VLASOV SIMULATIONS OF ION ACOUSTIC DOUBLE LAYERS

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ABSTRACT

We present a detailed description of a Vlasov code which has been used to study the formation of weak ion acoustic double layers and we discuss the physical process responsible for the formation of these structures.

1. INTRODUCTION

Recent spacecraft observations in the auroral regions (Temerin et al., 1982) have again focussed the attention of scientists on current driven double layers. Yet both the scarcity of the data and the experimental difficulties and limitations advocate the use of computer simulations to understand their formation in space. For a global presentation of auroral physics the reader is referred to the review article by Sato (1982). We will not discuss the double layers initiated by applying an external difference of potential to the plasma since we believe that such conditions are more interesting for laboratory than for space plasmas. Moreover we will not discuss the formation of double layers in the Buneman's regime (De Groot et al., 1977, Sigov, 1982; Singh and Schunk, 1982) but we will concentrate on the ion acoustic case.

Since the pioneer work of Sato and Okuda (1980), substantial progress has been made and a rather simple physical understanding of the problem has emerged. Recently Okuda and Ashour-Abdalla (1982) have proposed an interesting extension of the simulation studies in this field.

Section 2 is devoted to the statement of the problem and to a detailed description of the "Vlasov" code we have used to solve it numerically. Section 3 contains a presentation of our results, a comparison with similar results from particle simulations and a discussion of the formation of ion acoustic double layers.

2. DESCRIPTION OF THE "VLASOV" CODE

2.1 What is the problem?

We want to integrate numerically the following Cauchy problem (hereafter we use the rationalized MKS unit system):

$$\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} E \frac{\partial f_\alpha}{\partial v} = 0$$

for $\alpha = (e,i)$ with $q_\alpha = \text{sign} q_\alpha \cdot q$ ($q > 0$)

$$\frac{\partial E}{\partial x} = \frac{1}{e_\alpha} \sum_\alpha q_\alpha \int_{-\infty}^{+\infty} f_\alpha(x,v,t) dv$$

$$f_\alpha(x,v,t=0) = \frac{n(x)}{v_{\text{th},\alpha}/2\pi} \exp\left\{ - \frac{(v - v_{d,\alpha})^2}{2v_{\text{th},\alpha}} \right\}$$

with the periodic boundary conditions

$$f_\alpha(x,t,v) = f_\alpha(x,v,t)$$

The notations are obvious: let us just say that $v_{d,\alpha}$ and $v_{\text{th},\alpha}$ are respectively the drift and the thermal velocities of the species $\alpha$. The initial conditions need some comments. We use products of functions in $x$ and $v$ variables: this just reflects our inability to prepare an initial microscopic state with true modes of the system.

Next, the same density function $n(x)$ is used in $f_e$ and $f_i$ to avoid the excitation of undesirable large amplitude Langmuir waves at the beginning of the computation. Section 3 will present simulations corresponding to different choices of $n(x) = n_0 + \delta n(x)$.

Physical quantities of interest are, as usual: the temperatures $T_\alpha = m_\alpha v_{\text{th},\alpha}^2$, the plasma frequency $\omega_p = (n_0 q^2 / e_\alpha m_\alpha)^{1/2}$ and the Debye length $l_D = v_{\text{th},e}/\omega_p$. Let us define standard dimensionless variables:

$$\tilde{n}_\alpha = \frac{n_\alpha}{n_0}, \quad \tilde{v} = \frac{v}{v_{\text{th},e}}, \quad \tilde{x} = \frac{x}{l_D}, \quad \tilde{t} = \omega_p t, \quad \tilde{E} = \frac{E}{n_0 q l_D}$$

The above system of equations is now written in dimensionless form.

$$\frac{\partial f_\alpha}{\partial \tilde{t}} + \tilde{v} \frac{\partial f_\alpha}{\partial \tilde{x}} + \frac{q_\alpha}{m_\alpha} \tilde{E} \frac{\partial f_\alpha}{\partial \tilde{v}} = 0, \quad \alpha = (e,i) \quad (1a,b)$$

$$\frac{\partial \tilde{E}}{\partial \tilde{x}} = \sum_\alpha q_\alpha \int_{-\infty}^{+\infty} f_\alpha(\tilde{x},\tilde{v},t) dv \quad (1c)$$

$$f_\alpha(\tilde{x},\tilde{v},t=0) = \frac{\tilde{n}(\tilde{x})}{\sqrt{2\pi}} \exp\left\{ - \frac{(\tilde{v} - \tilde{v}_{d,\alpha})^2}{2} \right\} \quad \frac{m_{\alpha} T_e}{m_e l_D^2} \quad (1d,e)$$

$$f_\alpha(\tilde{x},\tilde{v}) = f_\alpha(\tilde{x},\tilde{v},\tilde{t}) \quad (1f,g)$$
We will now describe a finite difference scheme which approximates the above differential system (1). We will not make a mathematical study of the convergence and of the stability of this scheme, this still remains to be done and this is a formidable task. We will rather present this scheme step by step from a very empirical point of view. When the opportunity arises we refer the reader to recent related works in numerical analysis.

