{4}{3} s + \frac{1}{2} \]

The corresponding coefficient in Eq. (55) is \( 3s^2 \).

Therefore, for many practical purposes, adiabatic compression curves give a very good estimate of Hugoniot pressures. That is,

\[ p_H \equiv p_S = P_0 C_0^2 \left[\varepsilon + (\gamma + 1) \varepsilon^2\right] \quad (58) \]

where \( \varepsilon = 1 - V/V_0 \).

Using the preceding observation about Hugoniot pressures, one can develop similar approximations for temperature.

At a fixed volume,
\[ E_H - E_S = \int_{p_s}^{p_H} \left( \frac{\partial E}{\partial p} \right)_V dp = \int_{p_s}^{p_H} \left( \frac{\gamma}{V} \right)_V dp \]  
(59)

\[ = \int_{T_s}^{T_H} \left( \frac{\partial E}{\partial T} \right)_V dT = \int_{T_s}^{T_H} C_V dT \]  
(60)

If \((\gamma/V)\) and \(C_V\) were constant, then

\[ T_H - T_S = \frac{\gamma}{V} (P_H - P_S) \]

So, using Eq. (58)

\[ T_H \approx T_S \]  
(61)

The adiabatic temperature can be calculated using a thermodynamic identity.

\[ TdS = C_V dT + \left( \frac{\partial p}{\partial T} \right)_V TdV \]  
(62)

So,

\[ \frac{dT_S}{T_S} = -\frac{\gamma}{V} dV \]

Upon integration, the above equation yields

\[ T_S = T_0 \exp \left[ \frac{\gamma}{V} (V_0 - V) \right] \]

\[ \approx T_0 \exp \left[ \frac{\gamma}{V} (V_0 - V) \right] = T_0 \exp(\gamma_0 \epsilon) \]  
(63)

Improvement of the above approximation is self-explanatory. Figure 2-7 shows representative examples. Hugoniot temperatures are based on Eq. (63).

Alternatively, using Eqs. (18) and (62), one can show that the temperature and entropy changes along a Hugoniot curve are given by
Fig. 2-7. Hugoniot and residual temperature calculations for representative materials based on Eqs. (61) and (69).

\[ dT = \frac{V_0 - V}{2C_V} dp + \left[ \frac{P}{2C_V} - \frac{T\gamma_0}{V_0} \right] dV \]  
\[ dS = \frac{1}{2T} \left[ pdV + (V_0 - V) dp \right] \]  

where Eq. (46) is used for \( \gamma \). The integration can be written in a closed form for the linear \( U-u \) relationship (Carter, 1973). They are

\[ T = T_0 \left[ 1 + \frac{C_0^2}{C_V T_0} \int_1^s d\xi \frac{s(\xi - 1)^2}{[s - \xi(s - 1)]^3} \exp \left( -\gamma \frac{\xi - 1}{\xi} \right) \right] \exp \left( \gamma \frac{\xi - 1}{\xi} \right) \]
\[ S = S_0 + C_v \ln \left( 1 + \frac{C_0^2}{C_v T} \int_1^\xi d\xi \frac{s(\xi-1)^2}{s-\xi(s-1)} \exp \left( -\gamma \frac{\xi-1}{\xi} \right) \right) \tag{67} \]

where the first term in each of the big brackets represents isentropic compression, and \( \xi = V_0/V = 1/(1-\varepsilon) \). However, as we noted earlier, non-isentropic terms are not significant until the third term in compression \( \varepsilon \) becomes important.

Another important measure of temperature in shock compression is residual temperature immediately after the passage of a shock wave when the pressure in the material is again \( p_0 \) (or zero). This temperature is a measure of dissipation of mechanical energy during shock compression, and hence its calculation requires the entropy change given by Eq. (65). But, in this section we will adopt a simple formula, Eq. (22), which should be valid for shocks of moderate strength.

\[ S_H - S_0 \approx \frac{1}{12T_0} \left( V_0 - V \right)^3 \frac{d^2 p}{dv^2} \bigg|_{S,V=V_0} \]

Applying the Murnaghan equation, one finds

\[ S_H - S_0 = \frac{K_0(n+1)}{12T_0} V_0^3 \varepsilon^3 = \frac{C_0^2 s}{3T_0} \varepsilon^3 \tag{68} \]

Immediately after the passage of the shock, when the pressure in the material is unloaded adiabatically to \( p_0 \) (a zero pressure), the residual temperature is related to the entropy increase in Eq. (65) by

\[ T_{res} = T_0 \exp \left( \frac{S_H - S_0}{C_p} \right) \tag{69} \]

\[ = T_0 \exp \left( \frac{C_0^2 s \varepsilon^3}{3T_0 C_p} \right) \tag{70} \]

where \( C_p \) is the heat capacity at constant pressure and we used the thermodynamic relationship

\[ TdS = T \left( \frac{\partial S}{\partial T} \right)_p dT + T \left( \frac{\partial S}{\partial p} \right)_T dp \]
\[ = C_p dT, \quad p = \text{constant} \]

Figure 2-7 shows representative examples for \( T_H \) and \( T_{res} \). If necessary more accurate estimates can be obtained by use of Eqs. (65) and (69). For example, the entropy change can be calculated by use of the equation of state (37) and the thermodynamic relationship

\[
S_H - S_0 = \int_{p_s}^{p_H} \left( \frac{\partial S}{\partial p} \right)_V dp = \int_{p_s}^{p_H} \frac{C_V}{T} \left( \frac{\partial p}{\partial T} \right)_V dp \quad (71)
\]

2.3 Non-Hydrostatic Deformation of Solids

At very high pressures the hydrodynamic model discussed in the previous section captures the first order nature of the shock compression involved in the shock front in many situations. However, at low pressures, the effect of shear strength, which is the hallmark of solid behavior, can not be ignored. Unfortunately, the non-hydrodynamic response of solids is complex, particularly in an inelastic regime, arising from the fact the underlying basic mechanisms are inhomogeneous, discontinuous, anisotropic, and not in thermodynamic equilibrium. It is likely that no exact mathematical relationship can be developed that is applicable for all conditions of pressure, temperature, and loading rate. Therefore, in this section we will consider a simple model that describes the first order features of isotropic elastic-plastic response of solids. The model can be extended to deal with a variety of more advanced features than those discussed in this chapter. For instance, see Desai (1987).

2.3.1 Elastic-ideally-plastic solids

This model, dating as far back as the late 19th century, has been developed to describe the macroscopic behavior of metals which are assumed to be homogeneous and isotropic. The model incorporates certain key features of experimentally observed data under uniform loading. These features are (1) materials obey Hooke’s law in a certain range of small stress, (2) the limit of elastic behavior, called yield criterion, is given by a function of stress, (3) behavior beyond the elastic limit is complex, but the key feature is that the work due to plastic deformation under the cyclic application of stresses is positive, and (4) there is no volume change due to plastic deformation.