2.2 The discretization of the variables

First we introduce the velocity cut-offs and the phase space discretization. We define $v_{a,\text{min}}$ and $v_{a,\text{max}}$ for each species $a$ usually by $v_d,a=5v_{\text{th},a}$, but an a priori knowledge of the gross features of the physical evolution may be a guide to choose the velocity intervals. For example for the ion acoustic instability it is well known that the space-averaged velocity distribution of the ions develops a high energy tail towards the mean electron velocity, so we can take the ion velocity cut-offs at e.g. $-3v_{\text{th},i}$ and $7v_{\text{th},i}$. The velocity cut-offs could be a serious bias because the phase space fluid of the species $a$ is not allowed to escape out of the interval $(v_{a,\text{min}},v_{a,\text{max}})$ in order to have an exact conservation of the integral

$$\int_{v_{a,\text{min}}}^{v_{a,\text{max}}} \int_{x_0}^{x_L} f_a(x,v,t) dv dx.$$ 

Hence, when the species $a$ is accelerated locally towards a velocity cut-off the phase space fluid $a$ is artificially accumulated just above $v_{a,\text{min}}$ or just below $v_{a,\text{max}}$: during the course of the computation we have to check that these accumulations near the velocity cut-offs only concern negligible fractions of the local distribution functions, when they occur. Each interval $(v_{a,\text{min}},v_{a,\text{max}})$ is then divided into $N_V$ identical velocity cells (we choose the same number $N_V$ for the ions and for the electrons but this restriction may be released without difficulty if needed). the width $\Delta v_a$ of each cell being of the order of $0.1v_{\text{th},a}$. The length $L$ of the system is divided into $N_x$ cells of width $\Delta x=L/N_x$ for both species. This description of the phase spaces requires $2.N_X.N_V$ memory words, it may be huge and this is one of the main limitations of Vlasov codes.

Second, the time is discretized with a time step $\Delta t$ which has to be small enough to ensure the numerical stability of the computation. As we will see later we integrate the system (1) along the characteristics. hence we have no Courant's condition but nevertheless the time-step $\Delta t$ is limited, as usual for electrostatic codes by $\omega_p.\Delta t < \pi$ which guarantees a correct sampling of the plasma oscillations. To reduce the phase errors to an acceptable level we choose $\omega_p.\Delta t=0.5$. We have made two simulations of the ion acoustic instability with $V_d,i=0.8V_{\text{th},i}$ but with the same initial microscopic state. The ion and electron densities and the electric field for these two computations were exactly superposable after $\omega_pt=512$. 


2.3 The numerical treatment of the Poisson's equation

From now on we will use only the dimensionless quantities defined above and we drop the superscript "~" for the sake of simplification. Let us now introduce the following standard notations: \( t^n = n \Delta t \), \((x_1, v_m)\) coordinates of the center of the \((l, m)\)-th phase space cell

\[
E_l^n = \frac{1}{\Delta x} \left( \int \frac{\Delta x}{2} E(x + x_1, t^n) dx \right)
\]

\[
f_{a,l,m}^n = \frac{1}{\Delta x \cdot \Delta v_a} \left( \int \frac{\Delta x}{2} f_{a}(x + x_1, v + v_m, t^n) dv dx \right)
\]

The integration of Poisson's equation (1c) over the \(l\)-th space cell gives, at time \(t^n\), taking into account the velocity cut-offs and discretization

\[
E(x_1 + \frac{\Delta x}{2}, t^n) - E(x_1 - \frac{\Delta x}{2}, t^n) = \Delta x \sum_{a} \text{sign} q_a \cdot \Delta v_a \sum_{m=1}^{NV} f_{a,l,m}^n
\]

now we approximate

\[
E(x_1 \pm \frac{\Delta x}{2}, t^n) \text{ by } \frac{1}{2} \left( E_{l+\frac{1}{2}}^n + E_{l-\frac{1}{2}}^n \right)
\]

which leads to the discretized Poisson's equation

\[
E_{l+1}^n - E_{l-1}^n = 2 \Delta x \sum_{a} \text{sign} q_a \cdot \Delta v_a \sum_{m=1}^{NV} f_{a,l,m}^n
\]

The left member of the above equation recalls the second order approximation of \(2 \Delta x(\partial E/\partial x)^n_l\). For "particle codes" the effect of the spatial grid has already been discussed in great details (see e.g. Birdsall and Maron, 1980 and references therein), but for "Vlasov codes" the situation differs by the discretization of the whole phase space and not only of the space coordinate. In principle it should be possible to evaluate the approximation of Poisson's equation by equation (2) by considering an exact nonlinear solution of the Vlasov-Poisson system; Bernstein, Greene and Kruskal (1957) have shown how to construct such solutions which are now commonly designated as BGK solutions. This study is left for a future work.

2.4 The numerical treatment of the Vlasov equation

Let us come now to the main difficulty, that is the integration of the Vlasov equation itself. The scheme we describe pertains to a rather wide class proposed some years ago by Boris and Book (1976b). In such methods there are two basic ingredients which preserve separately the global conservation and positivity of \(f_e\) and \(f_1\). First the Vlasov equation is splitted up into two advection equations (one
over \( x \), one over \( v \) using a Strang splitting scheme (Strang, 1968) and second a "Flux Corrected Transport" algorithm (Boris and Book, 1976b) is used to integrate the advection equations.

2.4a The splitting scheme

We want to compute \( f_\alpha(x, v, t_0 + \tau) \) from \( f_\alpha(x, v, t_0) \) for \( \alpha = (e, i) \) using (1a, b, c). In this case the Strang splitting consists of the following cycle of integrations (Boris, 1970, Boris and Book, 1976, Cheng and Knorr, 1976).

- integrate

\[
\frac{\partial f_\alpha^*}{\partial t} + v \frac{\partial f_\alpha^*}{\partial x} = 0 \quad \text{from } t_0 \text{ to } t_0 + \frac{\tau}{2}
\]  

(3a, b)

with the initial condition

\[
f_\alpha^*(x, v, t_0) = f_\alpha(x, v, t_0)
\]  

(4a, b)

for \( \alpha = (e, i) \)

- compute the electric field at time \( t_0 + \tau/2 \) by substituting

\[
f_\alpha^*(x, v, t_0 + \frac{\tau}{2})
\]  

in equation (1c).

- integrate

\[
\frac{\partial f_\alpha^{**}}{\partial t} + \text{sign} \alpha \frac{me}{m_\alpha} E(x, t_0 + \frac{\tau}{2}) \frac{\partial f_\alpha^{**}}{\partial v} = 0
\]  

(5a, b)

from \( t_0 \) to \( t_0 + \tau \) with the initial condition

\[
f_\alpha^{**}(x, v, t_0) = f_\alpha^*(x, v, t_0 + \frac{\tau}{2})
\]  

(6a, b)

for \( \alpha = (e, i) \)

- integrate

\[
\frac{\partial f_\alpha^*}{\partial t} + v \frac{\partial f_\alpha^*}{\partial x} = 0 \quad \text{from } t_0 + \frac{\tau}{2} \text{ to } t_0 + \tau
\]  

(7a, b)

with the initial condition

\[
f_\alpha^*(x, v, t_0 + \frac{\tau}{2}) = f_\alpha^{**}(x, v, t_0 + \tau)
\]  

(8a, b)

for \( \alpha = (e, i) \)

- finally let an approximation of \( f_\alpha(x, v, t_0 + \tau) \) be \( f_\alpha^*(x, v, t_0 + \tau) \) for \( \alpha = (e, i) \).

Integrating formally the whole system (3a, b to 8a, b) we obtain
\[ f_\alpha^*(x, v, t_{0+\tau}) = f_\alpha^*(x-v\frac{T}{2}, v, t_{0+\frac{T}{2}}) = f_\alpha^*(x-v\frac{T}{2}, v, t_{0+\tau}) = f_\alpha^*(x-v\frac{T}{2}, v-E_\alpha t, t_0) = f_\alpha^*(x-vT, v-E_\alpha t, t_0+\frac{T}{2}) = f_\alpha^*(x-vT+E_\alpha\frac{T}{2}, v-E_\alpha t, t_0) \]

\[ f_\alpha^*(x, v, t_{0+\tau}) = f_\alpha^*(x-vT+E_\alpha\frac{T}{2}, v-E_\alpha t, t_0) \]

with \( E_\alpha = \text{sign} q_\alpha \frac{m_\alpha}{m_\alpha} E(x-v\frac{T}{2}, t_{0+\frac{T}{2}}) \)

To demonstrate that the proposed approximation of \( f_\alpha^*(x, v, t_{0+\tau}) \) is second order accurate with respect to \( \tau \) we can recognize with Cheng and Knorr (1976) that it is equivalent to take \( f_\alpha = \text{constant} \) along the approximate characteristics of the Vlasov-Poisson system obtained by a leap-frog integration which is known to be second order accurate in time. Another way, more systematic, is to compute the Taylor development of \( f_\alpha(x,v,t_{0+\tau}) \) up to the second order in \( \tau \) using

\[ \frac{\dot{f}_\alpha}{\dot{t}} = -\mathcal{L}_\alpha f_\alpha \quad \text{where} \quad \mathcal{L}_\alpha = \frac{\partial}{\partial x} \frac{\partial}{\partial v} + \text{sign} q_\alpha \frac{m_\alpha}{m_\alpha} E \frac{\partial}{\partial v} \]

and then to compare it with the Taylor development of the right member of (9a, b) up to the second order in \( \tau \); it appears that these two developments are identical. Of course during the implementation of the code the two adjacent integrations over \( x \) from the \( n \)-th and \( (n+1) \)-th cycles are combined together; hence we only need an half time step integration over \( x \) to start the computation as with the usual leapfrog algorithm. The above result concerning the second order accuracy of the splitting scheme is interesting, yet limited because it disregards totally the actual schemes used to integrate the advection equations; the above statement means only that the second order accuracy is the best we can expect with integration schemes of the advection equations which do not alter this accuracy. To evaluate the accuracy and convergence of a splitting scheme taking into account the full discretization of all variables (and not only of the time variable as above) and the particular numerical algorithms used for each fractional step is a difficult problem. Rigorous results have been demonstrated by Marchuk (1970) for the Crank-Nicholson schemes applied to various equations, and more recently by Crandall and Majda (1980) for conservation equations approximated by monotone schemes.
Lax-Wendroff schemes and the Glimm scheme combined with either Strang or Godounov splitting.