Development of the mathematical model will first be carried out in three dimensions. Then, the resulting equations will be specialized to plane shock compression. The introduction of Hooke’s law is based on thermodynamics (Landau and Lifshitz, 1970). We chose the method of a plastic potential (associ-
ated flow rule) to derive the stress-strain relationships in plastic regime because of its other application in Chapter 4.

1) Elastic response

The extension of the first law of thermodynamics to include small non-hydrodynamic loads is given by (Landau and Lifshitz, 1970)

\[ dE = TdS - pdV + s_{ij} d_{ij} \]  \hspace{1cm} (72)

where \( s_{ij} \) and \( d_{ij} \) are the deviatoric (distortional) stress and strain tensors, and related to the stress \( \sigma_{ij} \) and strain \( e_{ij} \) tensors by

\[ s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{KK} \delta_{ij} \]  \hspace{1cm} (73)

\[ e_{ij} = e_{ij} - \frac{1}{3} e_{KK} \delta_{ij} \]  \hspace{1cm} (74)

where \( \delta_{ij} \) is the Kronecker delta. Repeated indices signify summation. By definition,

\[ s_{ii} = 0 \text{ and } e_{KK} = 0 \]  \hspace{1cm} (75)

In tensor form hydrostatic pressure is defined by

\[ \sigma_{ij} = +p \delta_{ij} \ (\sigma_{ij} \text{ is positive for compression}) \]  \hspace{1cm} (76)

Obviously,

\[ \sigma_{KK} = +3p \]  \hspace{1cm} (77)

For small strain

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

\[ V = V_0 (1 - \epsilon_{KK}) \]  \hspace{1cm} (78)

where \( u_i \) is the displacement vector, \( x_i \) a coordinate system, and \( V_0 \) the volume of
the unstrained state.

It follows from (72) that

\[ p_s = -\left( \frac{\partial E}{\partial V} \right)_{s,e_{ij}} \quad (79) \]

\[ s_{ij} = \left( \frac{\partial E}{\partial e_{ij}} \right)_{s,V} \quad (80) \]

If \( e_{ij} \) is small and \( E \) is additive with respective to the contribution from \( e_{ij} \), then

\[ E_{\text{distortion}} = \mu e_{ij}^2 \quad (81) \]

where \( \mu \) can be identified as the shear modulus of the solid at constant \( S \) and \( V \).

Using Eqs. (73)--(81) we can now assemble the expression for stress tensors for isentropic (adiabatic) compression.

\[ \sigma_{ij} = p_s \delta_{ij} + 2\mu e_{ij} \quad (82) \]

\[ s_{ij} = 2\mu e_{ij} \quad (83) \]

Let us specialize the preceding equations to those for a plane shock propagating in the direction of \( x \)-axis (or \( x_1 \)). However, we will carry out the reduction in incremental form, to be in consort with the formulation of plastic response. By definition the strain system behind a plane shock is uniaxial strain. Hence, the only non-zero component of \( \varepsilon_{ij} \) is \( \varepsilon_x \). So, we will drop second indices from normal component in the subsequent analysis. Then, using Eq. (74),

\[ de_x = \frac{2}{3} d\varepsilon_x \]

\[ de_y = de_z = -\frac{1}{3} d\varepsilon_x \quad (84) \]

Substituting Eq. (84) into (82), one obtains
\begin{align}
  d\sigma_x &= -\left( \frac{\partial p}{\partial V} \right)_{S} dV + \frac{4\mu}{3} d\varepsilon_x \tag{85} \\
  d\sigma_y &= d\sigma_z = -\left( \frac{\partial p}{\partial V} \right)_{S} dV - \frac{2\mu}{3} d\varepsilon_x \tag{86}
\end{align}

Or using (78), Eqs. (85) and (86) become

\begin{align}
  d\sigma_x &= \left( K_S + \frac{4\mu}{3} \right) d\varepsilon_x \tag{87} \\
  d\sigma_y &= \left( K_S + \frac{2\mu}{3} \right) d\varepsilon_x \tag{88}
\end{align}

where \( K_S \) is the adiabatic bulk modules given by

\begin{equation}
  K_S = -\frac{1}{V_0} \left( \frac{\partial p}{\partial V} \right)_{S} \tag{89}
\end{equation}

and

\begin{equation}
  dp = +K_S d\varepsilon_x \tag{90}
\end{equation}

In plane shock loading, normal stresses are all principal stresses. Then, the maximum shear stress occurs on planes that make an angle of 45° with the x-axis. The differential increment of this magnitude is

\begin{equation}
  d\tau(\text{max}) = \frac{1}{2} d\left( \sigma_x - \sigma_y \right) = \mu d\varepsilon_x \tag{91}
\end{equation}

We can derive useful expressions by substituting Eqs. (90) and (91) into Eqs. (87) and (88). The results are

\begin{align}
  d\sigma_x &= dp + \frac{4}{3} d\tau \tag{92} \\
  d\sigma_y &= dp - \frac{2}{3} d\tau \tag{93}
\end{align}
Figure 2-8 illustrates a schematic of $\sigma_x$, $\sigma_y$, $p$ and $\tau$ in terms of $\varepsilon_x$.

It is apparent that if $\tau$ is zero, one recovers the equations for ideal fluids. If a fluid is viscous, then $\tau$ may be described by an equation such as

$$\tau = -\mu \frac{\partial u}{\partial x}$$

where $(\partial u/\partial x)$ is velocity gradient.

Like viscosity, plastic response of solids can be handled by providing a proper expression for $\tau$, but in what follows we will first formulate a formal three dimensional model and then we will return to Eqs. (92) and (93) to look at microscopic modeling based on dislocation dynamics.

2) Plastic response

This section will describe a potential theory of plastic stress-strain relationships based on four basic assumptions.

(1) The limit of elastic response is prescribed by a yield function $f(\sigma_{ij})$ such that

$$f(\sigma_{ij}) \leq 0 \quad \text{(94)}$$

The equality holds in plastic deformation.
(2) In plastic deformation, the total strain increment is the sum of elastic and plastic increments.

\[ d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p \]  

(95)

(3) Shear stress increments are functions of the elastic deviatoric strain increments.

\[ ds_{ij} = 2\mu \, d\epsilon_{ij}^e \]  

(96)

(4) Work due to plastic deformation under the cyclic application of stress cannot be negative. One of the consequences of the assumption is the existence of a "plastic potential" (Fung, 1965).

\[ d\epsilon_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}} \]  

(97)

where \( \lambda \) is a positive scalar function. If \( g = f \), then Eq. (97) is known as the associated flow rule.