2.4b The numerical integration of the advection equation

Equations (3a.b) and (5a.b) have a common property: the coefficient of the "spatial" derivative $\frac{\partial f^*}{\partial x}$ or $\frac{\partial f^{**}}{\partial v}$ does not depend upon the "space" variable (respectively $x$ or $v$); thus, in both cases, we have to integrate the advection equation with constant velocity

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = 0 \quad (10)$$

with $\rho(x, t) \geq 0$, positivity constraint \hspace{1cm} (11)

and $\int \rho(x, t) dx =$ constant, conservation constraint \hspace{1cm} (12)

The very simple equation (10) is a special case of

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0$$

which has been the subject of a huge amount of work in numerical analysis. What we want to do here is just to present in details one of the simplest FCT codes proposed by Boris and Book (1976b), that is an antidiffused donor-cell algorithm. The reader interested in the numerous developments of antidiffusion is referred to the following classical papers: Boris and Book (1973); Book, Boris and Hain (1975); Boris and Book (1976a, b). Until now the justification of these algorithms lay on rather intuitive ground, although rigorous results have been demonstrated, for example Ikeda and Nakagawa (1979) have studied the convergence and the stability of the SHASTA FCT algorithm (Boris and Book, 1973) (slightly modified). Starting from a very different point of view, Harten (1983) has proposed explicit second order finite difference schemes which, in some cases, recall the FCT algorithms.

The donor-cell algorithm is the simplest approximation to (10,11,12). Let $\rho^n_j$ be the value of the density in the $j$-th cell of width $\Delta x$ at a time $t^n = n \Delta t$. Suppose that during the time step $\Delta t$ the profile $(\rho^n_j)$ is shifted to the right ($v > 0$) or to the left ($v < 0$) by the amount $v \cdot \Delta t = (1 + \varepsilon) \Delta x$ where $l$ is the nearest integer to $v \Delta t / \Delta x$, so $-0.5 \leq \varepsilon \leq 0.5$. First the integral part $l \Delta x$ of the displacement is done exactly by shifting the profile $(\rho^n_j)$ of $l$ cells, and second the remaining fractional part is taken into account by

$$\rho^{n+1}_j = l \varepsilon \rho^n_j \cdot \text{sign} \varepsilon + (1-l \varepsilon) \rho^n_j \quad (13)$$

The boundary conditions have already been discussed. This scheme is obviously conservative (i.e. the discretized version of (11) holds) and preserves the positivity because $0 \leq l \varepsilon \leq 0.5$, that is $\rho^{n+1}_j \geq 0$ for any $j$ when $\rho^n_j \geq 0$ for any $j$: stability and monotonicity are achieved
under the same condition. It is a first order approximation to (10) in \(\Delta x\) and \(\Delta t\) which is obtained when approximating \((\partial p/\partial t)^N_j\) by \(1/\Delta t(p^{N+1}_j - p^N_j)\) and \((\partial p/\partial x)^N_j\) by \(1/\Delta x(p^{N+1}_j + 1/2 - 1/2\text{sign}(-\Delta x)\) - 1/2\text{sign}\). The smearing of the transported profile results in a undesirable strong diffusion. To look into this point, let us suppose that \(\phi^0_j = \delta_{j,0}\) (Kronecker symbol) and that \(\ell = 0\), then

\[p^N_j = C^N_j (1 - i\ell)^N - 1 \text{ for } 0 \leq j \leq n\]

and \(p^N_j = 0\) otherwise, where the \(C^N_j\) are the binomial coefficients. Asymptotically for large \(n\),

\[p^N_j \sim \frac{1}{2\pi n! |1 - i\ell|} \exp \{ - \frac{(j - n\ell)^2}{2 |1 - i\ell| n} \}\]

it appears that the initial pulse spreads out during the propagation: \(p^N_j\) is maximal for \(j = n\ell\). i.e., for \(x = vt\). thus the velocity of the pulse is exact and the spreading of the pulse is described by the diffusion coefficient

\[D = \frac{1}{2 |1 - i\ell|}.\]  

(14)

To remedy this diffusion we insert an unphysical stage in the computation: we consider \(p^{N+1}_j\) given by (13) as an intermediate value \(\bar{p}^N_j\) and we apply an antidiffusion operator to \((p^N_j\), that is the provisional values \(\bar{p}^N_j\) can be seen as the result of a normal diffusion process (with a coefficient \(\eta > 0\)) over the profile \((p^{N+1}_j)\). Approximating the diffusion equation

\[
\frac{\partial \bar{p}}{\partial t} = \eta \frac{(\Delta x)^2}{\Delta t} \frac{\partial^2 \bar{p}}{\partial x^2}
\]

by the explicit first order method (see Potter), we obtain

\[
\bar{p}^N_j = p^{N+1}_j + \eta(p^{N+1}_j - 2p^N_j + p^{N+1}_{j-1})
\]

(15)

this is known as implicit antidiffusion (for other types of antidiffusion and discussion of their respective advantages see the "classical" papers). From the above discussion of the diffusion associated with the donor cell transport algorithm we can guess that we have to choose (see 14) \(\eta = \ell^2 / 2(1 - \ell^2)\). This is confirmed by the analysis of the linear amplification factor. let us consider an harmonic wave \(p^N_j = e^{i(k\Delta x - \omega\Delta t)}\) where \(k\) is the wave number, \(\omega = \omega_0 + if\) the complex frequency and \(f = \omega_0^{/-1}\). Substituting for \(p^N_j\) in

\[i\ell \eta (p^N_j)\text{sign} + (1 - i\ell) p^N_j = p^{N+1}_j + \eta(p^{N+1}_j - 2p^N_j + p^{N+1}_{j-1})
\]