(5) There is no volume change due to plastic deformation. This means that

\[ d\epsilon_{KK}^p = 0 \]  

(98)

\[ d\epsilon_{KK} = d\epsilon_{KK}^e \]  

(99)

For pressure, Eqs. (78) and (90) continue to hold.

An immediate consequence of Eq. (99) is seen in the following new expression of Eq. (82).

\[ d\sigma_{ij} = K_S d\epsilon_{KK} \delta_{ij} + 2\mu \left( d\epsilon_{ij}^e - \frac{1}{3} d\epsilon_{KK} \delta_{ij} \right) \]

\[ = \left( K_S - \frac{2\mu}{3} \right) d\epsilon_{KK} \delta_{ij} + 2\mu d\epsilon_{ij} - 2\mu d\epsilon_{ij}^p \]  

(100)

where \( d\epsilon_{ij}^p \) is given by Eq. (97).

To complete the stress-strain relationships given by Eq. (100), we introduce
von Mises yield function given by Eq. (101) as the plastic potential.

\[
f(= g) = \frac{1}{2} \left( s_{ij} \right)^2 - k^2 = 0
\]  

(101)

Or,

\[
f = \frac{1}{6} \left[ \left( \sigma_1 - \sigma_2 \right)^2 + \left( \sigma_3 - \sigma_1 \right)^2 + \left( \sigma_2 - \sigma_3 \right)^2 \right] - k^2 = 0
\]

where \( k \) is \( Y/\sqrt{3} \). \( Y \) is the yield stress in simple tension, and \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) are principal stresses. Comparing Eq. (101) with Eq. (81), we recognize that the above criterion is equivalent to the condition of maximum shear energy. Putting Eq. (101) into (97) yields

\[
d\varepsilon_{ij}^p = \lambda \frac{\partial}{\partial \sigma_{ij}} \left[ \frac{1}{2} \left( \sigma_{K\ell} - \frac{1}{3} \sigma_{mm} \delta_{K\ell} \right) \left( \sigma_{K\ell} - \frac{1}{3} \sigma_{mm} \delta_{K\ell} \right) - k^2 \right]
\]

\[
= \lambda \left( \sigma_{K\ell} \rho \delta_{K\ell} \right) \left( \delta_{iK} \delta_{j\ell} - \delta_{K\ell} \delta_{ij} \right)
\]

\[
= \lambda s_{ij}
\]

(102)

Since \( d\varepsilon_{KK}^p = 0, \)

\[
s_{KK} = 0
\]

(103)

The problem of determining \( \lambda \) is solved by requiring that Eq. (101) must hold for plastic deformation. Thus,

\[
df = s_{ij} ds_{ij} = s_{ij} \left( 2\mu \ v_{ij}^p \right)
\]

\[
= 2\mu s_{ij} \left( v_{ij} - v_{ij}^p \right)
\]

\[
= 2\mu s_{ij} \left( v_{ij} - \lambda s_{ij} \right) = 0
\]

This equation yields
\[ \lambda = \frac{s_{ij} d e_{ij}}{s_{ij} s_{ij}} = \frac{s_{ij} d \varepsilon_{ij}}{s_{ij} s_{ij}} \]  

(104)

where Eq. (103) is used to get the second expression.

Specializing to uniaxial strain, one finds

\[ \lambda = \frac{s_x}{s_x^2 + 2s_y^2} d\varepsilon_x = \frac{2}{3s_x} d\varepsilon_x \]  

(105)

where Eq. (103) is again used to get the second equation.

Now, putting Eq. (105) into a one dimensional version of Eq. (102), we find

\[ d\varepsilon_x^p = \frac{2}{3} d\varepsilon_x \]  

(106)

This is what is required to complete the stress-strain relationships Eq. (100) for plane shock compression. The resulting equations are

\[ d\sigma_x = d\sigma_y = dp = K_x d\varepsilon_x \]  

(107)

This shows that above the elastic limit, not only the slopes of \( \sigma_x, \sigma_y, \) and \( p \) are equal, but also the differences that existed at the limit are maintained. Then, using Eqs. (92) and (93), it may be shown that in the plastic region

\[ \sigma_x = p + \frac{2}{3} Y \]

\[ \sigma_y = p - \frac{1}{3} Y \]  

(108)

Using similar arguments as those above, one can construct stress-strain relationships for initially elastic unloading and reverse yielding at the appropriate condition of \( \tau \). The task does not require any new equations. Figs. 2-9 and 2-10 illustrate the resulting composite pictures of stress and strain paths for the elastic-plastic solid under shock loading.

From the energetic point of view it is of interest to calculate work done through plastic deformation. By definition,

\[ dw^p = \sigma_x d\varepsilon_x^p + 2\sigma_y d\varepsilon_y^p \]
Using plastic incompressibility, Eq. (98), one finds
\[ dw^p = (\sigma_x - \sigma_y) d\varepsilon_x^p + Y d\varepsilon_x = \frac{2}{3} Y d\varepsilon_x \]  
\[ d\varepsilon = dw^{total} - dw^p \]
\[ = \sigma_x d\varepsilon_x - dw^p = \left( \sigma_x - \frac{2}{3} Y \right) d\varepsilon_x \]  
(109)

By contrast elastic work is given by

One can generalize the basic model to describe such effects as strain hardening and strain-rate by modifying the yield function and/or some of its assumptions. Sometimes the task has been carried out by directly referring to Eq. (100). That is, its specialization to uniaxial strain loading yields

\[ d\varepsilon = \left( K_s + \frac{4\mu}{3} \right) d\varepsilon_x - 2\mu d\varepsilon_x^p \]  
(111)
Fig. 2-10. Trajectory of shock loading in the $(\sigma_x, \sigma_y)$ plane for an elastic-plastic solid.

\[
d\varepsilon_x^p = \frac{2}{3} \left( d\varepsilon_x - \frac{d\tau}{\mu} \right)
\]

(112)

Since the maximum plastic shear strain $d\gamma^p$ on the plane of the maximum shear stress $\tau$ is given by

\[
d\gamma^p = \frac{3}{4} d\varepsilon_p
\]

Eqs. (111) and (112) can be written in the form,

\[
d\sigma_x = \left( K_s + \frac{4\mu}{3} \right) d\varepsilon_x - \frac{8}{3} d\gamma^p
\]

(113)

\[
d\tau = \mu d\varepsilon_x - 2\mu d\gamma^p
\]

(114)
The last term in Eq. (113) is called the relaxation function by Duvall (1964).

There are three often mentioned models of the relaxation function describing shear failure. The first, and probably the most extensively studied model is the dislocation dynamic interpretation of $d\dot{\gamma}_p$ using the Orowan relation (Taylor, 1965)

$$\dot{\gamma}_p = \psi b N_m V_d$$  \hspace{1cm} (115)

where the dot “.” signifies time derivative, $\psi$ is a constant of the order of one, $b$ the Burger’s vector, $N_m$ mobile dislocation density, $V_d$ mean dislocation density. $N_m$ and $V_d$ depend on such factors as strain rate, stress, and temperature. Typical forms of functions used for $V_d$ are

$$V_d = \left(\frac{\tau}{\tau_0}\right)^m$$

$$V_d = B(\tau - \tau_0)^n$$

$$V_d = V_0 \exp\left(-\frac{d}{\tau}\right)^n$$

Interested readers are referred to numerous references on $V_d$ cited in Davison and Graham (1979).

The second is a Malvern model in which

$$\dot{\gamma}_p = \frac{\sigma_x - \left(\frac{p + \frac{2}{3}Y}{8\mu}\right)}{\frac{3}{3}T_r}$$  \hspace{1cm} (116)

where $T_r$ is a relaxation time constant.

The third is a frictional model in which the maximum shear is prescribed by a static frictional relationship,

$$\tau \leq f_s p$$  \hspace{1cm} (117)

where $f_s$ is a coefficient of “friction.” This model may be useful in describing the loss of shear strength observed in some materials (e.g., Dandekar and Martin, 1981). In this model the onset of failure is characterized by
\[ \sigma_x = -\left(1 + \frac{4f_s}{3}\right)p \]  

(118)

After failure, \(f_s\) may drop to a smaller "dynamic" coefficient \(f_2\) or may disappear to zero in some fashion specified by an appropriate function of \(p\), temperature, etc.

2.3.2 Experimental observations of elastic-plastic behavior

The elastic limit under plane shock wave loading (called Hugoniot elastic limit, HEL) has been measured using the information that a single shock is unstable in the elastic-ideally plastic solid, and forms a two-wave structure with an elastic precursor in the front. Since similar instability occurs through a phase transition and even a chemical reaction, it is instructive to revisit the stability conditions discussed in Section 2.1 of this chapter. There, a shock wave is said to be stable if its motion is supersonic with respect to the material ahead and subsonic to the material behind. Mathematically it is expressed that

\[ C_s + u > U \]  

(119)

To simplify the calculation, we assume that the elastic-plastic stress-strain relation shown in Fig. 2-8 can be approximated by a bilinear function shown in Fig. 2-11. To test the condition, Eq. (119), we assume that there exists a single shock centered at \(V_0\) and terminated at point 1 on the second segment. Then, using the Hugoniot jump conditions, one finds

![Diagram showing stress-strain relationship](image)

Fig. 2-11. An hypothetical convex stress-strain relationship to illustrate shock wave instability.
\[ U = V_0 \left( \frac{\sigma}{V_0 - V} \right)^{\frac{1}{2}} = V_0 j \]  
(120)

\[ u = \left( \sigma (V_0 - V) \right)^{\frac{1}{2}} = \sigma / j \]  
(121)

where

\[ j^2 = \sigma / (V_0 - V) \]  
(122)

In the present case, the upper slope represents both the Hugoniot slope and that for the adiabatic sound speed (See Eq.(107)). Hence,

\[ C_s^2 = -V^2 \left( \frac{\partial p}{\partial V} \right)_S = -V^2 \left( \frac{\partial p}{\partial V} \right)_H \]  
(123)

Also, if the break at HEL is a convex cup, we can always find \( \alpha \) such that

\[ \left( \frac{\partial p}{\partial V} \right)_H = -\alpha^2 \frac{\alpha}{V_0 - V} = -\alpha^2 j^2, \quad 0 \leq \alpha \leq 1 \]  
(124)

Now putting Eqs. (120) into (124) yields the stability condition,

\[ \alpha Vj > V_0 j - \frac{\sigma}{j} \]

solving for \( \alpha \)

\[ \alpha > \frac{V_0}{V} - \frac{\sigma}{Vj^2} = \frac{V_0}{V} - \frac{1}{V} (V_0 - V) = 1 \]

This contradicts the assumption regarding \( \alpha \). Hence, the single shock is unstable.

The above argument can be extended to handle a concave plastic regime through modification of Eq. (123). The result is the same as that for a fluid and the shock is stable. But, there will be a third regime where a single shock becomes stable again because the Rayleigh line becomes steeper than the initial elastic slope. Figure 2-12 shows a schematic plane shock wave profile undergoing a full
cycle of compression and decompression in an elastic-plastic solid. Figure 2-13 is a measured profile in 6061-T6 Al (Jones, 1973):

Table 2-1 provides the values of HEL for a few selected materials. These values can be related to the yield stress in simple tension, $Y$. At HEL both Hooke’s law and the yield condition Eq. (108) are satisfied. Then assuming $K_s$ to be a constant,

$$\sigma_x = \frac{K_s + \frac{4\mu}{3}}{2\mu} \sigma_y$$

$$\sigma_x - \sigma_y = Y$$

Eliminating $\sigma_y$ from the above equations and using the identity equation (Landau...
Table 2-1. Yield strength of materials at the Hugoniot elastic limit.

<table>
<thead>
<tr>
<th>Material</th>
<th>Y (at HEL) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum (1100-0)</td>
<td>0.04</td>
</tr>
<tr>
<td>Aluminum (2024-T4)</td>
<td>0.26</td>
</tr>
<tr>
<td>Aluminum (6061-T6)</td>
<td>0.29</td>
</tr>
<tr>
<td>Copper (OFHC)</td>
<td>0.12</td>
</tr>
<tr>
<td>Lead</td>
<td>0.008</td>
</tr>
<tr>
<td>Nickel</td>
<td>0.14</td>
</tr>
<tr>
<td>Niobium</td>
<td>0.7</td>
</tr>
<tr>
<td>Platinum</td>
<td>0.03</td>
</tr>
<tr>
<td>Silver</td>
<td>0.05</td>
</tr>
<tr>
<td>Stainless Steel (304)</td>
<td>0.34</td>
</tr>
<tr>
<td>Tin</td>
<td>0.16</td>
</tr>
<tr>
<td>Titanium</td>
<td>0.85</td>
</tr>
<tr>
<td>Tungsten</td>
<td>2.2</td>
</tr>
<tr>
<td>Uranium</td>
<td>0.40</td>
</tr>
<tr>
<td>WC</td>
<td>5.1</td>
</tr>
</tbody>
</table>

(after Steinberg, 1991, UCRL-MA-106439)

and Lifshitz, 1970)

\[
K_s = \frac{2\mu(1 + \nu)}{3(1 - 2\nu)}
\]

where \(\nu\) is Poisson’s ratio, one obtains

\[
\sigma_s = \frac{(1 - \nu)}{(1 - 2\nu)} Y
\]

For a typical Poisson’s ratio of 0.3 for metals, this yields

\[
\sigma_s = 1.75Y
\]

2.4 Wave-body Interactions

In the previous sections we investigated the idealized nature of a steady shock transition and its key features using two basic material models. Now we will broaden our view and consider problems involving wave propagation and interactions. Since shock waves are generated by a sudden deposition of energy in a localized area by, e.g., impact of solid objects at high speed or explosive
detonation, wave propagation and interactions are essential aspects of application studies as well as basic studies of material response behavior under high rates of loading. In many studies wave interactions play a key role in the design of shock experiments. In this section, however, our emphasis is on a qualitative understanding of wave-body interactions. Therefore, extensive use will be made of graphical representations in planes where variables are \((p, u), (x, t), (p, x), (p, T)\), etc. Mathematically, they belong to a difficult class of non-linear initial and boundary value problems, and normally require numerical calculations of partial differential equations.