(16)

we obtain the following dispersion relation for the linearly and implicitly antidiffused donor cell algorithm:

\[i\ell e^{-i(k\Delta x)} \cdot \text{sign} + (1 - i\ell) = (1 - 2\eta(1 - \cos k\Delta x)) e^{-i\omega \Delta t}
\]

The square of the modulus of the amplification factor is
$e^{2\tau\Delta t} = \frac{1 - 2\tau\epsilon(1-\epsilon\epsilon)(1-\cos k\Delta x)}{(1 - 2\eta(1-\cos k\Delta x))^2}$

For $k\Delta x \ll 1$

$e^{2\tau\Delta t} = 1 - [2\eta-\epsilon\epsilon(1-\epsilon\epsilon)](k\Delta x)^2 + O((k\Delta x)^4)$

It appears that our choice $\eta=\frac{1}{2}\epsilon\epsilon(1-\epsilon\epsilon)$ reduces the diffusion (or damping) to terms of the order $(k\Delta x)^4$ for the longer wavelengths and guarantees $\tau \ll 0$ for any $k\Delta x$.

Another way of looking at this point is to check that (16) approximates (10) with second order accuracy when $\eta=\frac{1}{2}\epsilon\epsilon(1-\epsilon\epsilon)$. This is done with Taylor expansions like

$p_j^{\eta+1} = p_j^\eta + \Delta t \frac{\partial p}{\partial t}|^\eta + \Delta t \cdot \Delta x \left( \frac{\partial^2 p}{\partial x^2} \right)^\eta + \cdots$

$\cdots + \frac{\Delta t^2}{2} \left( \frac{\partial^2 p}{\partial t^2} \right)^\eta + O(\Delta x^3, \Delta t^3, \Delta x^2 \Delta t, \Delta x \Delta t^2)$

and with the help of $\partial / \partial t = -(v) \partial / \partial x$. The dispersion of the scheme is governed by (Boris and Book, 1973)

$\tau \left( \omega_r \cdot \Delta t \right) = \frac{\epsilon \sin(k\Delta x)}{1 - \epsilon\epsilon(1-\cos k\Delta x)}$

For $k\Delta x \ll 1$

$\omega_r = kv\{1 - \frac{1}{6}(1-\epsilon\epsilon-1)(2-\epsilon\epsilon-1)(k\Delta x)^2 + O((k\Delta x)^4)\}$

The relative phase error is of the order $(k\Delta x)^2$ for the longer wavelengths. It is possible to obtain a relative phase error of the order $(k\Delta x)^4$ by adding an appropriate diffusion term to the transport stage (13). Book, Boris and Hain (1975) have discussed the advantages of diffusive transport with regard to the phase properties of a scheme.

The constraint $0<\epsilon\epsilon<0.5$ gives $\eta_{max}=1/8$, hence we may consider it as a small parameter and we will look for an approximation to (16), accurate to second order in $\epsilon$ and $\eta$, to avoid the solution of a tridiagonal system. Looking for a solution like

$p_j^{\eta+1} = \xi_k^2 (\delta_k + b_k \epsilon + c_k \eta + d_k \epsilon \eta + e_k \epsilon^2 + f_k \eta^2) p_j^\eta + \kappa

we find that $p_j^{\eta+1}$ is given by the following sequence

$p_j^\eta = (1-\epsilon\epsilon) p_j^\eta - \epsilon \delta_j^\eta + (1-\epsilon\epsilon) p_j^\eta$

$p_j^\eta = p_j^\eta + \eta \delta_j^\eta \frac{1}{2} - \eta \delta_j^\eta \frac{1}{2}$

$p_j^{\eta+1} = p_j^\eta + \eta \delta_j^\eta \frac{1}{2} - \eta \delta_j^\eta \frac{1}{2}$

(17a)

(17b)

(17c)
with
\[ \Delta_{j+1}^\eta = \rho_{j+1}^\eta - \rho_j^\eta \]
\[ \Delta_{j+1}^\eta = \tilde{\rho}_{j+1}^\eta - \tilde{\rho}_j^\eta \]

We have reduced the diffusion but the positivity of \( \rho_j^{n+1} \) given by (17c) is no more guaranteed. To preserve this important property Boris and Book (1973) have introduced a limiter on the antidiffusion fluxes \( f_{j+1/2}^{\eta} = \eta \Delta_j^{n+1/2} \) (see section IV in Book. Boris and Hain, 1975) ; (17c) is replaced by

\[ \rho_j^{n+1} = \tilde{\rho}_j^\eta + f_{j+1/2}^c - f_{j-1/2}^c \]  \hspace{1cm} (18)

\( c \) stands for corrected. Whenever the transported profile \( (\tilde{\rho}_j^\eta) \) is not strictly increasing (or decreasing) over the four points \( j-1, j, j+1, j+2 \), the corrected flux \( f_{j+1/2}^c \) is equal to zero; in all other cases \( f_{j+1/2}^c \) is such that \( \rho_j^{n+1} \) given by (18) is increasing (respectively decreasing). In fact this prescription does more than preserve the positivity, it guarantees that the antidiffusion stage neither creates new extrema nor accentuates already existing ones. The corrected flux is expressed by (19)

\[ f_{j+1/2}^c = S_{j+1/2} \max\{0, \min(S_{j+1/2} \Delta_j^{n+1/2}, \eta \Delta_j^{n+1/2}, S_j^{n+1/2} \Delta_{j+1}^{n+1/2})\} \]

with
\[ S_{j+1/2} = \text{sign} \Delta_{j+1/2}, \Delta_{j+1/2} = \tilde{\rho}_{j+1}^\eta - \tilde{\rho}_j^\eta \]

Finally equation (10) with constraints (11) and (12) is approximated by (17a, 17b, 18, 19). The description of the code we have used is now complete. Let us say to close this section that this code could be made faster by implementing the electron subcycling algorithm proposed by Adam, Gourdin-Serveniere and Langdon (1982).