Before we consider specific interaction problems it will be useful to introduce a few preliminaries. An elegant discussion of the subject is found in Duvall (1971). We also benefited from class notes prepared by M. B. Boslough for his class “Shock Wave Physics,” which he taught at Sandia National Laboratories in 1985.

2.4.1 Preliminaries

With the linear \(U - u\) relationship, the momentum jump condition for a steady plane shock is given by

\[
p = p_0 U u = p_0 (C_0 + su)(u)
\]

(125)

So, the Hugoniot in a \((p, u)\) plane is a parabola. Figure 2-14 shows representative examples. If the coordinate is shifted to one moving with a velocity \(u_0\), then Eq. (125) yields

\[
p = p_0 \left[ C_0 + s(u - u_0) \right] (u - u_0)
\]

This means that the \((p - u)\) curve is the Galilean invariant and can be translated graphically to be recentered about \(u_0\). See Fig. 2-15. The product \(p_0 U\) is called shock impedance and is often represented by \(Z\). It is a material property, and independent of the choice of the coordinate. Shock impedance is a slowly varying function of pressure and reduces to acoustic impedance when \(u = 0\).

If a medium is homogeneous, then shock response of the material should be independent of its direction of propagation. Then, the Hugoniots for forward-moving (+) and backward-moving (−) shocks should be the mirror image of each other with respect to \(u\). The situation is illustrated in Fig. 2-16.

It is shown with Eq. (11), that a shock wave of infinitesimal strength is a sound wave. Then, replacing the jump in the Hugoniot equations, Eqs. (5)-(7), by the differential operator “\(d\)”, one can show that
Fig. 2.14. Hugoniots of representative materials in the \((p, u)\) plane. Adapted from Orava and Wittman (1975). The explanation of reflected transition curves is explained in section 5.

Fig. 2.15. Galilean translation of a Hugoniot curve in the \((p, u)\) plane.
Fig. 2-16. Forward- and backward-moving shocks in the \((p, u)\) plane.

\[ dp = \pm pC_s du = \pm \left[ dp(-dV) \right]^{1/2} \]  \hspace{1cm} (126)

\[ C_s = -V^2 \frac{dp}{dV} \]  \hspace{1cm} (127)

where signs \((\pm)\) correspond to forward and backward facing waves. This sign convention may be easily understood if one applies Eqs. (126) and (127) to an infinitesimal “shock.” Then, decompression (rarefaction) from a high-pressure state can be understood as a sequence of states whose characteristic wave speeds are given by Eqs. (126) and (127). Obviously, wave profiles are dispersive. Hodograph diagrams \((p, u)\) for rarefaction waves are given by

Fig. 2-17. Four possible transitions from a high pressure initial state in the \((p, u)\) plane.
\[ u - u_1 = \pm \int_{p_1}^{p} \frac{dp}{\rho_0 p C_s} \]  

(128)

Since sound velocity decreases in proportion to pressure drop in normal materials, forward facing rarefaction waves have final velocity \( u \) to the left of \( u_1 \) for \( p < p_1 \). By contrast backward facing rarefaction lies to the right of the initial state. Figure 2-17 shows four types of transition from an initial state \((p_0, u_0)\) to other states by either shocks or rarefactions.

In qualitative studies, Hugoniots from \((p_0, u_0)\) are often approximated by those centered at \( p_0 = 0 \). Also, release adiabats are represented by Hugoniots passing through \((p_0, u_0)\).

2.4.2 Planar impact of similar and dissimilar bodies

Figure 2-18 (a) shows a schematic of the physical problem in which a semi-infinite flyer plate impacts on a semi-infinite target at constant velocity \( u_0 \).

The impact produces two shocks: a forward moving \( (S^+) \) shock in the target and a backward moving \( (S^-) \) shock in the flyer. Figure 2-18 (b) illustrates the trajectories of these two shocks in the \((x-t)\) plane. It is assumed without loss of generality that \( Z_F > Z_T \) where \( F \) and \( T \) signify the flyer and the target respectively. The continuity of pressure and particle velocity at the impact surface will determine their impact conditions. In the \((p-u)\) plane \( S^+ \) is centered at \( u = 0 \) and increases to the left. On the other hand, \( S^- \) is centered at \( u_0 \) and increases to the right as illustrated in Fig. 17 (c). Therefore, they have an intersection with equal pressure and particle velocity.

As shown in Fig. 2-14, the \( p-u \) relationship can be approximated by a linear function: \( Z_F \) and \( Z_T \) are constant.

Then,

\[ p_T = Z_T u \]

\[ p_F = -Z_F (u - u_0) \]

After impact, \( p_T = p_F \) so,

\[ u = \frac{Z_F}{Z_F + Z_T} u_0 \]

\[ p_T = p_F = \frac{Z_T Z_F}{Z_F + Z_T} u_0 \]
If $Z_F = Z_T$, i.e., in the case of impact of similar materials,

$$u = \frac{u_0}{2}$$  \hspace{1cm} (129)

$$p = \frac{Z_F}{2} u_0$$

Equation (129), however, is independent of the assumption of the linearity.

2.4.3 Shock wave interaction with material boundaries

There are three commonly encountered situations. The first is a case where
a shock wave impacts on a contact discontinuity. The second, though this could be considered as a special case of the first class, is a situation where a shock wave strikes a free boundary. The third is a case where a detonation wave strikes an inert material surface. In this section we shall consider the first two. Since a detonation wave needs special preparation, the third case will be treated separately at the end of this chapter.