3. THE FORMATION OF ION ACOUSTIC DOUBLE LAYERS

Theoretical studies and computer simulations of the ion acoustic instability have been mainly devoted in the past either to cases with constant current or to cases where a constant electric field is imposed throughout the plasma. The effects of the dimensionality and of the magnetic field have been discussed in details (see the review by Dum, 1981, and references therein).

Ishihara and Hirose (1981), Appert and Vaclavik (1981) have studied the formation of the ion tail in the frame work of the quasilinear theory. This section is devoted to a study of the relaxation regime; the system is not influenced by any external factor, its evolution is governed by the initial condition only. We
will distinguish between weak and strong initial perturbations. The evolution in the latter case will be compared with the evolution of an assembly of particles.

3.1 Weak initial perturbations

We will at first describe the simulation results we have obtained for noisy initial states. that is when the initial density perturbation \( \delta n(x) \) is built as the superposition of \( NX/4 \) sinusoids having a common amplitude \( \delta n_k = 0.01 \), wave numbers ranging from \( 2\pi/L \) to \( NX/4 \), \( 2\pi/L \) and random phases. We use it as an approximation to white noise. At the very beginning of the computation (say until \( \omega_D t \sim 100 \)) the system builds up velocity perturbations consistent with the density perturbations; during this self-organization stage the electrostatic energy density decreases down to \( 10^{-5} - 10^{-6} T_e \) before increasing, due to the ion acoustic instability. This rearrangement in phase space is accompanied by electron diffusion around \( v=0: \delta n_k \) has to be chosen small enough to avoid the formation of a plateau around \( v=0 \), otherwise the instability could be quenched during this transient stage. Hence we are limited to weak initial perturbations. For comparison, the initial electrostatic energy density for a one-dimensional electrostatic particle code with 100 particles per \( L_0 \) is of the order of \( 10^{-3} T_e \). We have made a series of runs with \( T_e/T_i = 20 \), \( m_i/m_e = 100 \), \( L=512L_0 \) and \( 1024L_0 \) and with the drift speed of the electrons being equal to 0.6, 0.8 and 1.0 \( V_{th,e} \). In all these cases the system evolves qualitatively in the same way, which is somewhat different from that exhibited by similar particle simulations. In the relaxation regime, that is when the electric current is not sustained by a current generator or by an external inductive electric field, the particle simulations show an asymptotic decrease of the spatially averaged current density equal to a few tens of per cent. Greater is the drift speed, greater and faster is the current interruption (see for example Nishihara et al., 1983). The same tendency appears in our computations but the current interruption is only equal to a few percent. The saturation levels of the electrostatic energy \( W \) also differ by an order of magnitude or more; e.g. for \( V_d = 0.6 \), \( W \) reaches \( 4.10^{-5} \) instead of \( 2.7 \times 10^{-3} T_e \) in the particle case (Sato and Okuda, 1980). Nevertheless in all cases we have simulated, we have observed localized negative potential structures similar to those observed in particle simulations (Chanteur et al., 1980, Sato and Okuda, 1981, Kindel et al., 1981, Hudson and Potter, 1981, Nishihara et al., 1983).

In both types of computations these potential spikes grow from the ion acoustic noise but their growth is much slower in the Vlasov case, and moreover they do not produce important differences of potential between their upstream and downstream sides ("upstream" and "downstream" refer to the mean electron flow which is going from the left to the right, i.e. increasing \( x \), for all the relevant figures). They propagate in the direction of the mean electron flow with a velocity equal to the phase velocity of the ion acoustic waves in the beginning and later with a somewhat lower velocity. These potential
wells reflect the electrons and accelerate the ions, thereby producing characteristic signatures in the phase spaces. Figure 1 shows the electron phase space at $\omega_{p\alpha}=1024$ for $L=512\lambda_D$ and $V_d=0.8V_{th,e}$. A strong acceleration region is clearly visible near $x=130$ where the isodensity curves are squeezed. Especially the penetration towards $v=0$ of the curves related to the lower phase space density levels for $v<0$ indicates a pronounced dip in the electron density which is lowered down to 0.85 (unperturbed value $n_0=1$). Because of the drift velocity there are more electrons reflected on the upstream side than on the downstream side, and this results in a slight asymmetry of $n_e(x)$. The ion density looks like the electron density (a typical profile $n_i(x)$ is displayed by the upper part of Figure 4), but the space charge created by the reflection of the electrons is not exactly balanced by the massive ions, hence the upstream side having an excess of negative charge acquires a negative potential with respect to the downstream side where positive charges dominate. This rather weak difference of potential could be inferred from the electron phase space which shows that the domain of the reflected electrons is more inflated (in the velocity direction) on the downstream side, indicating that the potential barrier is higher seen from the downstream side than from the upstream one. Figure 2 represents the ion phase space at the same time but we have represented only the lower isodensity curves. The core of the ion distribution function represented by the white central band exhibits the sloshing motion of
Figure 2. Ion phase space (same case as Figure 1). $C_s$ is the ion acoustic velocity and only the lower isodensity levels are represented.