Figure 2-19 illustrates three graphical descriptions of the problem in the three planes: \((p, x)\), \((x, t)\) and \((p, u)\). Figure 2-19 (A) is a \((p, x)\) diagram where a shock in material I is about to strike the interface. As we shall see in Fig. 2-19 (C),

![Diagram](image)

**Fig. 2-19.** Interaction of a shock wave with a solid surface.
the nature of the transmitted and reflected waves depends on the relative magnitude of shock impedance of the two materials. Figure 2-19 (b) describes wave fronts in the \((x, t)\) plane. Slopes depend on wave speeds. Figure 2-19 (c) shows two positions of material \(II\) relative to that of material \(I\). Since the shock \((p_1, u_1)\) is propagating into undisturbed states, the transmitted wave in material \(II\) is a shock wave and centered at \((0, 0)\) in the \((p, u)\) plane.

If material \(II\) has a higher impedance than that of material \(I\), then to satisfy the continuity of pressure and particle velocity, the transition from \((p_1, u_1)\) to a point on the Hugoniot of material \(II\) must be done by means of \(S^\rightarrow\). Unloading from \((p_1, u_1)\) will not intersect the Hugoniot of material \(II\), because the slope of the adiabat at \((p_1, u_1)\) is steeper than that of its Hugoniot. Similarly, if the impedance of material \(II\) is lower, then the transition must be accomplished by rarefaction. This means that the unloading is dispersive.

Let us again assume that the two materials have linear shock impedance, and that \(Z_{II} > Z_I\). The reflected wave is now a shock, \(S^\rightarrow\), centered at \((p_1, u_1)\). Then,

\[
p_1 = Z_I u_1 \tag{130}
\]

\[
p_2 = Z_{II} u_2 \tag{131}
\]

Approximating the path from \((p_1, u_1)\) to \((p_2, u_2)\) by a line segment, and using \(Z_A\) centered at \((0, 0)\) as discussed in he previous section, one finds

\[
p_2 - p_1 = Z_I (u_1 - u_2) \tag{132}
\]

Putting Eq. (132) into Eqs. (130) and (131) yields

\[
p_2 = p_1 \frac{2Z_{II}}{Z_I - Z_{II}} \tag{133}
\]

\[
p_2 - p_1 = p_1 \frac{Z_{II} - Z_I}{Z_{II} + Z_I} \tag{134}
\]

If material \(II\) is a rigid material \((Z_{II} = \infty)\), then

\[p_2 = 2 p_1\]

But, this only holds for materials with linear impedances. For normal materials

\[p_2 > 2 p_1\]
If material II has a very low impedance or zero impedance (vacuum), then reflected waves form a rarefaction “fan” as shown in Fig. 2-20. Free surface velocity is given by Eq. (128) as

\[ u_{fs} = u_1 + \int_0^{p_s} \frac{dp}{\rho C_s} \quad \text{on } R^- \]  

(135)

If the unloading isentrope is approximated by the mirror image of the Hugoniot about \((p_1, u_1)\) then

\[ u_{fs} \approx 2u_1 \quad \text{(Rice, McQueen and Walsh, 1958)} \]

Fig. 2-20. A “fan” of rarefaction waves created at a free surface upon impact of a shock wave.
2.4.4 Wave-wave interactions

There is a variety of wave-wave interactions, but in this reaction we shall consider only the collision of two rarefaction waves as illustrated in Fig. 2-21. Such a problem will occur in the impact of two finite plates and play a key role in controlling dynamic fractures. In recovery experiments the control of both macro- and micro-crackings caused by such wave interaction is still an issue that has not been completely resolved. In the above figure two rarefactions move from both directions into a high pressure state $p_0$. It is assumed that $u = 0$, without loss

Fig. 2-21. Interaction of two rarefaction waves.
of generality. The unloadings from \((p_0, 0)\) to \((p_1, u_1)\) and \((p_2, u_2)\) are dispersive as shown in Fig. 2-21 (b). The corresponding paths in the \((p-u)\) plane are \(R^+\) and \(R^-\), respectively. The collision of rarefaction may be best illustrated by looking at it in terms of incremental unloadings, as discussed in section 3. For example, in Fig. 2-21 (c), the unloading from \(p_0\) to \(p_1\) and \(p_2\) are carried out in two and three steps, represented by points \((1, 2)\) and \((3, 4, 5)\), respectively. Then, the final state reached by the interaction of unloading waves represented by points 1 and 3, must lie at the intersection of wave transition curves (see Fig. 2-17) passing these points. This is point 6. Continuation of similar constructions will show that the final state is \((p_3, u_3)\) represented by point 11. For simplicity the diagram construction is based on the assumption of constant impedances. It can be easily seen that if \(u_1 = u_2 = 0\), then \(p_3(\leq 0)\) is always in tension.

2.4.5 Detonation wave and its interaction with a solid surface

Detonation is a form of combustion caused by compression and heating in a shock wave. By contrast the normal burning of combustible materials such as propellants is caused by the transfer of heat from the burning material to that which is still unreacted. Therefore, the propagation of the latter is, at most, of the order of several meters per second. But, the propagation of detonation is supersonic and typical velocities are of the order of several kilometers per second.

We are interested in detonation for two reasons. The first is that explosives have been used as a very efficient means of generating very high pressures in solids, so it would be useful to gain some basic understanding of their application. The second is its similarity to shock-induced chemical reaction of non-energetic materials. Since many of these reactions are exothermic, it would be of interest to highlight the interrelationship of the two phenomena. In fact, it has been proposed, (Boslough, 1990), that a form of detonation called "thermal detonation" takes place in a class of materials called thermitles, under shock loading. Therefore, in this section we shall present a brief description of the ZND model named after its independent authors, Zeldovich, von Neuman, and Doering. For a comprehensive discussion of the subject, the readers are referred to e.g., Fickett and Davis (1979) and references therein.

According to the ZND model the structure of a steady, plane detonation wave consists of three key components (or assumptions). The first is the leading shock front is called the von Neumann peak. There is no reaction in the shock front. The second is the narrow reaction zone that follows the leading front and is accompanied by a pressure drop due to expansion of the material. There is no transport effects such as viscosity and heat conduction in the reaction zone. Chemical equilibrium is reached at the rear boundary of the zone called the Chapman-Jouget point. The leading shock and the reaction zone are assumed to propagate at a constant velocity \(D\). If the rear boundary were not supported by, say, a piston, then the expansion behind the \(C-J\) point forms a centered rarefaction fan called Taylor wave. As we shall see shortly, this rarefaction wave will never overtake the shock
for the reason that the $C$–$J$ point is sonic relative to the reaction products. Figure 2-22 shows a schematic of the above described detonation wave followed by the Taylor wave. The ZND model is applicable to any medium, but the following description assumes a solid explosive.

In the $p$–$V$ diagram shown in Fig. 2-23, the peak of the von Neuman spike is a point, $p_{vn}$, on the Hugoniot of the unreacted explosive. The Rayleigh line to $p_{vn}$ indicates the detonation velocity $D$. Since the propagation of the detonation zone is assumed to be steady, the end of the reaction zone must be also on the Hugoniot of the reaction products that are assumed to be in local thermodynamic equilibrium. This means that the Rayleigh line to the von Neumann peak must intersect the product Hugoniot. There are obviously two possibilities, as shown in Fig. 2-23. The first case has two intersections labeled $S$ and $W$. The second has one intersection at a point called the $C$–$J$ point. However, it may also be seen from the figure that the former solution is not compatible with the assumption of the steady state. At point $S$, since the slope of the Rayleigh line is less than that of the local sound speed, rarefaction waves will overtake the leading shock, i.e., $u + C_s > D$. Similarly, at the lower point $W$, the situation is reversed, that is, $u + C_s < D$. In this

![Diagram](image_url)

Fig. 2-22. Schematic illustration of a steady, plane detonation wave based on the ZND model.
case the energy released by the chemical reaction cannot maintain the detonation because its effects will never reach the shock. The condition of steady state is only satisfied at the $C$–$J$ point where

$$ u + C_s = D $$

(136)

Mathematically, the above equality may be shown as follows. Since the $C$–$J$ point lies on the product Hugoniot, the slope satisfies the differential equation in Section 2.1.

$$ \frac{dp}{dV}_H = \left[ \frac{\left( \frac{dp}{dV} \right)_s + \frac{\gamma}{2V} \cdot p}{1 - \frac{\gamma}{2V} (V_0 - V)} \right] $$

(16)

At the $C$–$J$ point this slope is that of the Rayleigh line. Therefore

$$ \left( \frac{dp}{dV} \right)_H = \left( \frac{dp}{dV} \right)_R = - \frac{p}{V_0 - V} $$

(137)

Equating Eq. (137) with Eq. (16), one finds
\[
\left( \frac{dp}{dV} \right)_s = -\frac{p}{V_0 - V} = \left( \frac{dp}{dV} \right)_H
\]

\[
= -\left( \rho c_s \right)^2
\]

where the second relation is the definition of sound speed. Since the transition from \( V_0 \) to \( p_{cj} \) is a shock transition, the jump conditions yield

\[
D^2 = V_0 \left( \frac{p}{V_0 - V} \right)
\]

\[
u^2 = \left[ p(V_0 - V) \right]
\]

Hence,

\[
D - u = V \left( \frac{p}{V_0 - V} \right)^\frac{1}{2}
\]

By comparison, with Eq. (138), one obtains the equality relationship at the C–J point

\[
D - u = c_s
\]

Empirically, the reaction products (gases) from solid explosive are known to be clearly approximated by the isentropic polytropic equation

\[
p = p_{cj} \left( \frac{V_{cj}}{V} \right)^\gamma
\]

where \( \gamma \) is a parameter characteristic of the explosive. As shown in Table 2-2, \( \gamma \) for some common explosives is about 3, while for an ideal monoatomic gas \( \gamma \) is 5/3. Eq. (142) signifies that as a first approximation, an ideal gas equation of state can be used to describe the products with \( \gamma \) representing the ratio of heat capacities at constant pressure and volume.

The internal energy of this “ideal gas” is
Table 2-2. Properties of some common explosives (Deal, 1957).

<table>
<thead>
<tr>
<th></th>
<th>( \rho_0 )</th>
<th>( \rho_{ CJ } )</th>
<th>( P_{ CJ } )</th>
<th>( u_{ CJ } )</th>
<th>( D )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>RDX</td>
<td>1.767±0.011</td>
<td>2.375±0.012</td>
<td>337.9±3.1</td>
<td>2.213±0.029</td>
<td>8.639±0.041</td>
<td>2.904±0.047</td>
</tr>
<tr>
<td>TNT</td>
<td>1.637±0.003</td>
<td>2.153±0.006</td>
<td>189.1±1.0</td>
<td>1.664±0.011</td>
<td>6.942±0.016</td>
<td>3.172±0.029</td>
</tr>
<tr>
<td>64/36 Comp B</td>
<td>1.713±0.002</td>
<td>2.331±0.008</td>
<td>292.2±2.6</td>
<td>2.127±0.019</td>
<td>8.018±0.017</td>
<td>2.770±0.034</td>
</tr>
<tr>
<td>(64% RDX, 36% TNT)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>77/23 Cyclotol</td>
<td>1.743±0.001</td>
<td>2.366±0.009</td>
<td>312.5±2.9</td>
<td>2.173±0.020</td>
<td>8.252±0.017</td>
<td>2.798±0.034</td>
</tr>
<tr>
<td>(77% RDX, 23% TNT)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( D \) is the detonation velocity. Subscripts CJ refer to values at the Chapman-Jouget plane.

\( \rho_0 \) is initial density of the explosive.
\[ E = \frac{pV}{\gamma - 1} \]  \hspace{1cm} (143)

So, equating \( E \) with the energy in the jump condition, one obtains

\[ E - E_0 = \frac{1}{2} p(V_0 - V) + Q \]  \hspace{1cm} (144)

where \( Q \) is the heat of reaction, \( p_0 = 0 \) and \( E_0 \ll Q \). \( E_0 \) is typically less than 1% of \( Q \). By transferring \( Q \) to the left hand side, it could be thought of as stored energy in the initial state.

Now, putting Eqs. (140) and (143) into Eq. (144) yields

\[ \frac{pV}{\gamma - 1} = \frac{1}{2} u^2 + Q \]

\[ = \frac{p}{\gamma - 1} \left( V_0 - \frac{u^2}{p} \right) \]

Solving for \( p \),

\[ p = \left( \frac{\gamma - 1}{V_0} \right) Q + \left( \frac{\gamma + 1}{2V_0} \right) u^2 \]  \hspace{1cm} (145)

This is a parabola in the \((p-u)\) plane, illustrated in Fig. 2-24.

As demonstrated in the \((p-V)\) plane, it may be shown that the \( C-J \) point in the \((p = u)\) plane is also located at a point where the Rayleigh line \((dp/du = Z = \text{constant})\) is tangent to the Hugoniot curve given by Eq. (145).