Figure 3. Upper: same as Figure 1 but for $L = 1024 \lambda_D$ lower: same as Figure 2 but for $L = 1024 \lambda_D$ (heavy lines delineate the core of the distribution function).
the ions in the ion acoustic waves and displays clearly the main features of the acceleration region near $x=136$. In particular, the variation of the ion bulk velocity through the structure is consistent with the observations made for the electrons. Another interesting feature appears from the representation: as the potential well grows and propagates, it leaves an oscillating tail behind it. Two other acceleration regions are visible with their oscillating tails near $x=40\lambda_D$ and $x=270\lambda_D$ but they are less developed. The amplitude $-\psi_{\text{min}}$ of a potential spike is limited to $e\psi_{\text{min}} \approx m_e V_d^2$, the saturation occurs by electron trapping in the oscillating tail which grows to a level comparable to $\psi_{\text{min}}$. Varying the length of the system does not change this picture, as shown by Figure 3 which represents the electron and ion phase spaces for $L=1024\lambda_D$, the other parameters being left unchanged. Increasing the drift velocity of the electrons tends to decrease the separation between the acceleration regions. The long time behavior in these relaxation experiments is characterized as in the particle case by an absence of resistivity and a very slow decay of the wave trains produced: Figure 4 shows the ion density and the space charge density at $\omega_p t = 1280$ for the case $L=512\lambda_D$, $V_d = 0.6 V_{\text{th,e}}$. The situation does not evolve significantly until $\omega_p t = 1792$, the time

![Graph showing ion density and space charge density](image)

Figure 4. Ion density (upper) and space charge density (lower) for a Vlasov simulation with noisy initial condition: $L = 512\lambda_D$ and $V_d = 0.6 V_{\text{th,e}}$. 
at which we have stopped the computation. The space charge density plot shows that the field has been broken into a few coherent nonlinear wave packets which have a sharply defined front and a decreasing tail (see packet marked (1) and (2)). Moreover an examination of the phase spaces indicates that the electrons and the ions are trapped in these oscillations. It is likely that this behaviour is strongly related to the one dimensionality of the system.

3.2 Strong initial perturbations

The acceleration regions we have discussed above are so weak that they hardly deserve the name of double layers but basically they are driven by the same mechanism as we will see now. To get round the undesirable electron diffusion which occurs during the self-organization stage, we start the computation with a localized and smoothly shaped density depression having an amplitude of 0.1. The advantage of a localized density perturbation is the following: the velocity perturbations generated during the transient rearrangement of the system do not spoil the incoming plasma on the upstream side of the initial density dip. Due to the periodic boundary conditions, these ballistic perturbations reappear after some time on the upstream side of the density trough but the system is long enough to guarantee that the instability is well developed at that time. The length of the system is \( L = 512 \lambda_D \) and the drift speed \( V_d \) of the electrons is.

![Figure 5. Potential energy of the electrons versus space and time for the Vlasov simulation with smooth initial condition](image)
equal to 0.8 $V_{th,e}$, the other parameters having the values already mentioned. This simulation is discussed in detail by Chanteur et al. (1983). Let us recall the main features the evolution and the proposed interpretation. Figure 5 shows a three-dimensional plot of $-\phi$, the potential energy of the electrons versus $x$ (unit $\lambda_p$) and time (unit $\omega_p^{-1}$). A potential energy barrier for the electrons grows in the depressed density region and moves in close association with it. At the beginning they propagate in the direction of the main electron flow with a velocity equal to 0.8 $C_s$ ($C_s$ is the ion acoustic speed which is equal to 0.1 $V_{th,e}$ in our case), a value corresponding to the linear group velocity of waves having the same spatial scale as the present one. The most interesting and puzzling points are the following: first the potential barrier suffers a very strong slowing down until it stops and starts propagating backwards with an increasing speed (a similar behavior has been noticed in particle simulations, see below); second, as the barrier grows and interrupts the electron flow, a positive difference of potential $\phi$ downstream $-\phi$ upstream is built up by the unbalanced space charge; third, it could be seen that the upstream flank of the barrier becomes steeper than the downstream flank before the reversal of the motion of the

![Figure 6](image-url)  

*Figure 6.* Same part of the ion phase space at different times for the same case as for Figure 5

structure. Figure 6 shows the same part of the ion phase space in the vicinity of the electric potential well at different times. Until $\omega_p t=400$ the velocity of the structure is almost constant and the main feature in the ion phase space is the localized modification of the
mean velocity over the potential well. In the frame of reference moving with the localized wave the ions enter the potential region from the downstream side with a negative velocity, they are successively accelerated and decelerated as they fly above the well. This is the reason why the potential well produces a negative velocity perturbation in the laboratory frame. When the velocity of the wave is reversed we gain the opposite signature in the ion phase space (see below). Ion trapping occurs after $\omega_p t \approx 400$ when the velocity of the localized wave begins to decrease noticeably and an ion hole is forming after $\omega_p t \approx 550$ when the whole structure is almost stationary in the ion frame. The spatial variation of the ion bulk velocity through the structure also supports evidence for a localized difference of potential between the downstream and upstream sides.