Consider the derivative of Eq. (140) with respect to \( p \)

\[ 2u \frac{du}{dp} = (V_0 - V) - p \frac{dV}{dp} \]

Replacing the last derivative by Eq. (137), one finds

\[ 2u \left( \frac{du}{dp} \right)_{CJ} = 2(V_0 - V) = 2 \frac{u^2}{p} \]
where Eq. (140) is used to get the last equation. Simplifying the result, one finds the geometric relationship required for the tangency.

$$\left( \frac{dp}{du} \right)_{CJ} = \left( \frac{p}{u} \right)_{CJ}$$  \hspace{1cm} (146)

Also, the differentiation of Eq. (145) yields

$$\left( \frac{dp}{du} \right)_{CJ} = \frac{\gamma + 1}{V_0} (u)_{CJ}$$  \hspace{1cm} (147)

Putting Eq. (147) into Eq. (145) yields

$$p_{CJ} = \frac{\gamma + 1}{V_0} u_{CJ}^2$$

Using Eq. (145) again, one obtains

$$p_{CJ} = \frac{2(\gamma - 1)}{V_0} Q$$  \hspace{1cm} (148)
\[ u_{cJ}^2 = \frac{2(\gamma - 1)}{(\gamma + 1)} Q \]  \hspace{1cm} (149)

The detonation velocity can be calculated using the momentum jump condition

\[ D = V_0 \left( \frac{p}{u} \right)_{cJ} = \left[ 2(\gamma - 1)(\gamma + 1)Q \right]^{1/2} \]  \hspace{1cm} (150)

Also, the conditions at the $C-J$ point can be expressed in terms of $D$.

\[ p_{cJ} = \frac{D^2}{2V_0(\gamma + 1)} \]  \hspace{1cm} (151)

\[ u_{cJ} = \frac{D}{(\gamma + 1)} \]  \hspace{1cm} (152)

\[ \frac{V_{cJ}}{V_0} = \frac{\gamma}{(\gamma + 1)} \]  \hspace{1cm} (from the mass conservation equation)  \hspace{1cm} (153)

\[ c_{cJ} = \frac{\gamma}{(\gamma + 1)} D \]  \hspace{1cm} (154)

These relationships may be tested with the properties listed in Table 2-2.

Since the $C-J$ state is now fully characterized, one can determine the Taylor wave. To illustrate the procedure, let us imagine a semi-infinite explosive ($x \geq 0$) detonated at $t = 0$ and $x = 0$. The subsequent flow is illustrated in the ($x-t$) diagram shown in Fig. 2-25 where the thickness of the detonation zone is ignored.

The Taylor wave is a rarefaction wave centered at $x = 0$. If the unloading is viewed as a sequence of stepped decompression (see Section 2.4), each disturbance propagates at speed ($u + C_s$) relative to a stationary laboratory coordinate. Then, the trajectories of the disturbance called the characteristic curves are given by

\[ \frac{dx}{dt} = u \pm C_s \]  \hspace{1cm} (155)

where the sign indicates the direction of wave motion: “+” denotes forward-facing waves and “−” backward facing waves. The particle velocity and sound
Fig. 2-25. Diagram of the $x$–$t$ diagram for a steady, plane detonation wave. The detonation zone is assumed to be negligible.

The speed on the right-hand side are given by

$$\frac{dp}{du} = \pm \rho C_s$$  \hspace{1cm} (156)

$$C_s = \left( \frac{\partial p}{\partial \rho} \right)_s^{1/2}$$  \hspace{1cm} (157)

This means that for a polytropic gas, each wavelet of unloading propagates along a straight line passing the origin. So,

$$u \pm C_s = \frac{x}{t}$$  \hspace{1cm} (158)

With the polytropic equation of state, Eq. (142), the integration of Eqs. (156) and (157) is straightforward. The results are

$$p = p_{CJ} \left[ 1 + \frac{(\gamma - 1)(u - u_{CJ})}{2C_{CJ}} \right]^{2\gamma/(\gamma - 1)}$$  \hspace{1cm} (159)

$$u = u_{CJ} + \frac{2}{(\gamma - 1)}(C_s - C_{CJ})$$  \hspace{1cm} (160)
where

\[ C_s = C_{CJ} \left( \frac{P}{p_{CJ}} \right)^{\frac{\gamma-1}{2\gamma}} \]  \hspace{1cm} (161)

If the rear boundary is assumed to be free, then \( p = 0, C_s = 0, \) and

\[ u_{fs} = u_{CJ} - \frac{2}{(\gamma-1)} C_{CJ} \]  \hspace{1cm} (162)

Putting this into Eq. (160) yields

\[ C_s = \frac{\gamma-1}{2} \left( u - u_{fs} \right) \]  \hspace{1cm} (163)

Eliminating \( C_s \) from Eqs. (158) and (163) and choosing the forward-facing waves, one obtains

\[ u = \frac{2}{(\gamma+1)} \left( \frac{x}{t} \right) + \frac{\gamma-1}{\gamma+1} u_{fs} \]  \hspace{1cm} (164)

Let us assume, as shown in Fig. 2-26, that an explosive is in contact with an inert solid, and detonated at a point to the left of the interface. At the instant when the detonation wave strikes the interface, all the explosive is exhausted, and the explosion products are in the state at the \( C-J \) point. When the wave reflects from the interface, the state at the interface depends on the relative impedance of the two media: the inert solid and the reaction products. See Fig. 2-17. If the impedance of the solid is higher, then the reflected wave is a shock. In this case, the polytropic equation of state and Eq. (140) yield

\[ (u - u_{CJ})^2 = \left[ \frac{P}{p_{CJ}} - 1 \right] \left[ 1 - \left( \frac{P_{CJ}}{p} \right)^{\frac{1}{\gamma}} \right] \frac{p_{CJ}}{P_{CJ}} \]  \hspace{1cm} (165)

where the initial state in Eq. (140) is moved to the \( C-J \) point. In the other case the transition is a rarefaction wave. The reflected state is already obtained in Eq. (159) except for the sign. So,
Fig. 2-26. Reflection of a detonation wave at an inert solid surface.

\[ u - u_{\text{CJ}} = -\frac{2}{(\gamma - 1)} \left( \gamma \frac{V_{\text{CJ}}}{P_{\text{CJ}}} \right)^{1/2} \left[ \left( \frac{p}{P_{\text{CJ}}} \right)^{2\gamma} - 1 \right] \]  \hspace{1cm} (166)

The reflection curves, Eqs. (41) and (42), are plotted for several common explosives in Fig. 2-14.

Recovery experiments in shock synthesis often employ a technique in which a thin metallic flyer is accelerated by a high explosive, and impacts on a sample container at very high speed. The problem is similar to the one described above, but now the inert solid has a free boundary at the left end. The acceleration of the thin flyer is schematically illustrated in Fig. 2-27. Waves in the plate reverberate between the \( u \)-axis (\( p = 0 \)) and the unloading curve of the detonation products. From Eq. (166) the final velocity of the flyer may be approximated by

\[ u = u_{\text{CJ}} + \frac{2\gamma}{\gamma - 1} u_{\text{CJ}} = \left( \frac{3\gamma - 1}{\gamma - 1} \right) u_{\text{CJ}} \approx 4 u_{\text{CJ}} \]
Fig. 2-27. Schematic illustration of acceleration of a thin flyer by a high explosive.

This depends only on the characteristics of the explosive used. With real explosives, one cannot achieve such a high velocity, because energies are lost to real gas effects, dissipative mechanisms, etc.

REFERENCES