![Figure 7](image_url)

**Figure 7.** Same part of the electron phase space at different times for the same case as Figure 5.

Figure 7 displays the same part of the electron phase space at three different times during the forwards propagation of the potential barrier. The reflections of the particles are clearly visible on both sides of the acceleration region, together with the resulting asymmetry between the upstream and downstream sides. The similarity with Figure 1 is obvious and the localized difference of potential is also responsible for a more extended velocity range of the reflected particles on the downstream side. Before discussing these results let us recall a particle simulation (Chanteur et al., 1980) done with the same physical parameters and the same length of the system. The main difference is that the particle computation is started from the
Figure 8. Potential energy of the electrons for the particle simulation with $L=512 l_D$ and $V_d=0.8 V_{th,e}$

Figure 9. Same as Figure 6 for the particle simulation.
"thermal" density fluctuations (we have used 100 particles of each species per Debye length). The physical evolution of the system is dominated by the growth of a potential barrier for the electrons (see Figure 8 and compare with Figure 5), in particular the current interruption is mainly caused by this barrier. Figure 9 represents the same part of the ion phase space at different times and has to be compared with Figure 6. The first and second plots are taken when the localized wave propagates towards the right (notice the negative velocity dip). The third plot is for $\omega_p t = 640$ when the potential structure is still almost stationary in the ion frame and an ion hole could be seen forming. The two last plots are relative to the backwards propagation of the wave, and as was already discussed above, a positive velocity perturbation is apparent. This strange motion of the potential barrier was also noticed by Kinzel et al. (1981), Nishihara et al. (1983), and in a different way by Sato and Okuda (1981). Figures 12 and 13 in Sato and Okuda (1981) are particularly interesting in this respect. The spatial variation of the frequency spectrum of the turbulence discussed by Sato and Okuda is now easy to understand. Figure 10 is a three-dimensional plot of the electric field versus space and time in our particle simulation. The huge electric field of the localized wave is clearly identified as well as the boomerang motion of the wave. The vertical black arrows indicate four positions, namely $x = 90, 100, 120$ and 125 $\lambda_0$; the corresponding temporal variations of the electric field at these particular places
are shown on Figure 11 (which is very similar to Figure 13 of Sato and Okuda (1981). Thus we conclude that the spatial variation of the frequency spectrum of the electric field is mainly due to the slowing down of the coherent localized wave which has emerged from the turbulence.

![Figure 11](image)

Figure 11. Temporal variations of the electric field at four different places for the particle simulation. Arrows under the plots indicate the coherent localized wave.

3.3 Discussion and conclusion

Nishirara et al. (1983) and Chanteur et al. (1983) have proposed the same interpretation of the physical process discussed above. They have shown that the propagation, the growth and the distortion of the potential barrier could be explained by a modified Korteweg de Vries equation for the potential, taking into account the reflections of the electrons. A similar equation has been proposed by Karpman et al. (1979) and Lotko (1983) to interpret the damping of standard compressive ion acoustic solitons. Moreover, Nishihara et al. (1983) have integrated numerically both their modified KdV equation and the usual KdV equation for negative and localized initial conditions. In the pure KdV case which has also been studied numerically in detail by
Fornberg and Whitham (1978) and by Okutsu and Nakamura (1979) the negative perturbation is damped by the "radiation" of a dispersive tail while the localized wave is slowed down by the nonlinear convective term $\phi \partial \phi / \partial x$. This last term also acts to steepen the trailing edge of the potential well and to splay its leading edge. Similar conclusions have been obtained experimentally for rarefactive ion acoustic pulses propagating in a quiescent plasma by Okutsu and Nakamura (1979) and by Saxena et al. (1981). When the amplification term caused by the asymmetry of the electronic distribution function is taken into account the work of Nishihara et al. (1983) indicates, as expected theoretically, the amplification of the potential well but also a stronger slowing down of the solution. This slowing down is a nonlinear hydrodynamic effect as in the pure KdV case but is in some way enhanced by the reflections of the electrons which amplify the potential well. Both interpretations of Nishihara et al. (1983) and Chanteur et al. (1983) are based on a fluid treatment of the ions. In conclusion, both particle and "Vlasov" simulation results concerning the formation of ion acoustic double layers could be accounted for by a fluid description of the ions until the wave has sufficiently decreased its velocity to have resonant interaction with the core of the ion distribution. It is likely that the ion trapping which occurs then accentuates the slowing down and is the key for understanding the reversal of the motion of the structure, but this last point has still to be checked. It appears that ion trapping is important in the late evolution of the system; in the beginning ion trapping is insignificant and does not explain the slowing down, as recently proposed by Hudson et al. (1983) on the basis of time-stationary models (see also Hudson and Potter, 1981). We are still far from a good theoretical understanding of the space observations related to double layers because the real situation in auroral regions is much more complex, due to the interpenetration of ionospheric and magnetospheric populations of particles. The work of Lotko and Kennel (1981) on rarefactive ion acoustic solitons supported by two electron populations is interesting in this respect. Care must be taken of stationary models invoking ion trapping because —as demonstrated by simulations—the growth of the structures occurs on a time scale comparable to the transit time of the ions through these structures.

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